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Disjunctive Nearlattices and Semi-Boolean Algebras

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ABSTRACT

A distributive nearlattice S with 0 is *disjunctive* if $0 \le a < b$ implies the existence of $x \in S$ such that $x \wedge a = 0$ and $0 < x \le b$. A nearlattice S with 0 is *Semi-Boolean* if it is distributive and the interval $[0, x]$ is complemented for each $x \in S$.
In this paper , we establish the following fundamental results :
When S is a distributive nearlattice with a central element n, then $P_n(S)$ is disjunctive if and only if each dense n-ideal J is both join and meet-dense which is equivalent to the condition that the n-kernel of each skeletal congruence is an annihilator n-ideal. $P_n(S)$ is semi-Boolean if and only if for each n-ideal J, $(J^+) = (J)^*$ when n is a central element of S. When S is a distributive nearlattice with a central element n, $P_n(S)$ is semi-Boolean if and only if the map $\Theta \to Ker_n\Theta$ is a lattice isomorphism of $SC(S)$ onto $K_nSC(S)$ whose inverse is the map $J \to \Theta(J)$, J is an n-ideal of S.

Keywords: n-Kernels of a congruence, Dense subset, Disjunctive nearlattice, ssSemi-Boolean nearlattice.

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1. Introduction

Skeletal congruences on distributive lattices have been studied by Cornish[3]. Then Latif in [6] studied the n- Kernels of skeletal congruences on a distributive lattice. Disjunctive (sectionally semicomplemented) lattices have been studied by many authors including [3], Then [9] has extended the concept for nearlattices. On the other hand Latif in [6] has generalized the results of [3] for n-ideals in lattices. In this paper we have extended and generalized those results for nearlattices.

A nearlattice S is a meet semilattice with the property that any two elements possessing a common upper bound, have a supremum. Nearlattice S is distributive if for all $x, y, z \in S$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ provided $y \vee z$ exists. For detailed literature on nearlattices and its congruences and ideals we refer the reader to [7], [8] and [9]. Here $C(S)$ denotes the lattice of congruences of S. For any $\Theta \in C(S)$, Θ^* denotes the pseudocomplement of Θ. For a nearlattice S, we define the *skeleton*

$$SC(S) = \{\Theta \in C(S) : \Theta = \Phi^* \text{ for some } \Phi \in C(S)\}$$
$$= \{\Theta \in C(S) : \Theta = \Theta^{**}\}$$

The pseudocomplement J^* of an ideal J is the *annihilator ideal*

$$J^* = \{x \in S : x \wedge j = 0 \quad \textit{for all} \quad j \in J\}.$$

The kernel of congruence Θ

$$Ker\Theta = \{x \in S : x \equiv 0\Theta\}.$$

For an ideal J of a distributive nearlattice S, we define $\Theta(J)$ by $x \equiv y\Theta(J)$ if and only if $(x] \vee J = (y] \vee J$,which is the smallest congruence of S containing J as a class.

Of course $Ker\Theta(J) = J$.

For a fixed element $n \in S$, a convex subnearlattice of S containing n is called an n-ideal. For detailed literature on n-ideals see [2].

An element s of a nearlattice S is called *standard* if for all $t, x, y \in S$,

$t \wedge [(x \wedge y) \vee (x \wedge s)] = (t \wedge x \wedge y) \vee (t \wedge x \wedge s)$.

The element s is called *neutral* if

(i) s is standard and

(ii) for all $x, y, z \in S$, $s \wedge [(x \wedge y) \vee (x \wedge z)] = (s \wedge x \wedge y) \vee (s \wedge x \wedge z)$.

An element n of a nearlattice S is called *medial* if $m(x, n, y) = (x \wedge y) \vee (x \wedge n) \vee (y \wedge n)$ exists in S for all $x, y \in S$. An element n in a nearlattice S is called *sesquimedial* if for all $x, y, z \in S$, $\big([(x \wedge n) \vee (y \wedge n)] \wedge [(y \wedge n) \vee (z \wedge n)]\big) \vee (x \wedge y) \vee (y \wedge z)$ exists in S. An element n of a nearlattice S is called an *upper element* if $x \vee n$ exists for all $x \in S$. Every upper element is of course a sesquimedial element. An element n is called a *central element* of S if it is neutral, upper and complemented in each interval containing it.

When n is a medial element, then for any n-ideal J of a distributive nearlattice S,
we define

$$J^+ = \{x \in S : m(x, n, j) = n \quad \textit{for all} \quad j \in J\}.$$

Obviously J^+ is an n-ideal which we call, the annihilator n-ideal of J. We define the n-kernel of a congruence Θ by $Ker_n\Theta = \{x \in S : x \equiv n\Theta\}$, which is clearly an n-ideal.

