Journal of Physical Sciences, Vol. 19, 2014, 67-79
ISSN: 2350-0352 (print), www.vidyasagar.ac.in/journal
Published on 26 December 2014

On Almost Partially α–Compact Fuzzy Sets

M.A.M.Talukder[1] and D.M.Ali[2]

[1] Department of Mathematics, Khulna University of Engineering & Technology
Khulna–9203, Bangladesh
E–mail: mottalib_kuet@yahoo.com

[2] Department of Mathematics, University of Rajshahi, Rajshahi–6205, Bangladesh

Received 24August 2014; accepted 1 November 2014

ABSTRACT

In this paper, we introduce and discuss almost partially α–compact fuzzy sets in fuzzy topological spaces. We show that the continuous image of an almost partially α–compact fuzzy set is almost partially α–compact. Also we introduced α–level continuous mapping.

Keywords: Fuzzy topological spaces; almost partially α–compact fuzzy sets.

1. Introduction

Fuzzy compactness is very important in fuzzy topological spaces. Fundamental concept of fuzzy set was first introduced by Zadeh [16]. Chang developed the theory of fuzzy topological spaces and fuzzy compactness [3]. Gantner et al. introduced α–compactness in [6]. Talukder and Ali first introduced partially α–compact fuzzy set in [14]. Many other researchers discussed other fuzzy compactness in their different papers. α–almost compactness was first introduced by Mukherjee and Bhattacharyya [12]. Here we discuss almost partially α–compact fuzzy sets and obtain its several properties.

2. Preliminaries

In this section, we recall some basic definitions which are needed in the sequel. The followings are necessary in our discussion and can be found in the paper referred to.

Definition 2.1. [16] Let X be a non-empty set and I is the closed unit interval [0, 1]. A fuzzy set in X is a function u : X $\to$ I which assigns to every element x$\in$ X. u(x) denotes a degree or the grade of membership of x. The set of all fuzzy sets in X is denoted by I^X. A member of I^X may also be called a fuzzy subset of X.

Definition 2.2. [11] Let λ be a fuzzy set in X , then the set { x $\in$ X : λ(x) > 0 } is called the support of λ and is denoted by λ_0 or suppλ.

Definition 2.3. [3] Let X be a non-empty set and t $\subseteq I^X$ i.e. t is a collection of fuzzy sets in X. Then t is called a fuzzy topology on X if
(i) 0, 1 $\in$ t

(ii) if $u_i \in t$ for each $i \in J$, then $\bigcup_i u_i \in t$

(iii) if u, v ∈ t, then $u \cap v \in t$

The pair (X, t) is called a fuzzy topological space and in short, fts.

Definition 2.4. [11] Let λ be a fuzzy set in an fts (X, t). Then the closure of λ is denoted by $\bar{\lambda}$ and defined by $\bar{\lambda} = \cap\{ \mu : \lambda \subseteq \mu \text{ and } \mu \in t^c \}$.

Definition 2.5. [11] Let λ be a fuzzy set in an fts (X, t). Then the interior of λ is denoted by λ^0 or int λ and defined by $\lambda^0 = \cup\{ \mu : \mu \subseteq \lambda \text{ and } \mu \in t \}$.

Definition 2.6. [3] Let f : X $\to$ Y be a mapping and u be a fuzzy set in X. Then the image of u, written $f(u)$, is a fuzzy set in Y whose membership function is given by

$$f(u)(y) = \begin{cases} \sup\{ u(x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi \end{cases}.$$

Definition 2.7. [8] Let f be a mapping from a set X into Y and v be a fuzzy set of Y. Then the inverse of v written as $f^{-1}(v)$ is a fuzzy set of X and is drfined by $f^{-1}(v)(x) = v(f(x))$, for x∈ X.

Definition 2.8. [3] Let (X, t) and (Y, s) be two fuzzy topological spaces. A mapping $f : (X, t) \to (Y, s)$ is called an fuzzy continuous iff the inverse of each s–open fuzzy set is t–open.

Definition 2.9. [15] Let (X, t) and (Y, s) be two fuzzy topological spaces. Let f : $(X, t) \to (Y, s)$ be a mapping from an fts (X, t) to another fts (Y, s). Then f is called

(i) a fuzzy open mapping iff f (u) ∈ s for each u∈ t.

(ii) a fuzzy closed mapping iff f (v) is a closed fuzzy set of Y, for each closed fuzzy set v of X.

Definition 2.10. [11] Let (X, t) be an fts and $A \subseteq X$. Then the collection $t_A = \{u|A : u \in t\} = \{ u \cap A : u \in t \}$ is fuzzy topology on A, called the subspace fuzzy topology on A and the pair (A, t_A) is referred to as a fuzzy subspace of (X, t).

Definition 2.11. [5] Let (A, t_A) and (B, s_B) be fuzzy subspaces of fts's (X, t) and (Y, s) respectively. Then a mapping $f : (A, t_A) \to (B, s_B)$ is relatively fuzzy continuous iff for each $v \in s_B$, the intersection $f^{-1}(v) \cap A \in t_A$.

Definition 2.12. [2] Let $\lambda \in I^X$ and $\mu \in I^Y$. Then ($\lambda \times \mu$) is a fuzzy set in $X \times Y$ for which ($\lambda \times \mu$) (x , y) = min { λ(x) , μ(y) } , for every (x , y) $\in X \times Y$.

Definition 2.13. [13] An fts (X , t) is said to be fuzzy T_1–space iff for every x, y $\in$ X, x $\neq$ y, there exist u, v $\in$ t such that u(x) = 1, u(y) = 0 and v(x) = 0, v(y) = 1.

Definition 2.14. [4] An fts (X , t) is said to be fuzzy regular iff each open fuzzy set u of X is a union of open fuzzy sets u_i of X such that $\overline{u_i} \subseteq u$ for each i .

