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n-Distributive Lattice

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ABSTRACT

J. C. Varlet has given the concept of 0-distributive and 1-distributive lattices. In this paper the authors have generalized the whole concept and introduced the notion of n-distributive lattices. They show that for a neutral element of a lattice L, the n-annihilator of any subset of L is an n-ideal if and only if L is n-distributive. Then the authors study different properties of these lattices. Finally, using the n-annihilators they generalize the well known prime Separation theorem of distributive lattices with respect to annihilator n-ideal in a general lattice and produce an interesting characterization of n-distributive lattice.

***Keywords**:* Neutral element, 0-distributive lattice, n- annihilator, annihilator n-ideal, prime n-ideal, n-distributive lattice.

***AMS Mathematics Subject Classifications (2010)**:* 06A12, 06A99, 06B10

1. Introduction

In generalizing the notion of pseudocomplemented lattices, J. C. Varlet [6] introduced the notion of 0-distributive lattices. A lattice L with 0 is called 0-distributive if for all $a,b,c \in L$,with $a \wedge b = 0 = a \wedge c$ imply $a \wedge (b \vee c) = 0$. Similarly, lattice L with 1 is called 1-distributive if $a \vee b = 1 = a \vee c$ imply $a \vee (b \wedge c) = 1$ Of course every distributive lattice with 0 and 1 is both 0-distributive and 1-distributive. Pentagonal lattice of Figure1 is a non-distributive lattice which is both 0-distributive and 1-distributive. But the modular lattice of Figure2 is neither 0-distributive nor 1-distributive. Again Figure-3 is an example of a lattice which is 0-distributive but not 1-distributive, while Figure-4 is an example which is not 0-distributive but 1-distributive.

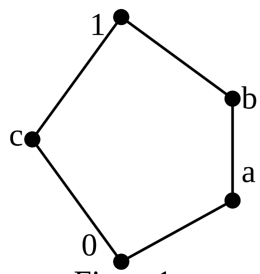

Figure1

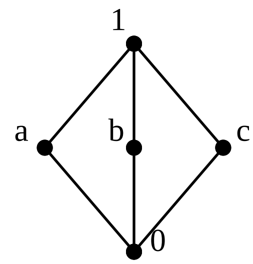

Figure2

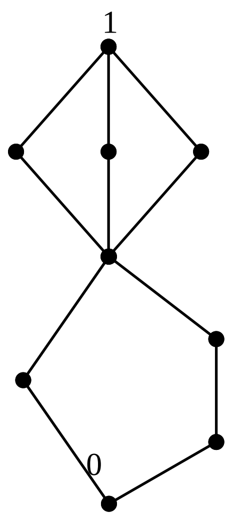

Figure3

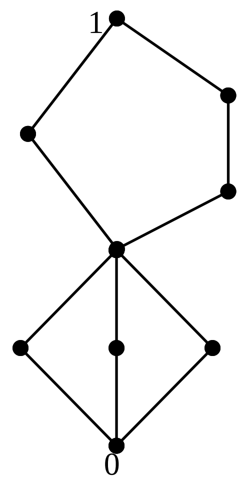

Figure4

A pseudocomplemented lattice L can be characterized by the fact that for each $a \in L$, the set of all elements which are disjoint with element a forms a principal ideal. But a 0-distributive lattice L says that for each $a \in L$ the set of all elements disjoint with a is simply an ideal but not necessarily a principal ideal. Hence every pseudocomplemented lattice is 0-distributive. For detailed literature on 0-distributive lattice we refer the readers to consult [6], [1], and [5]. In this paper we generalize the concept of 0-distributive and 1-distributive and give the notion of n-distributive lattice L where n is a neutral element of L.