Skeletal congruences in lattices have been studied by [3]. Then [9] have extended those results for nearlattices. Recently [1] have generalized some of their results for n-ideals.

$\Theta \in C(S)$ is called dense if $\Theta^* = \omega$, while an n-ideal J is called dense if $J^+ = \{n\}$. A non-empty subset T of a nearlattice S is called join-dense if each $z \in S$ is the join of its predecessors in T, while T is called meet-dense if each $z \in S$ is the meet of its successors in T.

A distributive nearlattice S with 0 is called *disjunctive* if $0 \le a < b$ implies the existence of $x \in S$ such that $x \wedge a = 0$ and $0 < x \le b$. A nearlattice S with 0 is *semi-Boolean* if it is distributive and the interval $[0, x]$ is complemented for each $x \in S$.

An n-ideal generated by a single element a is called a principal n-ideal, denoted by $< a >_n$. The set of principal n-ideals is denoted by $P_n(S)$. When $n \in S$ is standard and medial then for any $a \in S$

$$< a >_n = \{y \in S : a \wedge n \le y = (y \wedge a) \vee (y \wedge n)\}$$
$$= \{y \in S : y = (y \wedge a) \vee (y \wedge n) \wedge (a \wedge n)\}$$

When n is an upper element, then $< a >_n$ is the closed interval [$a \wedge n, a \vee n$]. By [7], for a medial and standard element n, $P_n(S)$ is a meet semilattice. Also, when n is neutral and sesquimedial, $P_n(S)$ is a nearlattice. Moreover, when n is central, then $P_n(S) \cong (n]^d \times [n)$.

In this paper, we generalize several results of [9] on disjunctive and semi-Boolean nearlattices in terms of $P_n(S)$. By [2] we know that for any n-ideal J of a distributive medial nearlattice S, R(J) denotes the largest congruence having J as its kernel, where $x \equiv yR(J)$ if and only if for each $r \in S$, $m(x,n,r) \in J$ if and only if $m(y,n,r) \in J$.

The following result is due to [9] which gives a description of disjunctive nerlattices.

Theorem 1.1. *For a distributive nearlattice S with 0, the following conditions are equivalent:*

(i) S *is disjunctive.*

(ii) *For all* $a \in S$, $(a] = (a]^{**}$.

(iii) $R((0]) = \omega$.

Following result is due to [7] which will be needed for the development of this paper.

Theorem 1.2. For a neutral element n of a nearlattice S, the following conditions are equivalent :

(i) n is central in S

(ii) n is upper and the map $\Phi : P_n(S) \to (n]^d \times [n)$ defined by $\Phi(< a >_n) = (a \wedge n, a \vee n)$ is an isomorphism, where $(n]^d$ represents the dual of the lattice (n].

Now we extend the above Theorem 1.1.

Theorem 1.3. Suppose S is a distributive medial nearlattice with a central element n. Then the following conditions are equivalent :

(i) $P_n(S)$ is disjunctive

(ii) For each $a \in S$, $< a >_n = < a >_n^{++}$.

(iii) $R(\{n\}) = \omega$

Proof. $(i) \Rightarrow (ii)$. Here n is central, and so it is upper.

Suppose $P_n(S)$ is disjunctive and suppose that $< a >_n \neq < a >_n^{++}$ for some $a \in S$. Since $< a >_n \subseteq < a >_n^{++}$, so there exists $t \in < a >_n^{++}$ but $t \notin < a >_n = [a \wedge n, a \vee n]$ which implies either $a \wedge n \not\leq t$ or $t \not\leq a \vee n$.

Suppose $a \wedge n > t$, then $t \wedge a \wedge n < a \wedge n$.

Thus, $[a \wedge n, n] \subset [t \wedge a \wedge n, n]$ and so $\{n\} \subseteq < a \wedge n >_n \subset < t \wedge a \wedge n >_n$.

Since $P_n(S)$ is disjunctive, so there exists $< b >_n$ such that $\{n\} \subset < b >_n \subseteq < t \wedge a \wedge n >_n$ and $< a \wedge n >_n \cap < b >_n = \{n\}$.