Definition 2.15. [6] Let (X , t) be an fts and $\alpha \in$ I. A collection M of fuzzy sets is called an α–shading (res. α^*–shading) of X if for each x $\in$ X there exists a u $\in$ M such that u(x) $> \alpha$ (res. u(x) $\geq \alpha$). A subcollection of an α–shading (res. α^*–shading) of X which is also an α–shading (res. α^*–shading) is called an α–subshading (res. α^*–subshading) of X.

Definition 2.16. [9] Let (X , t) be an fts and $0 \leq \alpha < 1$, then the family t_α = { α(u) : u $\in$ t } of all subsets of X of the form α(u) = { x $\in$ X : u(x) $> \alpha$ } is called α–level sets, forms a topology on X and is called the α–level topology on X and the pair (X , t_α) is called α–level topological space.

Definition 2.17. [14] Let (X , t) be an fts and $\alpha \in$ I. A family M of fuzzy sets is called a partial α–shading (respectively partial α^*–shading), in short, pα–shading (respectively pα^*– shading) of a fuzzy set λ in X if for each x $\in \lambda_0$, ($\lambda_0 \neq$ X) there exists a u $\in$ M with u(x) $> \alpha$ (respectively u(x) $\geq \alpha$). If each u is open, then M is called an open pα–shading (respectively open pα^*–shading) of λ in (X , t).
A subfamily of a pα–shading (respectively pα^*–shading) of λ which is also a pα–shading (respectively pα^*–shading) of λ is called a pα–subshading (respectively pα^*–subshading) of λ.
For definitions of other concepts, used here, cf.[1].

3. Characterizations of almost partially α–compact fuzzy sets
In this section, we define and discuss characterizations of almost partially α–compact fuzzy sets.

Definition 3.1. A family { u_i : i $\in$ J } is a proximate partial α–shading (respectively partial α^*–shading), in short ppα–shading (respectively ppα^*–shading) of a fuzzy

set λ in X when { $\overline{u_i}$: $i \in J$ } is a $p\alpha$–shading (respectively $p\alpha^*$–shading) of λ i.e. $\overline{u_i}$ (x) > α (respectively $\overline{u_i}$ (x) ≥ α) for each $x \in \lambda_0$.
A subfamily of { u_i : $i \in J$ } which is also a $pp\alpha$–shading (respectively $pp\alpha^*$–shading) of λ is said to be $pp\alpha$–subshading (respectively $pp\alpha^*$–subshading) of λ.

Definition 3.2. Let (X, t) be an fts and $\alpha \in I$. A fuzzy set λ is said to be almost partially α–compact, in short $ap\alpha$–compact (respectively almost partially α^*–compact, in short $ap\alpha^*$–compact) iff every open $p\alpha$–shading (respectively $p\alpha^*$–shading) of λ has a finite subfamily whose closure is $p\alpha$–shading (respectively $p\alpha^*$–shading) of λ or equivalently every open $p\alpha$–shading (respectively $p\alpha^*$–shading) of λ has a finite $pp\alpha$–subshading (respectively $pp\alpha^*$–subshading).
The ideas of the theorems 3.3, 3.6, 3.11, 3.12 and 3.13 are taken from [7, 10]

Theorem 3.3. Let (X, t) be an fts, $A \subset X$ and λ be a fuzzy set in X with $\lambda_0 \subseteq A$. Then
(i) λ is $ap\alpha$–compact in (X, t) iff λ is $ap\alpha$–compact in (A, t_A).
(ii) λ is $ap\alpha^*$–compact in (X, t) iff λ is $ap\alpha^*$–compact in (A, t_A).
Proof (i): Suppose λ is $ap\alpha$–compact in (X, t). Let { u_i : $i \in J$ } be an open $p\alpha$–shading of λ in (A, t_A), then { $(\overline{u_i})^0$: $i \in J$ } is also an open $p\alpha$–shading of λ in (A, t_A). So there exists $v_i \in t$ such that $u_i = A \cap v_i \subseteq v_i$. Therefore { v_i : $i \in J$ } is an open $p\alpha$–shading of λ in (X, t). But $\overline{u_i} = \overline{A \cap v_i} \subseteq \overline{A} \cap \overline{v_i} \subseteq \overline{v_i}$, so { $(\overline{v_i})^0$: $i \in J$ } is also an open $p\alpha$–shading of λ in (X, t). Since $(\overline{v_i})^0 \subseteq \overline{v_i}$ and λ is $ap\alpha$–compact in (X, t), then { $(\overline{v_i})^0$: $i \in J$ } has a finite $pp\alpha$–subshading, say { $\overline{v_{i_k}}$: $k \in J_n$ } such that $\overline{v_{i_k}}$ (x) > α for each $x \in \lambda_0$. Now, $(\overline{A} \cap \bigcup_{k \in J_n} \overline{v_{i_k}})(x) > \alpha \Rightarrow \bigcup_{k \in J_n} (\overline{A} \cap \overline{v_{i_k}})(x) > \alpha \Rightarrow \bigcup_{k \in J_n} \overline{u_{i_k}}(x) > \alpha$, as $\lambda_0 \subseteq A$ and hence it shows that { $\overline{u_{i_k}}$: $k \in J_n$ } is a finite $pp\alpha$–subshading of { u_i : $i \in J$ }. Therefore λ is $ap\alpha$–compact in (A, t_A).
Conversely, suppose λ is $ap\alpha$–compact in (A, t_A). Let { v_i : $i \in J$ } be an open $p\alpha$–shading of λ in (X, t), then { $(\overline{v_i})^0$: $i \in J$ } is also an open $p\alpha$–shading of λ in (X, t). Put $u_i = A \cap v_i$, then $A \cap (\bigcup_{i \in J} v_i) = \bigcup_{i \in J} (A \cap v_i) = \bigcup_{i \in J} u_i$. But

$u_i \in t_A$, so $\{ u_i : i \in J \}$ is an open pα–shading of λ in (A, t_A). Therefore $\{ (\overline{u_i})^0 : i \in J \}$ is also an open pα–shading of λ in (A, t_A). As $(\overline{u_i})^0 \subseteq \overline{u_i}$ and λ is apα–compact in (A, t_A), then $\{ (\overline{u_i})^0 : i \in J \}$ has a finite ppα–subshading, say $\{ \overline{u_{i_k}} : k \in J_n \}$ such that $\overline{u_{i_k}}(x) > \alpha$ for each $x \in \lambda_0$. But $\overline{u_i} = \overline{A \cap v_i} \subseteq \overline{A} \cap \overline{v_i} \subseteq \overline{v_i}$, then $\{ \overline{v_{i_k}} : k \in J_n \}$ is a finite ppα–subshading of $\{ v_i : i \in J \}$. Thus λ is apα–compact in (X, t).