Let L be a lattice and $n \in L$. Any convex sublattice of L containing n is called an n-ideal of L. An element $n \in L$ is called a standard element if for $a, b \in L,\ a \wedge (b \vee n) = (a \wedge b) \vee (a \wedge n)$, while n is called a neutral element if (i) it is standard and (ii) $n \wedge (a \vee b) = (n \wedge a) \vee (n \wedge b)$ for all $a, b \in L$. Set of all n-ideals of a lattice L is denoted by $I_n(L)$ which is an algebraic lattice; where

$\{n\}$and L are the smallest and largest elements. For two n-ideals I and J, $I \cap J$ is the infimum and
$I \vee J = \{x \in L / i_1 \wedge j_1 \leq x \leq i_2 \vee j_2,\ \mathit{for\ some}\ i_1, i_2 \in I\ \mathit{and}\ \ j_1, j_2 \in J\}$. The n-ideal generated by a finite numbers of elements $a_1, a_2, ..., a_m$ is called a finitely generated n-ideal denoted by $\langle a_1, a_2, ..., a_m \rangle_n$. Moreover,
$\langle a_1, a_2, ..., a_m \rangle_n = \{x \in L / a_1 \wedge a_2 \wedge ... \wedge a_m \wedge n \leq x \leq a_1 \vee a_2 \vee ... \vee a_m \vee n\}$
$= [a_1 \wedge a_2 \wedge .... \wedge a_m \wedge n, a_1 \vee a_2 \vee .... \vee a_m \vee n]$
Thus, every finitely generated n-ideal is an interval containing n. n-ideal generated by a single element $a \in L$ is called a principal n-ideal denoted by $\langle a \rangle_n$ and $\langle a \rangle_n = [a \wedge n, a \vee n]$. Moreover $[a,b] \cap [c,d] = [a \vee c, b \wedge d]$ and
$[a,b] \vee [c,d] = [a \wedge c, b \vee d]$. If n is a neutral element, then by [3], $\langle a \rangle_n \cap \langle b \rangle_n = \langle m(a,n,b) \rangle_n$, where $m(x,y,z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$. A proper convex sublattice M of a lattice L is called a maximal convex sublattice if for any convex sublattice Q with $Q \supseteq M$ implies either $Q = M$ or $Q = L$. A proper convex sublattice M is called a prime convex sublattice if for any $t \in M$, $m(a,t,b) \in M$ implies either $a \in M$ or $b \in M$. Similarly, an n-ideal P of L is called a prime n-ideal if $m(a,n,b) \in P$ implies either $a \in P$ or $b \in P$. Equivalently, P is prime if and only if $\langle a \rangle_n \cap \langle b \rangle_n \subseteq P$ implies either $\langle a \rangle_n \subseteq P$ or $\langle b \rangle_n \subseteq P$. Moreover, by [4], we know that every prime convex sublattice P of L is either an ideal or a filter. Let n be a neutral element of L. For $a \in L$,we define$\{a\}^{\perp_n} = \{x \in L / m(x,n,a) = n\}$, known as an n-annihilator of $\{a\}$. For $A \subseteq L$, $A^{\perp_n} = \{x \in L / m(x,n,a) = n,\ \mathit{for\ all}\ a \in A\}$. If L is a distributive lattice, then it is easy to check that $\{a\}^{\perp_n}$ and $A^{\perp_n}$are n-ideals. Moreover, $A^{\perp_n} = \bigcap_{a \in A} \{\{a\}^{\perp_n}\}$. If A is an n-ideal, then $A^{\perp_n}$ is called an annihilator n-ideal, which is obviously the pseudocomplement of A in $I_n(L)$ Therefore, for a distributive lattice L with n, $I_n(L)$ is pseudocomplemented. Let n be a neutral element of a lattice L.