This implies $[(a \wedge n) \vee (b \wedge n), n] = \{n\}$, and so $(a \wedge n) \vee (b \wedge n) = n.$

Now,

$$< a >_n \cap < b >_n = [(a \wedge n) \vee (b \wedge n), (a \vee n) \wedge (b \vee n)]$$
$$= [n, (a \vee n) \wedge (b \vee n)] = \{n\} \text{ as } b \leq n.$$

Hence $< b >_n \subseteq < a >_n^{+}$.

$$\begin{aligned}
\text{Now } < b >_n &= < b >_n \cap < t \wedge a \wedge n >_n \\
&= [(b \wedge n) \vee (t \wedge a \wedge n), n] \\
&= [((t \wedge n) \vee (b \wedge n)) \vee ((a \wedge n) \vee (b \wedge n)), n] \\
&= [((t \wedge n) \vee (b \wedge n)) \wedge n, n] \\
&= [(t \wedge n) \vee (b \wedge n), n] \\
&= < t \wedge n >_n \cap < b >_n \\
&= \{n\} \text{ as } t \wedge n \in < a >_n^{++} \text{ and } < b >_n \subseteq < a >_n^{+}.
\end{aligned}$$

Thus $< b >_n = \{n\}$, which is a contradiction.

Therefore, $<a>_n = <a>_n^{++}$ for all $a \in S$, which is (ii).

Again, suppose $t > (a \vee n)$.

Then $(t \vee n) > (a \vee n)$ and hence $t \neq (t \wedge a) \vee (t \wedge n)$.

That is, $(t \wedge a) \vee (t \wedge n) < t$ and so $(t \wedge a) \vee n < t \vee n$.

Thus, $\{n\} \subseteq <(t \wedge a) \vee n>_n \subset <t \vee n>_n$

Since $P_n(S)$ is disjunctive so there exists $<c>_n$ such that

$\{n\} \subset <c>_n \subseteq <t \vee n>_n$ and $<c>_n \cap <(t \wedge a) \vee n>_n = \{n\}$.

This implies $[c \wedge n, c \vee n] \cap [n, (t \wedge a) \vee n] = \{n\}$ and so

$[n, ((t \wedge a) \vee n) \wedge (c \vee n)] = \{n\}$.

Thus $((t \wedge a) \vee n)(c \vee n) = n$.

That is, $(t \wedge a \wedge c) \vee n = n$ and so $t \wedge a \wedge c \leq n$.

Also, $(t \wedge a \wedge c) \vee n = n$ implies $[(t \wedge c) \vee n] \wedge [a \vee n] = n$.

Hence, $<(t \wedge c) \vee n>_n \subseteq <a>_n^{+}$.

Now, $<c>_n = <c>_n \cap <t \wedge n>_n$

$= [c \wedge n, c \vee n] \wedge [n, t \vee n]$

$= [n, (t \wedge c) \vee n]$

$= <t \vee n>_n \cap <(t \wedge c) \vee n>_n$

$= \{n\}$ as $<(t \wedge c) \vee n>_n \subseteq <a>_n^{+}$

and $t \vee n \in <a>_n^{++}$.

Thus $<c>_n = \{n\}$, which is a contradiction.

Therefore $<a>_n = <a>_n^{++}$ for all $a \in S$.

Thus (ii) holds.

$(ii) \Rightarrow (i)$. Suppose $<a>_n = <a>_n^{++}$ for all $a \in S$.

Now let $n \leq a < b$. Then $\{n\} \subseteq <a>_n \subset <b>_n$ and $<a>_n = <a>_n^{++}$, $<b>_n = <b>_n^{++}$ implies $<a>_n^{+} \supset <b>_n^{+}$. So there exists $r \in <a>_n^{+}$ such that $r \notin <b>_n^{+}$.

This implies $m(r, n, a) = n$ and $m(r, n, x) \neq n$ for some $x \in <b>_n$.

Then $n = m(r, n, a) = (r \vee n) \wedge a$ and as $x \geq n$, $m(r, n, x) = (r \vee n) \wedge x$.

Then $\{n\} \subset <m(r, n, x)>_n \subseteq <b>_n$ and $n < (r \vee n) \wedge x \leq b$.

Moreover, $a \wedge (r \vee n) \wedge x = n \wedge x = n$. This implies [n) is disjunctive.

Similarly we can show that (n] is dual disjunctive.

Hence $(n]^d \times [n)$ is disjunctive.

Since by Theorem 1.2, $P_n(S) \cong (n]^d \times [n)$, so $P_n(S)$ is disjunctive which is (i).

$(i) \Rightarrow (iii)$. Suppose $P_n(S)$ is disjunctive.

Let $x \equiv yR(\{n\})$. If $x \neq y$, then either $x \wedge y < x$ or $x \wedge y < y$.