(ii) The proof is similar.

Corollary 3.4. Let (Y, t^*) be a fuzzy subspace of an fts (X, t) and $A \subset Y \subset X$ (proper subsets). Let λ be a fuzzy set in X with $\lambda_0 \subseteq$ A. Then λ is apα–compact (respectively apα^*–compact) in (X, t) iff λ is apα–compact (respectively apα^*–compact) in (Y, t^*).

Proof : Let t_A and t_A^* be the subspace fuzzy topologies on A. Then preceding theorem, λ is apα–compact in (X, t) or (Y, t^*) iff λ is apα–compact in (A, t_A) or (A, t_A^*). But $t_A = t_A^*$.

Similar proof of apα^*–compactness can be given.

Theorem 3.5. Let (X, t) and (Y, s) be two fts's and $f : (X, t) \to (Y, s)$ be fuzzy continuous and surjective. Then

(i) If λ is apα–compact in (X, t), then f (λ) is apα–compact in (Y, s).

(ii) If λ is apα^*–compact in (X, t), then f (λ) is apα^*–compact in (Y, s).

Proof (i): Let $\{ u_i : i \in J \}$ be an open pα–shading of f (λ) in (Y, s), then $\{ (\overline{u_i})^0 : i \in J \}$ is also an open pα–shading of f (λ) in (Y, s). As f is fuzzy continuous, then $f^{-1} (\overline{u_i})^0 \in t$ and hence $\{ f^{-1} (\overline{u_i})^0 : i \in J \}$ is an open pα–shading of λ in (X, t). For, let $x \in \lambda_0$, then f (x) $\in$ f (λ_0). So there exists $(\overline{u_{i_0}})^0 \in \{ (\overline{u_i})^0 : i \in J \}$ such that $(\overline{u_{i_0}})^0 (f(x)) > \alpha \Rightarrow f^{-1} (\overline{u_{i_0}})^0 (x) > \alpha$. As λ is apα–compact, then $\{ f^{-1} (\overline{u_i})^0 : i \in J \}$ has a finite subfamily, say $\{ f^{-1} (\overline{u_{i_k}})^0 : k \in J_n \}$ such that $\overline{f^{-1} (\overline{u_{i_k}})^0}(x) > \alpha$ for each $x \in \lambda_0$. But $(\overline{u_i})^0 \subseteq \overline{u_i}$ and fuzzy continuity of f, $f^{-1} (\overline{u_i})$ must be a closed fuzzy set in X such that $f^{-1} (\overline{u_i})^0 \subseteq f^{-1} (\overline{u_i})$ and then $\overline{f^{-1} (\overline{u_i})^0} \subseteq f^{-1} (\overline{u_i})$. Therefore f ($\overline{f^{-1} (\overline{u_i})^0}$) $\subseteq \overline{u_i}$ for each $i \in J$. Now, if $y \in f(\lambda_0)$, then y = f (x) for some $x \in \lambda_0$, as f is surjective. So there exists k such that $f^{-1}\left(\overline{u_{i_k}}\right)(x) > \alpha \Rightarrow \overline{u_{i_k}}(f(x)) > \alpha \Rightarrow \overline{u_{i_k}}(y) > \alpha$. Hence $f(\lambda)$ is apα–compact in (Y, s).

(ii) The proof is similar.

Theorem 3.6. Let (X, t) and (Y, s) be two fts's and $f : (X, t) \to (Y, s)$ be open, closed and bijective . Then

(i) If λ is apα–compact in (Y, s), then $f^{-1}(\lambda)$ is apα–compact in (X, t).

(ii) If λ is apα^*–compact in (Y, s), then $f^{-1}(\lambda)$ is apα^*–compact in (X, t).

Proof (i): Let $\{u_i : i \in J\}$ be an open pα–shading of $f^{-1}(\lambda)$ in (X, t), then $\{\left(\overline{u_i}\right)^0 : i \in J\}$ is also an open pα–shading of $f^{-1}(\lambda)$ in (X, t). Since f is open, then $f\left(\overline{u_i}\right)^0 \in s$ and hence $\{f\left(\overline{u_i}\right)^0 : i \in J\}$ is an open pα–shading of λ in (Y, s). For, let $y \in \lambda_0$, then $f^{-1}(y) \in f^{-1}(\lambda_0)$. So there exists $\left(\overline{u_{i_0}}\right)^0 \in \{\left(\overline{u_i}\right)^0 : i \in J\}$ such that $\left(\overline{u_{i_0}}\right)^0 (f^{-1}(y)) > \alpha \Rightarrow f\left(\overline{u_{i_0}}\right)^0 (y) > \alpha$. As λ is apα–compact in (Y, s), then $\{f\left(\overline{u_i}\right)^0 : i \in J\}$ has a finite subfamily, say $\{f\left(\overline{u_{i_k}}\right)^0 : k \in J_n\}$ such that $\overline{f\left(\overline{u_{i_k}}\right)^0}(y) > \alpha$ for each $y \in \lambda_0$. But $\left(\overline{u_i}\right)^0 \subseteq \overline{u_i}$ and f is closed, $f\left(\overline{u_i}\right)$ must be a closed fuzzy set in Y such that $f\left(\overline{u_i}\right)^0 \subseteq f\left(\overline{u_i}\right)$ and then $\overline{f\left(\overline{u_i}\right)^0} \subseteq f\left(\overline{u_i}\right)$. Therefore $f^{-1}(\overline{f\left(\overline{u_i}\right)^0}) \subseteq \overline{u_i}$ for each $i \in J$. For, if $x \in f^{-1}(\lambda_0)$, then $x = f^{-1}(y)$ for some $y \in \lambda_0$, as f is bijective. So we can obtain, there exists k such that $f\left(\overline{u_{i_k}}\right)(y) > \alpha \Rightarrow \overline{u_{i_k}}(f^{-1}(y)) > \alpha \Rightarrow \overline{u_{i_k}}(x) > \alpha$. Hence $f^{-1}(\lambda)$ is apα–compact in (X, t).

