European Option Pricing in Fractional Jump Diffusion Markets

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ABSTRACT

Previous option pricing research typically assumes that the stock volatility is constant during the life of the option. In this study, we assume the stock volatility in our option valuation model is function of time and stock price. The stock price process numerically is simulated by using the Monte Carlo method. Then, the numerical option pricing method for European option is hold. Finally, we compare our results with the known results in the linear case, the results show that our method is effective.

Keywords: fractional Brownian motion; Poisson process; incomplete markets; Monte Carlo method

AMS Mathematics Subject Classification (2010): 60H10; 90A06;

1. Introduction

The interest in pricing financial derivatives – including pricing options – arises from the fact that financial derivatives can be used to minimize losses caused by price fluctuations of the underlying assets. This process of protection is called hedging. There is a variety of financial products on the market, such as futures, forwards, swaps and options. In this paper we will concentrate on European Call and Put options.

We recall that a European Call option is a contract where at a prescribed time in the future, known as the expiry date \( T \), the owner of the option, known as the holder, may purchase a prescribed asset, known as the underlying asset \( S \), for a prescribed amount, known as the exercise or strike price \( K \). The opposite party, or the writer, has the obligation to sell the asset if the holder chooses to exercise his right. Therefore, the value of the option at expiry, known as the pay-off function, is \( C(S_T, T) = (S_T - K)^+ \). Reciprocally, a European Put option is the right to sell the asset with the pay-off function \( P(S_T, T) = (K - S_T)^+ \). While European options can only be
exercised in T, American options can be exercised at any time until the expiration, which complicates their pricing process significantly.

Option pricing theory has made a great leap forward since the development of the Black–Scholes option pricing model by Fischer Black and Myron Scholes in [2] in 1973 and previously by Robert Merton in [3]. The solution of the famous (linear) Black–Scholes equation

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 ,
\]

provides both the price for a European option and a hedging portfolio that replicates the option assuming that (see [4]):

(a) The price of the asset price or underlying derivative \( S_t \) follows a Geometric Brownian motion \( W(t) \), meaning that \( S_t \) satisfies the following stochastic differential equation (SDE): \( dS_t = \mu S_t dt + \sigma S_t dW_t \).

(b) The trend or drift \( \mu \) (measures the average rate of growth of the asset price), the volatility \( \sigma \) (measures the standard deviation of the returns) and the riskless interest rater are constant for \( 0 \leq t \leq T \) and no dividends are paid in that time period.

(c) The market is frictionless, thus there are no transaction costs (fees or taxes), the interest rates for borrowing and lending money are equal, all parties have immediate access to any information, and all securities and credits are available at any time and any size. That is, all variables are perfectly divisible and may take any real number.

Moreover, individual trading will not influence the price.

(d) There are no arbitrage opportunities, meaning that there are no opportunities of instantly making a risk-free profit.

Under these assumptions the market is complete, which means that any asset can be replicated with a portfolio of other assets in the market (see [5]). Then, the linear Black–Scholes equation (1) can be transformed into the heat equation and analytically solved to price the option [1].

One can argue that these restrictive assumptions never occur in reality. Due to transaction costs (see [6–8]), large investor preferences (see [9–11]) and incomplete markets [12] they are likely to become unrealistic and the classical model results in strongly or fully nonlinear, possibly degenerate, parabolic diffusion-convection equations, where both the volatility \( \sigma \) and the drift \( \mu \) can depend on the time \( t \), the stock price \( S_t \) or the derivatives of the option price \( C \) or \( P \) itself.

Recently, some articles have focused on the valuation of European options when the underlying value follows a jump diffusion process or Levy processes which are a fairly large class of continuous time processes with stationary independent increments. For jump diffusion process or Levy processes and their application in finance (see[13-17]).

On the other hand, fractional Brownian motion has been considered to replace Brownian motion in the usual financial models as it has better behaved tails and exhibits long-term dependence while remaining Gaussian. For details about the
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stochastic analysis theory of fractional Brownian motion, see (18-19). The fractional Brownian motion is applied in finance, such as Ref.[20-22].

In this paper we will be concerned with the nonlinear Black-Scholes model for European options with a non-constant volatility \( \sigma = \sigma(t,S_t) \) under the fractional jump-diffusion Environment. The remaining of the paper is organized as follow. The model and some theoretical results are presented in Section 2. In Section 3, we perform numerical simulations for the premium. Concluding remarks are given in Section 4.