Theorem 1. *If the intersection of all prime n-ideals of a lattice L is* $\{n\}$, *then L is n-distributive.*
Proof. Let $\langle a \rangle_n \cap \langle b \rangle_n = \{n\}$and $\langle a \rangle_n \cap \langle c \rangle_n = \{n\}$. Let P be any prime n-ideal. If $a \in P$, then $\langle a \rangle_n \subseteq P$and so $\langle a \rangle_n \cap (\langle b \rangle_n \vee \langle c \rangle_n) \subseteq P$. If $a \notin P$, then

$\langle b\rangle_n, \langle c\rangle_n \subseteq P$ as P is prime, and so $\langle b\rangle_n \vee \langle c\rangle_n \subseteq P$. Thus $\langle a\rangle_n \cap (\langle b\rangle_n \vee \langle c\rangle_n) \subseteq P$. That is in either case, $\langle a\rangle_n \cap (\langle b\rangle_n \vee \langle c\rangle_n) \subseteq P$ for all prime n-ideals P. Therefore, $\langle a\rangle_n \cap (\langle b\rangle_n \vee \langle c\rangle_n) = \{n\}$, and so L is n-distributive.

Lemma 2. *Every convex sublattice not containing n is contained in a maximal convex sublattice not containing n.*

Proof: Let F be a convex sublattice such that $n \notin F$. Let F be the set of all convex sublattice containing F but not containing n . F is non-empty as $F \in$ F. Let C be a chain in F and $M = \cup(X / X \in C)$. Let $x, y \in M$. Then $x \in X$ and $y \in Y$ for some $X, Y \in C$. Since C is a chain, so either $X \subseteq Y$ or $Y \subseteq X$. Suppose $X \subseteq Y$. Then $x, y \in Y$. Hence $x \wedge y, x \vee y \in X$, and so $x \wedge y, x \vee y \in M$.Thus M is a sublattice of L containing F . Also it is convex as each $X \in C$ is convex. Clearly $n \notin M$. Hence M is a maximum element of C. Therefore, by Zorn's Lemma, F has a maximal element.

Lemma 3. *Let $n \in L$ be neutral. A convex sublattice M not containing n is maximal if and only if for all $a \notin M$ there exists $b \in M$ such that m(a,n,b)= n.*

Proof. Suppose M is maximal and $n \notin M$. Let $a \notin M$. Suppose for all $b \in M$, $m(a,n,b) \neq n$.

Set $M_1 = \{y \in L / y \wedge n \leq (a \vee b) \wedge n \leq (a \wedge b) \vee n \leq y \vee n\}$. It is easy to check that M_1 is a convex sublattice as n is neutral. Moreover, $n \notin M_1$.For otherwise $n \wedge n \leq (a \vee b) \wedge n \leq (a \wedge b) \vee n \leq n \vee n$ implies $m(a,n,b) = n$, which gives a contradiction to the assumption. Now for $b \in M$,
$b \wedge n \leq (a \vee b) \wedge n \leq (a \wedge b) \vee n \leq b \vee n$ implies $b \in M_1$, and so $M \subseteq M_1$. Also, $a \wedge n \leq (a \vee b) \wedge n \leq (a \wedge b) \vee n \leq a \vee n$ implies $a \in M_1$ but $a \notin M$. Hence $M \subset M_1$. Thus we have a contradiction to the maximality of M. Hence there exists some $b \in M$ such that $m(a,n,b) = n$. Conversely, if M is not maximal and $n \notin M$, then by Lemma-2, M is properly contained in a maximal convex sublattice N not containing n. For any element $a \in N - M$, there is an element $b \in M$ such that $m(a,n,b) = n$. Hence $a, b \in N$ and $a \wedge b \leq n \leq a \vee b$ imply $n \in N$, by convexity, and which is a contradiction. Thus M must be maximal.

Following Lemmas give some information on $\{x\}^{\perp_n}$.

Lemma 4. $p \in \{x\}^{\perp_n}$ *if and only if* $p \wedge x \leq n \leq p \vee x$.