(ii) The proof is similar.

Theorem 3.7. Let $f : (A, t_A) \to (B, s_B)$ be relatively fuzzy continuous and bijective. If λ is apα–compact (respectively apα^*–compact) in (A, t_A), then $f(\lambda)$ is apα–compact (respectively apα^*–compact) in (B, s_B).

Proof : We have f (A) = B, as f is bijective. Let $\{v_i : v_i \in s_B\}$ be an open pα–shading of $f(\lambda)$ in (B, s_B) for every $i \in J$, then $\{\left(\overline{v_i}\right)^0 : v_i \in s_B\}$ is also an open pα–shading of $f(\lambda)$ in (B, s_B) for every $i \in J$. Since f is fuzzy relatively continuous, then $f^{-1}(\left(\overline{v_i}\right)^0) \cap A \in t_A$ and hence $\{f^{-1}(\left(\overline{v_i}\right)^0) \cap A : i \in J\}$ is an open pα–shading of λ in (A, t_A). For, let $x \in \lambda_0$, then $f(x) \in f(\lambda_0)$. So there exists $\left(\overline{v_{i_k}}\right)^0 \in \{\left(\overline{v_i}\right)^0 : i \in J\}$ ($k \in J_n$) such that $\left(\overline{v_{i_k}}\right)^0 (f(x)) > \alpha \Rightarrow (f^{-1}(\left(\overline{v_{i_k}}\right)^0) \cap A)(x) > \alpha$. But λ is apα–compact in (A , t_A), then $\{ f^{-1}((\overline{v_i})^0) \cap A : i \in J \}$ has a finite subfamily, say $\{ f^{-1}((\overline{v_{i_k}})^0) \cap A : k \in J_n \}$ such that $\overline{f^{-1}((\overline{v_{i_k}})^0) \cap A}$ (x) $> \alpha$ for each $x \in \lambda_0$. Since $(\overline{v_i})^0 \subseteq \overline{v_i}$ and fuzzy relatively continuity of f, $f^{-1}(\overline{v_i}) \cap A$ must be a closed fuzzy set in A such that $f^{-1}((\overline{v_i})^0) \cap A \subseteq f^{-1}(\overline{v_i}) \cap A$ and then $\overline{f^{-1}((\overline{v_i})^0) \cap A} \subseteq f^{-1}(\overline{v_i}) \cap A$. Therefore $f(\overline{f^{-1}((\overline{v_i})^0) \cap A}) \subseteq \overline{v_i} \cap f(A) = \overline{v_i} \cap B \subseteq v_i$ for every $i \in J$. Now, if $y \in f(\lambda_0)$, then y = f (x) for $x \in \lambda_0$, as f is bijective. Then there exists k such that $\overline{v_{i_k}}(f(x)) > \alpha \Rightarrow \overline{v_{i_k}}(y) > \alpha$ for each $y \in f(\lambda_0)$. Thus $f(\lambda)$ is apα–compact in (B , s_B).

Similar proof of apα^*–compactness can be done.

Definition 3.8. A mapping $f : (X, t_\alpha) \to (X, t)$ is said to be α–level continuous iff $\alpha(f^{-1}(\mu)) \in t_\alpha$ for every $\mu \in t$.

Theorem 3.9. Let $f : (X, t_\alpha) \to (X, t)$ be α–level continuous, bijective and λ be a fuzzy set in in X. If λ_0 is compact in (X, t_α), then $f(\lambda)$ is apα–compact in (X, t).

Proof: Suppose λ_0 is compact in (X, t_α). Let $\{ u_i : i \in J \}$ be an open pα–shading $f(\lambda)$ in (X, t), then $\{ (\overline{u_i})^0 : i \in J \}$ is also an open pα–shading of $f(\lambda)$ in (X, t). Since f is α–level continuous, then $\alpha(f^{-1}(u_i)) \in t_\alpha \Rightarrow \alpha(f^{-1}((\overline{u_i})^0)) \in t_\alpha$ and hence $\{ \alpha(f^{-1}((\overline{u_i})^0)) : i \in J \}$ is an open cover of λ_0 in (X, t_α). As λ_0 is compact in (X, t_α), then $\{ \alpha(f^{-1}((\overline{u_i})^0)) : i \in J \}$has a finite subcover, say $\{ \alpha(f^{-1}((\overline{u_{i_k}})^0)) : k \in J_n \}$. Now, we have f (x) = y foe some $y \in f(\lambda_0)$, as f is bijective. But from $(\overline{u_i})^0 \subseteq \overline{u_i}$ and $\{ \alpha(f^{-1}((\overline{u_{i_k}})^0)) : k \in J_n \}$ is a finite subcober of $\{ \alpha(f^{-1}((\overline{u_i})^0)) : i \in J \}$, there exist some k such that $\overline{u_{i_k}}(f(x)) > \alpha \Rightarrow \overline{u_{i_k}}(y) > \alpha$ for each $y \in f(\lambda_0)$. Thus $\{ \overline{u_{i_k}} : k \in J_n \}$ is a finite ppα–subshading of $\{ u_i : i \in J \}$. Therefore is $f(\lambda)$ is apα–compact in (X, t).