2. The model

In this paper we study the pricing problem for an underlying asset price with jumps which is governed by the following stochastic differential equation:

\[
\frac{dS_t}{S_t} = r(t) dt + \sigma(t,S_t)[a_1(t)\circ dW^H_t + b_1(t) d(N_t - \lambda t)], \quad \{S_t\}, t \in [0,T], \quad S_0 > 0 \]

where, \( r, a, b \) are deterministic functions with respect to \( t \) and \( S_t \) such that \( 1 + \sigma(t,S_t)b_t > 0 \). Here \( \{N_t\}, t \in [0,T] \) is a Poisson process with deterministic intensity \( \lambda \) and \( \{W^H_t\}, t \in [0,T] \) is a fractional Brownian motion with Hurst parameter \( H \). \( \int_0^t a_1(t) \circ dW^H_t \) is the wick product for the fractional Brownian motion. Note that the process \( M \) defined by \( M_t = N_t - \lambda t \) for \( t \in [0,T] \) is the compensated process associated to \( N \). We consider a market with two assets: the risky asset \( S_t \) given by the Eq. (1.1) to which is related a European call option and a risk-free asset given by

\[
dA_t = rA_t dt, \quad t \in [0,T], \quad A_0 = 1.
\]

We work on a probability space \((\Omega,F,P)\). \( M_t \) and \( W^H_t \) are independent and we denote by \( \{F_t\}, t \in [0,T] \) the filtration generated by \( \{N_t\}, t \in [0,T] \) and \( \{W^H_t\}, t \in [0,T] \). We assume that (1.1) is the price of the asset under the risk-neutral probability \( P \). Recall that a stochastic process is a function of two variables the time \( t \in [0,T] \) and the event \( \varepsilon \in \Omega \), but in the literature it is common to write \( S_t \), while it means \( S_t = S_t(\varepsilon) \). The same interpretation is true for \( W^H_t \), \( N_t \) and \( M_t \) or any other stochastic process in this paper. To the authors knowledge, it is impossible to find an explicit formula for the solution of the pricing problem. However, the premium can be determined and expressed in the following expectation form (see Ref.[23])

\[
C = \exp\{-\int_0^T r_t dt \} E_\varepsilon[(S_{T_t} - k)]^+, \quad (1.2)
\]

where \( E_\varepsilon \) denotes the expected value in a risk-neutral world. Here \( P \) is called the equivalent martingale measure. Note that, when \( \sigma(t,S_t) = \sigma_t \) Eq. (1.1) is reduced to the well-known fractional jump modle

\[
S_t = S_0 \exp\{\int_0^t (r_s - \sigma_s b_s \lambda_s)ds - H \int_0^t \sigma_s^2 s^{2H-1}ds + \int_0^t a_s \sigma_s \circ dW^H_s\} \times \prod_{k=1}^{t \in N}(1 + \sigma_k b_k).
\]
Therefore, the expectation in (1.2) can be calculated by integrating over the normal distribution which gives the European-call pricing formula \( C \) by Sun, Xue 2009 and Xue, Sun 2010, when \( \sigma, a, b \) are constants.

\[
C = S_e^{(1 + \sigma b)} \sum_{i=0}^{\infty} \frac{[1 + \sigma b] \lambda (T - t)}{i!} e^{-\lambda (T - t)} \Phi(d_i^2) - K \exp[-\int_T^t r ds] \sum_{i=0}^{\infty} \frac{[\lambda (T - t)]^i}{i!} e^{-\lambda (T - t)} \Phi(d_i^2),
\]

where \( \Phi(\cdot) \) is the cumulative distribution function of the standard normal distribution, and

\[
d_i^1 = \ln\left(\frac{S_e^{(1 + \sigma b)}}{K}\right) + \int_t^T r ds - \lambda b \sigma (T - t) - 0.5 \sigma^2 (T^2 - t^2),
\]

\[
d_i^2 = -a \sigma \sqrt{T - t}.
\]

Similarly, the pricing formula of the European put option \( P \) can be written as

\[
P = K \exp[-\int_T^t r ds] \sum_{i=0}^{\infty} \frac{[\lambda (T - t)]^i}{i!} e^{-\lambda (T - t)} \Phi(-d_i^2)
\]

\[
-S_e^{(1 + \sigma b)} \sum_{i=0}^{\infty} \frac{[1 + \sigma b] \lambda (T - t)}{i!} e^{-\lambda (T - t)} \Phi(-d_i^2).
\]

However, \( \sigma(t, S_t) \) is the function of \( t \) and \( S_t \), the expectation function cannot be calculated to have an explicit formula because the random variable \( S_t \) does not have a known probability density. To surmount this problem, we use Monte Carlo techniques to simulate the premium. The Monte Carlo method is a very effective tool to simulate the prices of financial derivatives that do not have closed explicit formulas. The use of this method in options pricing was initiated by Boyle (1977). Since then it has been used by many researchers in finance. In this paper, we compute the premium and the price of the option at any time \( t \in [0, T] \), using the Monte Carlo method.