Proof. $p \in \{x\}^{\perp_n}$ if and only if $m(p, n, x) = n$ if and only if

$(p\wedge x)\vee(p\wedge n)\vee(x\wedge n)=(p\vee x)\wedge(p\vee n)\wedge(x\vee n)=n$, as n is neutral. This implies $p\wedge x\le n\le p\vee x$.

Lemma 5. $p\in\{x\}^{\perp_n}$ *if and only if* $p\vee n\in\{x\vee n\}^{\perp}$ *in* $[n)$ *and* $p\wedge n\in\{x\wedge n\}^{\perp^d}$ *in* $(n]$.

Proof. Let $p\in\{x\}^{\perp_n}$. Then $p\wedge x\le n\le p\vee x$ and so $(p\vee n)\wedge(x\vee n)=(p\wedge x)\vee n=n$ and $(p\wedge n)\vee(x\wedge n)=(p\vee x)\wedge n=n$ as n is neutral. Hence $p\vee n\in\{x\vee n\}^{\perp}$ in $[n)$ and $p\wedge n\in\{x\wedge n\}^{\perp^d}$ in $(n]$.

Conversely, let $p\vee n\in\{x\vee n\}^{\perp}$ in $[n)$ and $p\wedge n\in\{x\wedge n\}^{\perp^d}$ in $(n]$. Then using neutrality of n, $(p\vee n)\wedge(x\vee n)=n$ implies $(p\wedge x)\vee n=n$,and so $p\wedge x\le n$. Also $(p\wedge n)\vee(x\wedge n)=n$ implies $(p\vee x)\wedge n=n$, and so $n\le p\vee x$. Thus $p\wedge x\le n\le p\vee x$ and so $p\in\{x\}^{\perp_n}$ by Lemma 4.

Now we include some characterizations n-distributive lattices.

Theorem 6. *For a lattice L with a neutral element n, the following conditions are equivalent.*

(i) *L is n-distributive.*

(ii) *For every $a\in L$, $\{a\}^{\perp_n}$ is an n-ideal.*

(iii) *For any $A\subseteq L$, $A^{\perp_n}$ is an n-ideal.*

(iv) *$I_n(L)$ is pseudocomplemented.*

(v) *$I_n(L)$ is 0-distributive.*

(vi) *Every maximal convex sublattice not containing n is prime.*

Proof. (i) $\Rightarrow$(ii). Let $x,y\in\{a\}^{\perp_n}$. Then $a\wedge x\le n\le a\vee x$ and $a\wedge y\le n\le a\vee y$. Since L is n-distributive, so we have $a\wedge(x\vee y)\le n\le a\vee(x\wedge y)$. Then $a\wedge(x\vee y)\le n\le a\vee x\vee y$ and $a\wedge x\wedge y\le n\le a\vee(x\wedge y)$ imply $x\wedge y, x\vee y\in\{a\}^{\perp_n}$ by Lemma 4. Since $m(n,n,a)=n$,so $n\in\{a\}^{\perp_n}$. Finally let $x\le t\le y$ and $x,y\in\{a\}^{\perp_n}$. Then $a\wedge x\le n\le a\vee x$ and $a\wedge y\le n\le a\vee y$ and so, $a\wedge t\le n\le a\vee t$, which implies $t\in\{a\}^{\perp_n}$. Therefore, $\{a\}^{\perp_n}$ is an n-ideal.

(ii) $\Rightarrow$ (iii). Since $A^{\perp_n}=\bigcap_{a\in A}\{\{a\}^{\perp_n}\}$, so $A^{\perp_n}$ is an n-ideal.

(iii) $\Rightarrow$(iv) is trivial as for any n-ideal $A\in I_n(L)$, $A^{\perp_n}$ is the pseudocomplement of A in $I_n(L)$.