The following example will show that the apα–compact (respectively apα^*–compact) fuzzy set in an fts need not be closed.

Example 3.10. Let X = { a, b, c }, I = [0 , 1] and $\alpha \in I_1$. Let u, v $\in I^X$ defined by u(a) = 0.3, u(b) = 0.2, u(c) = 0.4 and v(a) = 0.4, v(b) = 0.5, v(c) = 0.6. Choose t = { 0, u, v, 1 }, then (X, t) is an fts. So we have $\overline{u}$(a) = 0.6, $\overline{u}$ (b) = 0.5, $\overline{u}$(c) = 0.4 and $\overline{v}$(a) = 0.7, $\overline{v}$(b) = 0.8, $\overline{v}$(c) = 0.6. Again, let $\lambda \in I^X$ with λ(a) = 0, λ(b) = 0.9, λ(c) = 0.7. Take α = 0.4. Then clearly λ is apα–compact in (X, t). But λ is not closed, as its complement λ^c is not open in (X, t).
This example is also applicable for apα^*–compactness.

Theorem 3.11. Let (X, t) be an fts and let every family of closed fuzzy sets in X with empty intersection has a finite subfamily with empty intersection. Then any fuzzy set λ in X is apα–compact (respectively apα^*–compact). The converse is not true in general.
Proof : Let λ be any fuzzy set in X and let { u_i : $i \in J$ } be an open pα – shading of λ, then { $(\overline{u_i})^0$: $i \in J$ } is also an open pα–shading of λ. By the first condition of the theorem, we have $\bigcap_{i\in J} u_i^c = 0_X$. Therefore $\bigcup_{i\in J} u_i = 1_X$ and hence $\bigcup_{i\in J} (\overline{u_i})^0 = 1_X$. Again, by the second condition of the theorem, $\bigcap_{k\in J_n} u_{i_k}^c = 0_X$. So, we have $\bigcup_{k\in J_n} u_{i_k} = 1_X$ and hence $\bigcup_{k\in J_n} (\overline{u_{i_k}})^0 = 1_X$. But from $u_i \subseteq (\overline{u_i})^0 \subseteq \overline{u_i}$, then we get $\bigcup_{k\in J_n} \overline{u_{i_k}} = 1_X$ and consequently we have $\overline{u_{i_k}}$ (x) $> \alpha$ for each x $\in \lambda_0$. Therefore { $\overline{u_{i_k}}$: $k \in J_n$ } is a finite ppα–subshading of { u_i : $i \in J$ }. Hence λ is apα–compact. The proof for apα^*–compactness is similar.
Now consider the following example.
Let X = { a, b, c }, I = [0 , 1] and $\alpha \in I_1$. Let u, v $\in I^X$ defined by u(a) = 0.3, u(b) = 0.2, u(c) = 0.4 and v(a) = 0.4, v(b) = 0.3, v(c) = 0.5. Choose t ={ 0, u, v, 1 }, then (X, t) is an fts. So we have $\overline{u}$(a) = 0.6, $\overline{u}$ (b) = 0.7, $\overline{u}$(c) = 0.5 and $\overline{v}$(a) = 0.6, $\overline{v}$(b) = 0.7, $\overline{v}$(c) = 0.5. Again, let $\lambda \in I^X$ with λ(a) = 0, λ(b) = 0.3, λ(c) = 0.8. Take α = 0.2. Then clearly λ is apα–compact in (X, t). But $u^c \cap v^c \neq$ 0. Therefore the converse is not true in general. This example also works for apα^*–compactness.

Theorem 3.12. Let (X, t) be fuzzy T_1–space and λ be an apα–compact (respectively apα^*–compact) fuzzy set in X with $\lambda_0 \subset X$ (proper subset). Let x $\notin \lambda_0$ (λ(x) = 0), then there exist u, v $\in$ t such that $\overline{u}$(x) = 1 and $\lambda_0 \subseteq (\overline{v})^{-1}$ (0 , 1].

Proof : Let $y \in \lambda_0$. So clearly we have $x \neq y$. As (X, t) is fuzzy T_1–space, there exist u_y, $v_y \in t$ such that $u_y(x) = 1$, $u_y(y) = 0$ and $v_y(x) = 0$, $v_y(y) = 1$. Let us assume that $\alpha \in I_1$ such that $v_y(y) > \alpha > 0$ (as $v_y(y) = 1$). Thus we see that $\{ v_y : y \in \lambda_0 \}$ is an open $p\alpha$–shading of λ. Also we have $(\overline{u_y})^0(x) = 1$, $(\overline{v_y})^0(y) = 1$, then $\{ (\overline{v_y})^0 : y \in \lambda_0 \}$ is also an open $p\alpha$–shading of λ. Since λ is $ap\alpha$–compact, then $\{ (\overline{v_y})^0 : y \in \lambda_0 \}$ has a finite $pp\alpha$–subshading, say $\{ \overline{v_{y_k}} : k \in J_n \}$ such that $\overline{v_{y_k}}(y) > \alpha$ for each $y \in \lambda_0$. Now, let $(\overline{v})^0 = (\overline{v_{y_1}})^0 \cup (\overline{v_{y_2}})^0 \cup \ldots \cup (\overline{v_{y_n}})^0$ and $(\overline{u})^0 = (\overline{u_{y_1}})^0 \cap (\overline{u_{y_2}})^0 \cap \ldots \cap (\overline{u_{y_n}})^0$. Hence $(\overline{v})^0$ and $(\overline{u})^0$ are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. $(\overline{v})^0$, $(\overline{u})^0 \in t$. But we have $(\overline{v_y})^0 \subseteq \overline{v_y}$, $(\overline{u_y})^0 \subseteq \overline{u_y}$. Moreover, $\lambda_0 \subseteq (\overline{v})^{-1}(0, 1]$ and $\overline{u}(x) = 1$, as $\overline{u_{y_k}}(x) = 1$ for each k.
The proof of $ap\alpha^*$–compactness can similarly be done.