Suppose $x \wedge y < x$. Since S is distributive, so either

$$x \wedge y \wedge n < x \wedge n \text{ or } (x \wedge y) \vee n < x \vee n$$

If $x \wedge y \wedge n < x \wedge n$, then $< x >_n \subset < x >_n \vee < y >_n$ and so $< x >_n \cap < y >_n \subset < y >_n$.

If $(x \wedge y) \vee n < x \vee n$, then $< x >_n \cap < y >_n \subset < x >_n$.

Thus $x \neq y$ implies either $< x >_n \cap < y >_n \subset < x >_n$ or $< x >_n \cap < y >_n \subset < y >_n$.

Without loss of generality suppose that $< x >_n \cap < y >_n \subset < x >_n$.

Since $P_n(S)$ is disjunctive, there exists $< t >_n$ such that $\{n\} \subset < t >_n \subseteq < x >_n$ and $< t >_n \cap < x >_n \cap < y >_n = \{n\}$ and so $< t >_n \cap < y >_n = \{n\}$.

That is $m(y,n,t) = n$. Since $x \equiv yR(\{n\})$, so $m(x,n,t) = n$ and so $< x >_n \cap < t >_n = \{n\}$.

This implies $< t >_n = \{n\}$, which is a contradiction. Therefore , $x = y$.

Thus $R(\{n\}) = \omega$, which is (iii).

Finally, we show that $(iii) \Rightarrow (i)$. Let $R(\{n\}) = \omega$.

Consider the interval $[n,b]$. If $[n,b]$ is not disjunctive, then there exists $x \in S$ with $n \leq x < b$ such that $x \wedge t > n$ for all t with $n < t \leq b$.

Choose any $r \in S$. Then m(x,n,r) = m(x,n,(r $\wedge$ b) $\vee$ n)= (x $\wedge$ r) $\vee$ n.

Also m(b,n,r) = m(b,n,(r $\wedge$ b) $\vee$ n)= (b $\wedge$ r) $\vee$ n.

If $m(b,n,r) = n$, then $n \leq (x \wedge r) \vee n \leq (b \wedge r) \vee n = n$ implies $m(x,n,r) = n$.

Again $m(x,n,r) = n$ implies n = m(x,n,(r $\wedge$ b) $\vee$ n) = n $\vee$ (x $\wedge$ [(r $\wedge$ b) $\vee$ n]).

This implies $x \wedge [(r \wedge b) \vee n] = n$ as $x \geq n$.

Since $n \leq (r \wedge b) \vee n \leq b$, so by above condition $(r \wedge b) \vee n = n$.

Thus m(b,n,r) = m(b,n,(r $\wedge$ b) $\vee$ n)
= m(b,n,n)
= n.

Therefore , $m(x,n,r) = n$ if and only if $m(b,n,r) = n$ for any $r \in S$.

This implies $x \equiv bR(\{n\})$, and so $x = b$, which is a contradiction to our assumption. Hence $[n,b]$ must be disjunctive.

A dual proof of above shows that each interval $[a,n]$, $a \in S$ is a dual disjunctive.

Therefore, by Theorem 1.2, $P_n(S)$ is disjunctive. □

The following result is an extension of [9, Theorem 2.7], which is also a generalization of a result in [6].

Recall that an n-ideal J is dense if $J^{+} = \{n\}$. Recently [1] have shown that an n-ideal J is both meet and join dense if and only if $\Theta(J)$ is dense in $C(S)$, that is $\Theta(J)^{*} = \omega$.

Theorem 1.4. Let S be a distributive nearlattice and $n \in S$ be a central element, then the following conditions are equivalent :

(i) $P_n(S)$ is disjunctive.

(ii) Each dense n- ideal J is both join and meet-dense.

(iii) For each dense n- ideal J, $\Theta(J^{+}) = \Theta(J)^{*}$.

(iv) For each dense n- ideal J, $\Theta(J^{++}) = \Theta(J)^{**}$.

Proof. $(i) \Rightarrow (ii)$. Suppose $P_n(S)$ is disjunctive.

Suppose J is a dense n-ideal. Then $J^{+} = \{n\}$.

Let $x \wedge j = y \wedge j$ for all $j \in J$, ($x, y \in S$).

If $x \neq y$, then either $x \wedge y < x$ or $x \wedge y < y$.

Without loss of generality, suppose $x \wedge y < x$.

Then either $x \wedge y \wedge n < x \wedge n$ or $(x \wedge y) \vee n < x \vee n$.

Since $n \in J$, so $x \wedge n = y \wedge n$. So $x \wedge y \wedge n = x \wedge n$. Thus $(x \wedge y) \vee n < x \vee n$.

Since $P_n(S)$ is disjunctive, so by Theorem 1.2, [n) is disjunctive.