(iv) $\Rightarrow$ (v) is trivial as every pseudocomplemented lattice is 0-distributive.
(v) $\Rightarrow$ (vi). Suppose F is a maximal convex sublattice not containing n. Since $F=(F]\cap[F)$ and $n\notin F$, so either $n\notin(F]$ or $n\notin[F)$. Hence by the maximality of F , either F is an ideal or a filter. Let $x\notin F$ and $y\notin F$. Then by Lemma 3, there exists $a,b\in F$ such that $m(x,n,a)=n=m(y,n,b)$. This implies $x\wedge a\le n\le x\vee a$ and $y\wedge b\le n\le y\vee b$. Thus, $x\wedge a\wedge b\le n,\ y\wedge a\wedge b\le n$ and $x\vee a\vee b\ge n,\ y\vee a\vee b\ge n$ and $a\wedge b, a\vee b\in F$ .Then $\langle x\vee n\rangle_n\cap\langle a\wedge b\rangle_n=[n,x\vee n]\cap[a\wedge b\wedge n,(a\wedge b)\vee n]=[n,(x\wedge a\wedge b)\vee n]=[n,n]=\{n\}$ as n is neutral. Similarly, $\langle y\vee n\rangle_n\cap\langle a\wedge b\rangle_n=\{n\}$. Since $I_n(L)$ is 0-distributive, so $\langle a\wedge b\rangle_n\cap(\langle x\vee n\rangle_n\vee\langle y\vee n\rangle_n)=\{n\}$.

This implies $[n,(a\wedge b\wedge(x\vee y))\vee n]=\{n\}$, and so $a\wedge b\wedge(x\vee y)\le n$. Dually, $< x\wedge n >_n\cap< a\vee b >_n=< x\wedge n >_n\cap< a\vee b >_n=$ $\{n\}$. Without loss of generality suppose F is a filter. If $x\vee y\in F$, then $a\wedge b\wedge(x\vee y)\le n$ imply $n\in F$, which is a contradiction. Hence $x\vee y\notin F$. Therefore, F is a prime filter. Similarly, if F is an ideal, then it is a prime ideal.
(vi) $\Rightarrow$ (i). Let $a\wedge b\le n\le a\vee b$ and $a\wedge c\le n\le a\vee c$. We need to prove that $a\wedge(b\vee c)\le n\le a\vee(b\wedge c)$. If not, without loss of generality, let $a\wedge(b\vee c)\not\le$n.

Consider $F=[a\wedge(b\vee c))$. Here $n\notin F$. Then by Lemma-2, there exists a maximal convex sublattice $M\supseteq F$ but $n\notin M$. But a convex sublattice containing a filter is itself a filter. Then by (vi), M is a prime filter. Now $a\in M$ and $b\vee c\in M$ imply $a\wedge b\in M$ or $a\wedge c\in M$ as M is prime. This implies $n\in M$ which is a contradiction. Hence $a\wedge(b\vee c)\le n\le a\vee(b\wedge c)$, and so L is n-distributive.

Corollary 7. *In an n-distributive lattice every filter not containing n is contained in a prime filter.*
Proof. This is trivial by lemma-2 and Theorem 6.

Theorem 8. *Let L be an n-distributive lattice. If* $\{n\}\ne A=\cap\{I; I\ is\ an\ n-ideal\ne\{n\}\}$, *then* $A^{\perp_n}=\{x\in L/\{x\}^{\perp_n}\ne\{n\}\}$.
Proof. Let $x\in A^{\perp_n}$. Then $m(x,n,a)=n$ for all $a\in A$. Since $A\ne\{n\}$, so $\{x\}^{\perp_n}\ne\{n\}$. Thus, $x\in R.H.S$, and so $A^{\perp_n}\subseteq R.H.S.$ Conversely, let $x\in R.H.S$. Since L is n-distributive, $\{x\}^{\perp_n}$ is an n-ideal and $\{x\}^{\perp_n}\ne\{n\}$. Then $A\subseteq\{x\}^{\perp_n}$ and so $A^{\perp_n}\supseteq\{x\}^{\perp_n\perp_n}$. This implies $x\in A^{\perp_n}$, which completes the proof.