3. Numerical computing of option prices

In this section we discuss the simulation of the premium (1.2) using the Monte Carlo method. The main steps are summarized below:

**Step 1.** Simulation of \( S_t \): We select an integer \( L > 0 \), then we simulate \( S_t(i) \) for \( i \in \{1, 2, \ldots, L\} \).

**Step 2.** Monte Carlo solution for the premium: The simulation of the premium via the Monte Carlo method involves the following steps:

1. For each path \( S_t(i) \), compute the payoff \( \max\{S_t(i) - K, 0\} \).

2. Calculate the mean of the resulting payoffs \( \frac{1}{L} \sum_{i=1}^{L} \max(S_t(i) - K, 0) \).

3. Estimate the price of the option by discounting the mean payoff at the risk-free rate \( \frac{1}{L} \sum_{i=1}^{L} \max(S_t(i) - K, 0) \exp(-\int_0^T r ds) \).

In the proceeding subsections, we give the details of the above steps.
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3.1. Simulation of \( S_T \)

We seek \( L \) realizations of \( S_T \):

\[
S_T(\omega_1), \ldots, S_T(\omega_i), \ldots, S_T(\omega_L),
\]

where \( \omega_1, \ldots, \omega_i, \ldots, \omega_L \) are chosen randomly from \( \Omega \). We follow the following algorithm:

1. Simulate \( L \) trajectories for \( \{S_t, 0 \leq t \leq T\} \):

\[
S_t(1), \ldots, S_t(i), \ldots, S_t(L),
\]

where \( S_t(i) \) is a simulation of \( S_T(\omega_i) \) and \( i = 1,2,\ldots, L \).

2. For each \( i \) (\( i = 1,2,\ldots, L \)), take the value of \( S_t(i) \) at the terminal time: \( S_T(i) \).

First, we select an integer \( M > 0 \), then we discretize the time interval \([0, T]\) into \( M \) steps \( t_j = j \Delta t \), where \( j = 0,1,2,\ldots, M \) of identical duration \( \Delta t = \frac{T}{M} \):

\[
S_{t_j}(i), \ldots, S_{t_j}(i), \ldots, S_{t_j}(i)
\]

and thus we get \( L \) approximations of \( S_T \):

\[
S_{t_j}(1), \ldots, S_{t_j}(2), \ldots, S_{t_j}(L)
\]

Let \( i \) be fixed in \( \{1,\ldots, L\} \). We start by simulating a trajectory \( S_t(i) \) of the Brownian motion and a trajectory \( N_t(i) \) and then we use Eq. (2.2) to find the approximation \( S_{t_j}(i) \) of \( S_T(i) \).

We implement the following steps:

(a). Simulation of the Brownian motion and the Brownian integral.

We simulate \( \{W_t(i), i = 1,2,\ldots,L, j = 0,1,\cdots,H\} \) noting the fact that the Brownian motion fulfills:

\[
W_t^H(i) = 0, \quad W_t^H(i) = W_{t+\Delta t}^H(i) + \Delta t^H Z_j(i), i = 1,2,\ldots,L, j = 0,1,\cdots,M
\]

where \( Z_j(i) \) follows a normal distribution \( N(0,1) \). We simulate \( 2L \) uniform random variable \( U_j(i) \) and \( V_j(i) \), and we use the Box–Muller method

\[
Z_j(i) = \sqrt{-\log(U_j(i))} \cdot \cos(2\pi V_j(i)).
\]

(b). Simulation of the Poisson Process and the Poisson part.

Regarding the Poisson process, we simulate first the jump times \( \{T_k, k \geq 0\} \) of \( \{N_t, 0 \leq t \leq T\} \) with intensity \( \lambda \) by \( \{T_{n_j}(i), i = 1,2,\ldots,L, j = 0,1,\cdots,M\} \). We are using the following properties of the Poisson process:

\[
T_{n_j}(i) = 0, \quad T_{n_j}(i) = T_{n_j}(i) + \text{ExpLaw}(\lambda \Delta t), i = 1,2,\ldots,L, j = 0,1,\cdots,M
\]

where \( \text{ExpLaw}(\lambda) \) is an exponential random variable which can be written as

\[
\text{ExpLaw}(\lambda \Delta t) = -\frac{1}{\lambda \Delta t} \log(U), \quad \text{and } U \text{ is a uniform random variable. A trajectory of the Poisson process } N_t(i), i = 1,2,\ldots,L, j = 0,1,\cdots,M \text{ is then determined by using:} \]
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\[ N_{t_0} = 1, N_{t_i} (i) = \sum_{k=0}^{j} I_{(t_i(t_i))s(t_i)}, i = 1, 2, \cdots, L, j = 0, 1, \cdots M. \]