Theorem 3.13. Let (X, t) be fuzzy T_1–space and λ, μ be disjoint ($\lambda \cap \mu = 0$) $ap\alpha$–compact (respectively $ap\alpha^*$–compact) fuzzy sets in X with λ_0, $\mu_0 \subset X$ (proper subsets). Then there exist $u, v \in t$ such that $\lambda_0 \subseteq (\overline{u})^{-1}(0, 1]$ and $\mu_0 \subseteq (\overline{v})^{-1}(0, 1]$.

Proof : Let $y \in \lambda_0$. Then we have $y \notin \mu_0$, as λ and μ are disjoint. As μ is $ap\alpha$–compact, then by preceding theorem, there exist u_y, $v_y \in t$ such that $\overline{u_y}(y) = 1$ and $\mu_0 \subseteq (\overline{v_y})^{-1}(0, 1]$. Assume that $\alpha \in I_1$ such that $\overline{u_y}(y) > \alpha > 0$. Since $\overline{u_y}(y) = 1$, then we have $\{ (\overline{u_y})^0 : y \in \lambda_0 \}$ is an open $p\alpha$–shading of λ. But λ is $ap\alpha$–compact, so $\{ (\overline{u_y})^0 : y \in \lambda_0 \}$ has a finite $pp\alpha$–subshading, say $\{ \overline{u_{y_k}} : k \in J_n \}$ such that $\overline{u_{y_k}}(y) > \alpha$ for each $y \in \lambda_0$. Again, μ is $ap\alpha$–compact, then $\{ (\overline{v_y})^0 : x \in \mu_0 \}$ has a finite $pp\alpha$– subshading, say $\{ \overline{v_{y_k}} : k \in J_n \}$ such that $\overline{v_{y_k}}(x) > \alpha$ for each $x \in \mu_0$ and $\mu_0 \subseteq (\overline{v_{y_k}})^{-1}(0, 1]$ for each k. Now, let $(\overline{u})^0 = (\overline{u_{y_1}})^0 \cup (\overline{u_{y_2}})^0 \cup \ldots \cup (\overline{u_{y_n}})^0$ and $(\overline{v})^0 = (\overline{v_{y_1}})^0 \cap (\overline{v_{y_2}})^0 \cap \ldots \cap (\overline{v_{y_n}})^0$. But from $(\overline{u_y})^0 \subseteq \overline{u_y}$ and $(\overline{v_y})^0 \subseteq \overline{v_y}$, we see that $\lambda_0 \subseteq (\overline{u})^{-1}(0, 1]$ and $\mu_0 \subseteq (\overline{v})^{-1}(0, 1]$. Also $(\overline{u})^0$ and $(\overline{v})^0$

are open fuzzy sets, as they are the union and finite intersection of open fuzzy sets respectively i.e. $(\overline{u})^0$, $(\overline{v})^0 \in t$.

Similar proof of $ap\alpha^*$–compactness can be given.

Note : If $\lambda(x) \neq 0$ for all $x \in X$ i.e. $\lambda_0 = X$, then the above two theorems are not at all true.

The following example will show that the $ap\alpha$–compact (respectively $ap\alpha^*$–compact) fuzzy set in fuzzy T_1–space need not be closed.

Example 3.14. Let X = { a, b, c }, I = [0 , 1] and $\alpha \in I_1$. Let $u_1, u_2, u_3, u_4, u_5, u_6 \in I^X$ defined by $u_1(a) = 1$, $u_1(b) = 0$, $u_1(c) = 0$; $u_2(a) = 0$, $u_2(b) = 1$, $u_2(c) = 0$; $u_3(a) = 0$, $u_3(b) = 0$, $u_3(c) = 1$; $u_4(a) = 1$, $u_4(b) = 1$, $u_4(c) = 0$; $u_5(a) = 1$, $u_5(b) = 0$, $u_5(c) = 1$; $u_6(a) = 0$, $u_6(b) = 1$, $u_6(c) = 1$. Put t = { 0, u_1, u_2, u_3, u_4, u_5, u_6, 1 }, then (X, t) is a fuzzy T_1 – space. So we have $\overline{u_1}(a) = 1$, $\overline{u_1}(b) = 0$, $\overline{u_1}(c) = 0$; $\overline{u_2}(a) = 0$, $\overline{u_2}(b) = 1$, $\overline{u_2}(c) = 0$; $\overline{u_3}(a) = 0$, $\overline{u_3}(b) = 0$, $\overline{u_3}(c) = 1$; $\overline{u_4}(a) = 1$, $\overline{u_4}(b) = 1$, $\overline{u_4}(c) =0$; $\overline{u_5}(a) = 1$, $\overline{u_5}(b) =0$, $\overline{u_5}(c) = 1$ and $\overline{u_6}(a) = 0$, $\overline{u_6}(b) = 1$, $\overline{u_6}(c) =1$. Again, let $\lambda \in I^X$ with $\lambda(a) = 0.3$, $\lambda(b) = 0.6$, $\lambda(c) = 0$. Take $\alpha = 0.8$. Then clearly λ is $ap\alpha$–compact in (X, t). But λ is not closed, as its complement λ^c is not open in (X, t). This example is also applicable for $ap\alpha^*$–compactness.

Theorem 3.15. An $ap\alpha$–compact (respectively $ap\alpha^*$–compact) fuzzy set in a fuzzy regular space is $p\alpha$–compact (respectively $p\alpha^*$–compact).