We conclude the paper by giving another characterization of n-distributive lattices in terms of annihilator n-ideals which is related to the prime Separation Theorem for n-ideals in a distributive lattice given in [2].

Theorem 9. *Let L be a lattice and n be a neutral element of L. L is n-distributive if and only if for a convex sublattice F disjoint with $\{x\}^{\perp_n}$ $(x \in L)$. There exists a prime convex sublattice $Q \supseteq F$ and disjoint with $\{x\}^{\perp_n}$.*

Proof. Let L be n-distributive and F be a convex sublattice disjoint from $\{x\}^{\perp_n}$. Then applying Zorn's Lemma there exists a maximal convex sublattice Q disjoint from $\{x\}^{\perp_n}$. Since $Q = (Q] \cap [Q)$, so either $(Q] \cap \{x\}^{\perp_n} = \varphi$ or $[Q) \cap \{x\}^{\perp_n} = \varphi$. Hence by the maximality of Q, it is either an ideal or a filter. Without loss of generality, let Q be a filter. We claim that $x \in Q$. If not $Q \vee [x) \supset Q$. Then by the maximality of Q, $(Q \vee [x)) \cap \{x\}^{\perp_n} \neq \varphi$. Let $t \in (Q \vee [x)) \cap \{x\}^{\perp_n}$. Then $t \geq q \wedge x$ for some $q \in Q$ and $t \wedge x \leq n \leq t \vee x$. Thus $q \wedge x \leq t \wedge x \leq n$. Then $m(q \vee n, n, x) = n$, which implies $q \vee n \in \{x\}^{\perp_n}$. But $q \vee n \in Q$ as Q is a filter. This gives a contradiction to the fact that $Q \cap \{x\}^{\perp_n} = \varphi$. Therefore $x \in Q$. Now let $z \notin Q$. Then $(Q \vee [z)) \cap \{x\}^{\perp_n} \neq \varphi$. Let $y \in (Q \vee [z)) \cap \{x\}^{\perp_n}$. Then $y \wedge x \leq n \leq y \vee x$ and $y \geq q_1 \wedge z$ for some $q_1 \in Q$. Thus $q_1 \wedge x \wedge z \leq y \wedge x \leq n$. Then $m(z, n, (q_1 \wedge x) \vee n) = n$, where $(q_1 \wedge x) \vee n \in Q$ as it is a filter. Therefore, by Lemma 3, Q is a maximal filter not containing n. Hence by

Theorem 6, Q is prime.

Conversely, let $\langle x \rangle_n \cap \langle y \rangle_n = \{n\}$ and $\langle x \rangle_n \cap \langle z \rangle_n = \{n\}$. We need to prove that $\langle x \rangle_n \cap (\langle y \rangle_n \vee \langle z \rangle_n) = \{n\}$. That is $x \wedge (y \vee z) \leq n \leq x \vee (y \wedge z)$. If not, let $x \wedge (y \vee z) \not\leq$ n. Then $[y \vee z) \cap \{x\}^{\perp_n} = \varphi$. For otherwise $t \in [y \vee z) \cap \{x\}^{\perp_n}$, implies $t \wedge x \leq n \leq t \vee x$ and $t \geq y \vee z$. Which implies $x \wedge (y \vee z) \leq t \wedge x \leq n$, a contradiction. So, there exists a prime filter Q containing $[y \vee z)$ disjoint with $\{x\}^{\perp_n}$. As $y, z \in \{x\}^{\perp_n}$, so $y, z \notin Q$. Thus $y \vee z \notin Q$, as Q is prime. This implies $[y \vee z) \not\subset Q$, a contradiction. Dually by taking $x \vee (y \wedge z) \not\geq$ n, we would have another contradiction. Therefore, $x \wedge (y \vee z) \leq n \leq x \vee (y \wedge z)$, and so L is n-distributive.

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