Hence there exists b with $n < b \leq x \vee n$ such that $((x \wedge y) \vee n) \wedge b = n$.

Then for all $j \in J$,

$$\begin{aligned} n &= n \wedge (j \vee n) \\ &= [(x \wedge y) \vee n] \wedge b \wedge (j \vee n) \\ &= b \wedge [(x \wedge y) \vee n] \wedge (j \vee n) \\ &= b \wedge [(x \wedge y \wedge j) \vee n] \\ &= b \wedge [(x \wedge j) \vee n] \\ &= b \wedge (x \vee n) \wedge (j \vee n) \\ &= b \wedge (j \vee n) \\ &= m(b,n,j) \end{aligned}$$

which shows that $b \in J^{+} = \{n\}$ implies $b = n$ which is a contradiction.

Thus, $x = y$, and so J is join-dense.

Similarly, we can show that J is also meet-dense. Hence (ii) holds.

$(ii) \Rightarrow (i)$. For any $a \in S$, $<a>_n \vee <a>_n^{+}$ is always a dense n-ideal.

Since (ii) holds, so $<a>_n \vee <a>_n^{+}$ is both meet and join-dense.

Then by [1, Theorem 1.9], $\Theta(<a>_n \vee <a>_n^{+})$ is dense.

That is, $\omega = \Theta(<a>_n \vee <a>_n^{+})^{*}$

$$= (\Theta(< a >_n) \vee \Theta(< a >_n^+))^*$$
$$= \Theta(< a >_n)^* \cap \Theta(< a >_n^+)^*$$

Thus $\Theta(< a >_n^+)^* \subseteq \Theta(< a >_n)^{**} = \Theta(< a >_n)$.

Taking the n-kernels on both sides we have $< a >_n^{++} \subseteq < a >_n$ due to [1, Theorem 1.4 (ii)]. It follows that $< a >_n^{++} = < a >_n$.

Then by Theorem 1.3, $P_n(S)$ is disjunctive. Hence (i) holds.

Since $J^+ = \{n\}$ if and only if $J^{++} = S$ and by [1, Theorem 1.9], J is both meet and join-dense if and only if $\Theta(J)^* = \omega$, so obviously, (ii), (iii) and (iv) are equivalent. □

The following theorem is a generalization of [9, Theorem 2.8].

Theorem 1.5. Let S be a distributive nearlattice with a central element n. Then the following conditions are equivalent :

(i) $P_n(S)$ is disjunctive

(ii) For each congruence Φ, $\Phi^* = \Theta(Ker_n\Phi)^*$.

(iii) For each n- ideal J, $R(J)^* = \Theta(J)^*$.

(iv) For each congruence Φ, $Ker_n(\Phi^*) = (Ker_n\Phi)^+$.

(v) For each congruence Φ, $Ker_n(\Phi^{**}) = (Ker_n\Phi)^{++}$.

(vi) The n- kernel of each skeletal congruence is an annihilator n- ideal.

Proof. $(i) \Rightarrow (ii)$. Suppose (i) holds.

Since $\Theta(Ker_n\Phi) \subseteq \Phi$, so we have $\Phi^* \subseteq \Theta(Ker_n\Phi)^*$.

So it is sufficient to prove that $\Phi \cap \Theta(Ker_n\Phi)^* = \omega$.

Suppose $x \leq y$ and $x \equiv y(\Phi \cap \Theta(Ker_n\Phi)^*)$ implies $x \equiv y\Phi$ and $x \equiv y\Theta(Ker_n\Phi)^*$.

If $x < y$, then either $x \wedge n < y \wedge n$ or $x \vee n < y \vee n$.

Suppose $x \vee n < y \vee n$. Since $P_n(S)$ is disjunctive, so by Theorem 1.2, [n) is also disjunctive. So there exists $n < a \leq y \vee n$ such that $a \wedge (x \vee n) = n$.

Now , $n = a \wedge (x \vee n) \equiv a \wedge (y \vee n) = a(\Phi)$ and so, $a \in Ker_n\Phi$.