Proof : Let $\{ u_i : i \in J \}$ be an open $p\alpha$–shading of a fuzzy set λ in X i.e. $u_i(x) > \alpha$ for each $x \in \lambda_0$. Since (X, t) is fuzzy regular, then we have $u_i = \bigcup_{i \in J} v_{ij}$, where, v_{ij} is a open fuzzy set such that $\overline{v_{ij}} \subseteq u_i$ for each i. But $u_i(x) > \alpha \Rightarrow \bigcup_{i \in J} v_{ij}(x) > \alpha$ for each $x \in \lambda_0$. Therefore $v_{ij}(x) > \alpha$ for each $x \in \lambda_0$ and for some $i \in J$. So $\{ v_{ij} : i \in J \}$ is an open $p\alpha$–shading of λ. As λ is $ap\alpha$–compact, then $\{ v_{ij} : i \in J \}$ has a finite $pp\alpha$– subshading, say $\{ \overline{v_{i_k j}} : k \in J_n \}$ such that $\overline{v_{i_k j}}(x) > \alpha$ for each $x \in \lambda_0$. But we have $\overline{v_{i_k j}} \subseteq u_{i_k}$, then $u_{i_k}(x) > \alpha$ for each $x \in \lambda_0$. Thus we see that $\{ u_{i_k} : k \in J_n \}$ is a finite $p\alpha$–subshading of $\{ u_i : i \in J \}$ and hence λ $p\alpha$–compact.

Similar proof of $ap\alpha^*$–compactness can be given.

Theorem 3.16. Let λ and μ be apα–compact fuzzy sets in an fts (X, t). Then $(\lambda \times \mu)$ is also apα–compact in $(X\times X, t\times t)$.

Proof: Let $\{ u_i \times v_i : i \in J \}$ be an open pα– shading of $(\lambda \times \mu)$ in $(X\times X, t\times t)$ i.e. $(u_i \times v_i)(x, y) > \alpha$ for each $(x, y) \in (\lambda \times \mu)_0$. Therefore, we have $u_i(x) > \alpha$ for each $x \in \lambda_0$ and $v_i(y) > \alpha$ for each $y \in \mu_0$. Hence $\{ u_i : i \in J \}$ and $\{ v_i : i \in J \}$ are open pα– shading of λ and μ respectively. Thus $\{ (\overline{u_i})^0 : i \in J \}$ and $\{ (\overline{v_i})^0 : i \in J \}$ are also open pα–shading of λ and μ respectively. Now we have $(\overline{u_i})^0 \subseteq \overline{u_i}$ and $(\overline{v_i})^0 \subseteq \overline{v_i}$. As λ and μ are apα–compact, then $\{ (\overline{u_i})^0 : i \in J \}$ and $\{ (\overline{v_i})^0 : i \in J \}$ have finite ppα–subshading, say $\{ \overline{u_{i_k}} : k \in J_n \}$ and $\{ \overline{v_{i_r}} : r \in J_n \}$ such that $\overline{u_{i_k}}(x) > \alpha$ for each $x \in \lambda_0$ and $\overline{v_{i_r}}(y) > \alpha$ for each $y \in \mu_0$ respectively. Hence we can write $(\overline{u_{i_k}} \times \overline{v_{i_r}})(x, y) > \alpha$ for each $(x, y) \in (\lambda \times \mu)_0$. Therefore $(\lambda \times \mu)$ is apα–compact in $(X\times X, t\times t)$.

Theorem 3.17. Let (X, T) be a topological space, $(X, \omega(T))$ be an fts and λ be a fuzzy set in X with $\lambda_0 \subset X$ (proper subset). If λ_0 is compact in (X, T), then λ is apα–compact (respectively apα^*–compact) in $(X, \omega(T))$. The converse is not true in general.

Proof: Suppose λ_0 is compact in (X, T). Let $\{ u_i : i \in J \}$ be an open pα–shading of λ in $(X, \omega(T))$, then $\{ (\overline{u_i})^0 : i \in J \}$ is also an open pα–shading of λ in $(X, \omega(T))$. Thus we have $u_i^{-1}(a, 1] \in T$ and hence $\{ u_i^{-1}(a, 1] : i \in J \}$ is an open cover of λ_0 in (X, T). As λ_0 is compact in (X, T), then $\{ u_i^{-1}(a, 1] : i \in J \}$ has a finite subcover, say $\{ u_{i_k}^{-1}(a, 1] : k \in J_n \}$ such that $\lambda_0 \subseteq u_{i_1}^{-1}(a, 1] \cup u_{i_2}^{-1}(a, 1] \cup \ldots.. \cup u_{i_n}^{-1}(a, 1]$. But $u_i \subseteq (\overline{u_i})^0 \subseteq \overline{u_i}$, thus we observe that there exists $(\overline{u_{i_k}})^0 \in \{ (\overline{u_i})^0 : i \in J \}$ $(k \in J_n)$ such that $\overline{u_{i_k}}(x) > \alpha$ for each $x \in \lambda_0$. Hence it is clear that $\{ \overline{u_{i_k}} : k \in J_n \}$ is a finite ppα–subshading of $\{ u_i : i \in J \}$. Hence λ is apα–compact in $(X, \omega(T))$.

Now, we consider the following example.

Let X = { a, b, c } and T = { ϕ, {b}, {c}, {b, c}, X }. Then (X, T) is a topological space. Let $u_1, u_2, u_3 \in I^X$ with $u_1(a) = 0$, $u_1(b) = 0.3$, $u_1(c) = 0$; $u_2(a) = 0$, $u_2(b) = 0$, $u_2(c) = 0.4$ and $u_3(a) = 0$, $u_3(b) = 0.3$, $u_3(c) = 0.4$. Then $\omega(T) = \{ 0, u_1, u_2, u_3, 1 \}$ and $(X, \omega(T))$ is an fts. Again, let $\lambda \in I^X$ defined by $\lambda(a) = 0.2$, $\lambda(b) = 0.3$,

λ(c) = 0. Then we have λ_0 = { a, b }. Again, we have $\overline{u_1}$(a) = 1, $\overline{u_1}$(b) = 0.7, $\overline{u_1}$(c) = 0.6 ; $\overline{u_2}$(a) = 1, $\overline{u_2}$(b) = 0.7, $\overline{u_2}$(c) = 0.6 and $\overline{u_3}$(a) = 1, $\overline{u_3}$(b) = 0.7, $\overline{u_3}$(c) = 0.6. Take α = 0.3. Then clearly λ is apα–compact in $(X, \omega(T))$. But λ_0 is not compact in (X, T).