Since, from the last equation, and recalling the properties of the fractional Brownian motion, we get

\[ S_{t_j} (i) - S_{t_{j-1}} (i) = \Delta t \cdot r_{t_j} S_{t_{j-1}} (i) + \sigma(t_{j-1}, S_{t_{j-1}} (i)) a_{t_j} \Delta t Z_j (i) \]

\[ + \sigma(t_{j-1}, S_{t_{j-1}} (i)) b_{t_j} (N_{t_j} (i) - N_{t_{j-1}} (i)). \]

### 3.2. Monte Carlo solution for the premium

We have from the previous subsections L realizations for \( S_r \), so we can apply the Monte Carlo method to compute the premium numerically using

\[ \exp\{-\int_0^T r ds\} \cdot \frac{1}{L} \sum_{i=1}^{\hat{L}} \max(S_{t_i} (i) - K, 0) \]

To reduce the computational time we reduce the variance by using the antithetic variable method. This technique consists of computing two values of the premium \( C \). The first value \( C_1 \) is calculated as described above and the second value \( C_2 \) is calculated similarly as \( C_1 \) with changing the sign of all the random samples from the standard normal distribution. Then \( C \) is obtained by taking the average of \( C_1 \) and \( C_2 \).

The standard error of the estimate premium is then \( \frac{s_C}{L} \), where \( s_C \) is the standard deviation of the estimate premium and \( L \) is the number of trials. A 95% confidence interval for the premium is

\[ \mu_c - \frac{1.96s_C}{\sqrt{L}} < C < \mu_c + \frac{1.96s_C}{\sqrt{L}} \]

where \( \mu_c \) is the mean of the estimated premium.

![Fractional Brownian motion and Poisson process](image)

**Figure 1.** Fractional Brownian motion and Poisson process

Now, we present the numerical results of the premium by the Monte Carlo simulation when
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\[ T = 1, \ \lambda = 0.01t + 3; \ \ \ \ r = 0.01(2 + 8\sin(\pi t) + \cos(\frac{\pi t}{2})), \]
\[ a = 0.1, \ b = 0.3, \ \ \sigma = 0.1 + \sqrt{t}, \ S_0 = 7 \ \ \text{and} \ \ K = 7.5. \]

Notice that, the parameters \( T \) and \( \lambda \) are used to simulate trajectories for the Brownian motion and for the Poisson process with number of realizations \( H = 500 \) (see Fig. 1). Then, we simulate trajectories for the stock price at

![Figure 2](image2.png)

**Figure 2.** Realizations of the asset price for \( H = 0.53, T = 1, \lambda = 0.01t + 3, \)
\[ r = 0.1 + \sin(\frac{\pi t}{2}), \ a = 0.25, \ b = 0.3, \ \ \sigma = \sqrt{t} + 0.01 \ \ \text{and} \ \ S_0 = 7 \]

![Figure 3](image3.png)

**Figure 3.** Realizations of the asset premium for \( d \ K = 7.5 \)

terminal time \( T = 1 \) with \( H = 500 \) (see Fig. 2) and for the premium with number of realizations \( L = 500 \) (see Fig. 3). It is found that, the standard error of the estimate
premium is $0.237 \times 10^{-3}$. A 95% confidence interval for the premium is therefore given by

$$5.469 \times 10^{-2} < C < 5.562 \times 10^{-2}.$$  

We also provide the premium as a function of the stock price at $t=0$ for two different values of the strike $K = 7.5$ and $K = 9$ with number of realizations $L=500$, see Fig. 4.

We now obtain the results by using Monte Carlo method to deal with the linear model. For all remain calculations we used the following parameters:

$S_0 = 100$, $K = 100$, $T = 1$, $a = 1$, $b = 0$, $H = 0.5$.

In this linear case, the call option pricing formula can hold by the following closed form.

We compare our method with the real value given by the closed form.

We see that in the linear case the Mont Carlo method yield a very accurate result (see Table 1).

### Table 1. Numerical values by Mont Carlo method

<table>
<thead>
<tr>
<th>r, $\sigma$</th>
<th>$L=10^4$</th>
<th>$L=10^5$</th>
<th>$L=10^6$</th>
<th>$L=10^7$</th>
<th>Real value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1, 0.1</td>
<td>10.2100</td>
<td>10.2816</td>
<td>10.3165</td>
<td>10.3052</td>
<td>10.3082</td>
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<tr>
<td>0.1, 0.15</td>
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<td>11.6360</td>
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<td>11.6657</td>
<td>11.6691</td>
</tr>
<tr>
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<td>13.2332</td>
<td>13.2837</td>
<td>13.2659</td>
<td>13.2697</td>
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<td>14.9921</td>
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</tr>
<tr>
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</tr>
</tbody>
</table>

4. Conclusion
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In this paper, a fractional jump diffusion model is considered for option pricing. The pricing problem for such a model does not have a closed formula since the market is incomplete. However, since it imitates financial crashes, it is a more realistic approach. The price of a European option is simulated numerically by using the Monte Carlo method.

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