Since $x \equiv y\Theta(Ker_n\Phi)^*$, so $x \vee n \equiv y \vee n\Theta(Ker_n\Phi)^*$

and since $a \in Ker_n\Phi$, so by [1, Theorem 1.4], $m(x \vee n, n, a) = m(y \vee n, n, a)$, i.e.

$$((x \vee n) \wedge n) \vee (a \wedge (x \vee n)) \vee (n \wedge a) = ((y \vee n) \wedge n) \vee (a \wedge (y \vee n)) \vee (n \wedge a)$$

and so $n \vee (a \wedge (x \vee n)) = n \vee a$.
This implies, $n = a$, which is a contradiction.
Therefore $x = y$ and so $\Phi \cap \Theta(Ker_n\Phi)^* = \omega$.
Thus $\Theta(Ker_n\Phi)^* \subseteq \Phi^*$. Hence $\Phi^* = \Theta(Ker_n\Phi)^*$.
$(ii) \Rightarrow (iii)$ holds since J is the n-kernel of R(J) and $\Theta(J)$.
$(iii) \Rightarrow (i)$. Suppose (iii) holds. Since $\Theta(\{n\}) = \omega$ and since (iii) holds,
so $R(\{n\})^* = \Theta(\{n\})^* = \iota$ implies that $R(\{n\})^{**} = \omega$.
Then by Theorem 1.3, we have $P_n(S)$ is disjunctive.
Since by [1, Theorem 1.4 (ii)], $\Theta(J)^*$ and $\Theta(J^+)$ have J^+ as their n-kernels,
so $(ii) \Rightarrow (iv)$ is obvious.
$(iv) \Rightarrow (v)$ and $(v) \Rightarrow (vi)$ are obvious.
Finally we need to prove that $(vi) \Rightarrow (i)$.
Suppose (vi) holds. Let $n \le a < c$.
Then by [1,Theorem 1.4 (iii)], $< c, a >$ is the n-kernel of a
skeletal congruence. Since (vi) holds, so there is an annihilator n-ideal K
such that $< c, a >= K = K^{++}$.
As $a \wedge c \le a$ implies $a \in < c, a >= K = K^{++}$.
Also since $a < c$ implies $c \notin < c, a >= K = K^{++}$.
So there exists $e \in K^+$ such that $m(c, n, e) \ne n$.
But $m(a, n, e) = n$ implies $(a \wedge e) \vee n = n$.
That is, $a \wedge (e \vee n) = n$ and so $a \wedge ((e \vee n) \wedge c) = n$.
Also $m(c, n, e) \ne n$ implies $(e \vee n) \wedge c > n$ and so
$n < (e \vee n) \wedge c \le c$ with $a \wedge ((e \vee n) \wedge c) = n$
Therefore [n) is disjunctive.
A dual proof of this gives that (n] is dual disjunctive and so by Theorem 1.2, $P_n(S)$ is disjunctive. □

Recall that a nearlattice S with 0 is *semi-Boolean* if it is distributive and the interval $[0, x]$ is complemented for each $x \in S$.
The following result is an extention of [9, Theorem 2.9].

Theorem 1.6. Let S be a distributive nearlattice with a central element n. Then the following conditions are equivalent :

(i) $P_n(S)$ is semi-Boolean.

(ii) For each congruence Φ, $\Phi^* = \Theta(Ker_n\Phi^*)$.

(iii) For each n- ideal J, $\Theta(J^+) = \Theta(J)^*$.

(iv) For each n- ideal J, $\Theta(J^{++}) = \Theta(J)^{**}$.

Proof. $(i) \Rightarrow (ii)$. Suppose (i) holds.

Let Ψ be any congruence on S. Then by [2, Theorem 2.6], $\Psi = \Theta(Ker_n \Psi)$.

Thus with $\Psi = \Phi^*$, we see that (i) implies (ii).

$(ii) \Rightarrow (iii)$ follows from [1, Theorem 1.4] and $(iii) \Rightarrow (iv)$ is obvious.

$(iv) \Rightarrow (i)$. Suppose (iv) holds. Put $J = <a>_n \vee <a>_n^+$.

Since $J^{++} = S$, (iv) implies $\Theta(<a>_n \vee <a>_n^+)^{**} = \iota$

It follows that $\Theta(<a>_n)^* \cap \Theta(<a>_n^+)^* = \omega$

and so $\Theta(<a>_n^+)^* \subseteq \Theta(<a>_n)^{**} = \Theta(<a>_n)$.

Now by [1, Theorem 1.4], $<a>_n^+ = Ker_n \Theta(<a>_n)^*$.

Then, $\Theta(<a>_n^+) \subseteq \Theta(<a>_n)^*$ and so

$\Theta(<a>_n) = \Theta(<a>_n)^{**} \subseteq \Theta(<a>_n^+)^*$.

Therefore, $\Theta(<a>_n) = \Theta(<a>_n^+)^*$.

But $<a>_n^+ = <a>_n^{+++}$, so by (iv)

$\Theta(<a>_n)^* = \Theta(<a>_n^+)^{**} = \Theta(<a>_n^{+++}) = \Theta(<a>_n^+)$.