Similar proof of apα^*–compactness can be given.

Theorem 3.18. Let (X, t) be an fts and λ be a fuzzy set in X with $\lambda_0 \subset X$ (proper subset). If λ_0 is compact in (X, t_α), then λ is apα–compact in (X, t). The converse is not true in general.

Proof: Suppose λ_0 is compact in (X, t_α). Let { u_i : i $\in$ J } be an open pα–shading λ in (X, t), then { $(\overline{u_i})^0$: i $\in$ J } is also an open pα–shading of λ in (X, t). So the family { $\alpha((\overline{u_i})^0)$: i $\in$ J } is an open cover of λ_0 in (X, t_α). For let x $\in \lambda_0$, so there exists a $(\overline{u_{i_0}})^0 \in$ { $(\overline{u_i})^0$: i $\in$ J } such that $(\overline{u_{i_0}})^0(x) > \alpha$. Hence x $\in \alpha((\overline{u_{i_0}})^0)$ and thus $\alpha((\overline{u_{i_0}})^0) \in$ { $\alpha((\overline{u_i})^0)$: i $\in$ J }. But λ_0 is compact in (X, t_α), so { $\alpha((\overline{u_i})^0)$: i $\in$ J } has a finite subcover, say { $\alpha((\overline{u_{i_k}})^0)$: k $\in$ J_n }. So { $\alpha((\overline{u_{i_k}})^0)$: k $\in$ J_n } forms a finite subfamily of { $(\overline{u_i})^0$: i $\in$ J } such that $\overline{u_{i_k}}(x) > \alpha$ for each x $\in \lambda_0$ i.e. { $\overline{u_{i_k}}$: k $\in$ J_n } is a finite ppα–subshading of { u_i : i $\in$ J }. Hence λ is apα–compact in (X, t).

Conversely, let X = { a, b, c }, I = [0 , 1] and $\alpha \in I_1$. Let u, v $\in I^X$ defined by u(a) = 0.2, u(b) = 0.3, u(c) = 0.4 and v(a) = 0.3, v(b) = 0.4, v(c) = 0.5. Put t = { 0, u, v, 1 }, then (X, t) is an fts. So we have $\overline{u}$(a) = 0.7, $\overline{u}$ (b) = 0.6, $\overline{u}$(c) = 0.5 and $\overline{v}$(a) = 0.7, $\overline{v}$(b) = 0.6, $\overline{v}$(c) = 0.5. Again, let $\lambda \in I^X$ with λ(a) = 0, λ(b) = 0.6, λ(c) = 0.8. Take α = 0.4. Then clearly λ is apα–compact in (X, t). Again, we have λ_0 = { b, c } and $t_{0.4}$ ={ ϕ,{c}, X }. Then it is clear that λ_0 is not compact in $(X, t_{0.4})$.

REFERENCES

1. D.M. Ali, On Certain Separation and Connectedness Concepts in Fuzzy Topology, Ph. D. Thesis, Banaras Hindu University, Varanasi, 1990.
2. K.K.Azad, On fuzzy semi–continuity, fuzzy almost continuity and fuzzy weakly continuity, *J. Math. Anal. Appl.*, 82(1981) 14–32.
3. C.L.Chang, Fuzzy topological spaces, *J. Math. Anal. Appl.*, 24 (1968) 182–190.
4. A. Di Concilio and G. Gerla, Almost compactness in fuzzy topological spaces, *Fuzzy Sets and Systems*, 13 (1984) 187–192.

5. D.H. Foster, Fuzzy topological groups, *J. Math. Anal. Appl.*, 67 (1979) 549–564.
6. T.E. Gantner, R.C. Steinlage and R.H. Warren, Compactness in fuzzy topological spaces, *J. Math. Anal. Appl.*, 62 (1978) 547–562.
7. I.M. Hanafy, Fuzzy β–compactness and fuzzy β–closed spaces, *Turk J. Math*, 28 (2004) 281–293.
8. M.S.Hossain and D.M.Ali, Fuzzy normality and some of its α-forms, *J. Phy. Sciences*, 14 (2010) 87–94 .
9. A.J.Klein, α–closure in fuzzy topology, *Rocky Mountain Journal of Mathematics*, 11 (1981) 553–560.
10. S. Lipschutz, Theory and Problems of General Topology, Schaum's Outline Series, McGraw-Hill Book Publication Company, Singapore, 1965.
11. P.P. Ming and L. Y. Ming, Fuzzy topology i. neighborhood structure of a fuzzy point and moore–smith convergence, *J. Math. Anal. Appl.*, 76 (1980) 571–599.
12. M.N. Mukherjee and A. Bhattacharyya, α–almost compactness for crisp subsets in a fuzzy topological spaces, *J. Fuzzy Math*, 11(1) (2003) 105–113.
13. R.Srivastava, S.N.Lal and A.K.Srivastava, On fuzzy T_0 and R_0 topological spaces, *J. Math. Anal. Appl.*, 136 (1988) 66–73.
14. M.A.M. Talukder and D.M. Ali, Certain features of partially α–compact fuzzy sets, *J. Mech. Cont. & Math. Sci.*, (Accepted).
15. C.K.Wong, Fuzzy topology: product and quotient theorems, *J. Math. Anal. Appl.*, 45 (1974) 512–521.
16. L.A.Zadeh, Fuzzy sets, *Information and Control*, 8 (1965) 338–353.