Now, let $n \le a \le b$. Then for all $j \in <a>_n = [n,a]$, $m(a,n,j) = m(b,n,j) = j$.

Thus by [1, Theorem 1.4], $a \equiv b\Theta(<a>_n)^* = \Theta(<a>_n^+)$.

Then $(a] \vee (<a>_n^+] = (b] \vee (<a>_n^+]$ implies that

$b = (a \wedge b) \vee (b \wedge r_1) \vee \cdots \vee (b \wedge r_s)$ for some $r_1, \cdots, r_s \in <a>_n^+$.

That is, $b = a \vee (b \wedge r_1) \vee \cdots \vee (b \wedge r_s)$.

Again, $r_i \in <a>_n^+$ implies $m(a,n,r_i) = (a \wedge n) \vee (a \wedge r_i) \vee (r_i \wedge n) = n$,

and so $a \wedge r_i \le n$. Thus $a \wedge r = a \wedge r \wedge n = r \wedge n$.

Now, put $p_i = (b \wedge r_i) \vee n$ and $p = p_1 \vee \cdots \vee p_s$. Then $n \le p \le b$.

Again, $p \wedge a = (a \wedge b \wedge r_1) \vee \cdots \vee (a \wedge b \wedge r_s) \vee (a \wedge n) = n$.

and $p \vee a = (b \wedge r_1) \vee \cdots \vee (b \wedge r_s) \vee a \vee n = b \vee n = b$.

Hence $[n,b]$ is complemented for each $b \in S$.

Similarly a dual proof of above shows that $[e,n]$ is also complemented for each $e \le n$.

Hence by [2, Corollary 1.10], $P_n(S)$ is semi-Boolean. □

For a nearlattice S, the skeleton

$$SC(S) = \{\Theta \in C(S) : \Theta = \Phi^* \quad \text{for some} \quad \Phi \in C(S)\}$$

$= \{\Theta \in C(S) : \Theta = \Theta^{**}\}$ is a complete Boolean lattice.

The meet of a set $\{\Theta_i\} \subseteq SC(S)$ is $\cap \Theta_i$; as in C(S), while the join is given by $\vee \Theta_i = (\vee \Theta_i)^{**} = (\cap \Theta_i^*)^*$ and the complement of $\Theta \in SC(S)$ is Θ^*.

The fact that SC(S) is complete follows from the fact that SC(S) is precisely the set of closed elements associated with the closure operation $\Theta \to \Theta^{**}$ on the complete lattice C(S) and SC(S) is Boolean because of Glivenko's theorem, c.f. Grätzer [4, Theorem 4, p.58].

The set $KSC(S) = \{Ker\Theta : \Theta \in SC(S)\}$ is closed under arbitrary set-theoretic intersections and hence is a complete lattice.

Also , for any $n \in S$, $K_n SC(S) = \{ker_n \Theta : \Theta \in SC(S)\}$ is a complete lattice.

We also denote $A(S) = \{J : J \in I(S) ; J = J^{**}\}$, which is a complete Boolean lattice.

The following theorems are due to [9]. In fact Cornish proved these results for lattices in [3, Theorem 2.4 and Theorem 2.5], which are extensions of the classical theorem of Hashimoto [4, Theorem 8, p.91].

Theorem 1.7. Let S be a distributive nearlattice with 0. Then the following conditions are equivalent :

(i) S is disjunctive

(ii) The map $\Theta \to Ker\Theta$ of $SC(S)$ onto KSC(S) is one-to-one.

(iii) The map $\Theta \to Ker\Theta$ of $SC(S)$ onto KSC(S) preserves finite joins.

(iv) The map $\Theta \to Ker\Theta$ is a lattice isomorphism of $SC(S)$ onto KSC(S) whose inverse is the map $J \to \Theta(J)^{**}$

Theorem 1.8. Let S be a distributive nearlattice with 0. Then the nearlattice S is semi-Boolean if and only if the map $\Theta \to Ker\Theta$ is a lattice isomorphism of $SC(S)$ onto KSC(S) whose inverse is the map $J \to \Theta(J)$.

We conclude this paper with the following generalizations of the above theorems.

Theorem 1.9. Let S be a distributive nearlattice with a central element n. Then the following conditions are equivalent :

(i) $P_n(S)$ is disjunctive

(ii) The map $\Theta \to Ker_n\Theta$ of $SC(S)$ onto $K_n SC(S)$ is one-to-one and so is a one-to-one correspondence.

(iii) The map $\Theta \to Ker_n\Theta$ of $SC(S)$ onto $K_n SC(S)$ preserves finite joins.

(iv) The map $\Theta \to Ker_n\Theta$ is a lattice isomorphism of SC(S) onto

$K_nSC(S)$ whose inverse is the map $J \to \Theta(J)^{**}$ for any n- ideal J in S.

Proof. Firstly, we show that $(i) \Rightarrow (iv)$. Suppose (i) holds.

That is, $P_n(S)$ is disjunctive.

Then by Theorem 1.5 (vi), we have

$K_nSC(S) = \{J : J = J^{++}, J \ is\, n-ideal\}$.

Also, by Theorem 1.5 (ii), for any $\Phi \in SC(S)$, $\Phi = \Phi^{**} = \Theta(Ker_n\Phi)^{**}$.

Thus the map $\Theta \to Ker_n\Theta$ of SC(S) onto $K_nSC(S)$ is one-to-one.

Clearly this map preserves meets and it is also preserves joins since for any Θ , $\Phi \in SC(S)$, $\Theta \vee \Phi = (\Theta^* \cap \Phi^*)^*$ and

$$
\begin{aligned}
Ker_n(\Theta \vee \Phi) &= Ker_n(\Theta^* \cap \Phi^*)^* \\
&= [Ker_n(\Theta^* \cap \Phi^*)]^+ \\
&= [(Ker_n\Theta)^+ \cap (Ker_n\Phi)^+]^+ \\
&= (Ker_n\Theta)^{++} \vee (Ker_n\Phi)^{++} \\
&= (Ker_n\Theta^{**}) \vee (Ker_n\Phi^{**}) \\
&= Ker_n\Theta \vee Ker_n\Phi
\end{aligned}
$$

Thus, $\Theta \to Ker_n\Theta$ is a lattice isomorphism.

Also, note that, $Ker_n(\Theta(J)^{**}) = (Ker_n\Theta(J))^{++} = J^{++} = J$ for any n-ideal $J \in K_nSC(S)$, while $\Theta(Ker_n\Phi)^{**} = \Phi^{**} = \Phi$ for any $\Phi \in SC(S)$.

Thus $J \to \Theta(J)^{**}$ is the inverse of $\Theta \to Ker_n\Theta$. Hence (iv) holds.

$(iv) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (iii)$. Suppose (ii) holds, i.e., $\Theta \to Ker_n\Theta$ is one-to-one.

Then it is a meet isomorphism of the lattice SC(S) onto the lattice $K_nSC(S)$. It follows that $\Theta \to Ker_n\Theta$ is a lattice isomorphism and so (iii) holds.

Finally, we shall show that (iii) implies (i). Suppose (iii) holds.

Then $\Theta \to Ker_n\Theta$ is a lattice isomorphism of SC(S) onto $K_nSC(S)$. Hence $K_nSC(S)$ must be Boolean. It is not hard to see that $P_n(S)$ is a join-dense subnearlattice of $K_nSC(S)$. Since $K_nSC(S)$ is Boolean, so $P_n(S)$ is disjunctive. Hence (i) holds. □

Theorem 1.10. Let S be a distributive nearlattice with a central element n. Then $P_n(S)$ is semi-Boolean if and only if the map $\Theta \to Ker_n\Theta$ is a lattice isomorphism of $SC(S)$ onto $K_nSC(S)$ whose inverse is the map $J \to \Theta(J)$, J is an n- ideal of S.

Proof. Suppose $P_n(S)$ is semi-Boolean. Then of course $P_n(S)$ is disjunctive and so by Theorem 1.9, the inverse of $\Theta \rightarrow Ker_n\Theta$ is $J \rightarrow \Theta(J)^{**}$.

Now, by Theorem 1.6, $\Theta(J)^{**} = \Theta(J^{++})$ for any $J \in K_nSC(S)$.

So due to Theorem 1.5, $J = J^{++}$.

Hence $J \rightarrow \Theta(J)$ is the inverse of $\Theta \rightarrow Ker_n\Theta$.

Conversely, let $J \rightarrow \Theta(J)$ is the inverse of $\Theta \rightarrow Ker_n\Theta$.

Then by Theorem 1.9, $P_n(S)$ is disjunctive and so by Theorem 1.5,

$Ker_n(\Theta(J)^{**}) = [Ker_n(\Theta(J))]^{++} = J^{++}$ for any n-ideal J of S.

Then by [1, Theorem 1.4], we have $J^{++} \in K_nSC(S)$.

Also we must have, $\Theta(J^{++}) = \Theta(Ker_n(\Theta(J))^{**}) = \Theta(J)^{**}$.

Then by Theorem 1.6, $P_n(S)$ is semi-Boolean. □

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