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# Invariants and Flatness Criteria on an Almost Lagrangian Supermanifolds

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**ABSTRACT**

We studied Bianchi identities on distinguished ssuperconnections [4] and superconnections on an almost Lagrangian supermanifolds [3]. In this paper we studied to set up the necessary machinery to determine when an almost Lagrangian supermanifolds is flat with respect to a given scale and defined bilinear form which is a scale invariant.

***Keywords:***

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## 1. Introduction

Since the conception of supersymmetry [5] and super geometry has come to play an increasingly important role in the theoretical physics and is essential part of almost every attempt to go beyond the standard model in particle physics [7]. On any almost Lagrangian supermanifolds there exists a spinor connection [3] in superconnection consistent in Riemannian geometry [6] with some scale. In this paper we investigate the properties of such connections and aim to set up the necessary machinery to determine when an almost Lagrangian supermanifolds is flat with respect to a scalar and finally defined a bilinear form is a scale invariant.

## 2. Supertwistor Bundle

In [3], we gave a definition of a spinor connection as a map,

$$D: \sqcap S \otimes \sqcap S \to \sqcap S$$

Equivalently we may define a spinor connection to be a map $D$, which splits the following exact sequence,

$$0 \to \sqcap S^* \otimes \sqcap S \to J_r^1 \sqcap S \to \sqcap S \to 0$$

where $J_r^1 \sqcap S$ is called the reduced 1$^{st}$ jet bundle [1].

We now define the 1$^{st}$ jet bundle of a general bundle $E \sqcap S$ in the usual manner.

**Definition 1.** Let $E$ be a vector bundle over a supermanifold [2] $\mathcal{M}$. We define the 1st Jet Bundle of $E$ by the following exact sequence,

$$0 \to \Omega\mathcal{M} \otimes E \to J^1E \to E \to 0$$

In particular we are interested in the case where $\mathcal{M}$ is an almost Lagrangian supermanifold where $E = \sqcap S$. In this situation, taking the dualization of the Frobenious form gives the exact sequence,

$$0 \to S^* \odot S^* \to \Omega\mathcal{M} \to \sqcap S \to 0$$

We now choose a spinor connection [3], provides a splitting of the above exact sequence, giving,

$$\Omega\mathcal{M} \cong \sqcap S^* \oplus S^* \odot S^*$$

Using this splitting and the properties of the Parity change functor, we find that the first jet bundle of $\sqcap S$ is defined by the exact sequence,

$$0 \to (\sqcap S \otimes \sqcap S) \oplus (\sqcap S^* \Lambda \sqcap S^* \otimes \sqcap S) \to J^1\sqcap S \to \sqcap S \to 0$$

Choosing a scale $\in_{\alpha_{1,}\alpha_2 \ldots,\alpha_n} = \phi\eta_{\alpha_{1,}\alpha_2 \ldots,\alpha_n}$ and its distinguished superconnection $D$, we define maps $\Delta$ and $\Lambda$ in the following way,

$$\Delta \;:\; J^1\sqcap S \;\to \sqcap S^* \otimes \sqcap S$$

$$[\omega^\alpha] \longmapsto \frac{1}{n} D_\beta \omega^\alpha \equiv \theta^\alpha_\beta$$

and

$$\Lambda \;\;:\;\; J^1\sqcap S \;\to \sqcap S^* \Lambda \sqcap S^* \otimes \sqcap S$$

$$[\omega^\alpha] \longmapsto \frac{1}{n}\left[D_\mu, D_v\right]\omega^\alpha - \frac{1}{n(n+1)} 2\delta^\alpha_{[\mu, D_v]}\{(D_\in log\emptyset)\omega^\in\} \equiv \theta^\alpha_{\mu v}$$

We may now use the map $\Delta \oplus \Lambda$ to split the exact sequence defining the 1st jet bundle of $\sqcap S$.

Defining a mapping which we call the *trace*, in the following way

$$tr \;: \to (\sqcap S^* \otimes \sqcap S) \oplus \sqcap S^* \Lambda \sqcap S^* \otimes \sqcap S \;\to \mathcal{O}_\mathcal{M} \oplus \sqcap S^*$$

$$\left(\theta^\gamma_\alpha, \theta^\gamma_{\alpha\beta}\right) \qquad \longmapsto \qquad \left(\theta^\alpha_\alpha, \theta^\alpha_{\alpha\beta}\right)$$

leads us to the following definition.

**Definition 2.** Let $\mathcal{M}$ be an almost Lagrangian supermanifold. We define the supertwistor bundle of $\mathcal{M}$ to be the subbundle $\mathcal{T}$ of $J^1\sqcap S$ which makes the following diagram with exact rows commute,

$$\begin{tikzcd}
0 \arrow[r] & (\sqcap S \otimes \sqcap S) \oplus (\sqcap S^* \Lambda \sqcap S^* \otimes \sqcap S) \arrow[r] \arrow[d] & J^1\sqcap S \arrow[r] \arrow[d] & \sqcap S \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_\mathcal{M} \oplus \sqcap S^* \arrow[r] & \mathcal{T} \arrow[r] & \sqcap S \arrow[r] & 0
\end{tikzcd}$$

As a choice of distinguished super connection splits the top row of the above diagram, it may also be used to split the bottom row by the commutivity property of the diagram. Hence,

$$\mathcal{T} \cong \mathcal{O}_{\mathcal{M}} \oplus \Pi S \oplus \Pi S^{*}$$
$$= (\theta_{\in}^{\in},\ \omega^{\alpha}, \theta_{\alpha\in}^{\in})$$

Using the transformation rules we find the following,

$$\hat{\theta}_{\in}^{\in} = \frac{1}{n}\widehat{D}_{\in}\omega^{\in}$$
$$= \frac{1}{n}D_{\in}\omega^{\in} + \frac{1}{n}\delta_{\in}^{\in}(\gamma_{\in}\omega^{\in})$$
$$= \theta_{\in}^{\in} + \gamma_{\in}\omega^{\in}$$

Now defining $\zeta \equiv \theta_{\in}^{\in}$ we obtain, $\hat{\zeta} = \zeta + \gamma\delta\omega^{\delta}$

Also,

$$\widehat{\theta_{\alpha\in}^{\in}} = \frac{1}{n}\left[\widehat{D}_{\alpha}, \widehat{D}_{\in}\right]\omega^{\in} - \frac{1}{n(n+1)}2\delta_{[\alpha}^{\in}\widehat{D}_{\in]}\left\{\left(\widehat{D}_{\in}log\widehat{\emptyset}\right)\omega^{\in}\right\}$$
$$= \frac{1}{n}[D_{\alpha}, D_{\in}]\omega^{\in} - \frac{1}{n(n+1)}2\delta_{[\alpha}^{\in}\widehat{D}_{\in]}\{(D_{\in}log\emptyset)\omega^{\in}\} - \frac{1}{n}\gamma_{\alpha}D_{\in}\omega^{\in}$$
$$-\frac{1}{n}\gamma_{\in}D_{\in}\omega^{\alpha} + \frac{n+1}{n}(D_{\alpha}\gamma_{\in})\omega^{\in} - \frac{1}{n}D_{\alpha}(\gamma_{\in}\omega^{\in})$$
$$= \theta_{\alpha\in}^{\in} + (D_{\alpha}\gamma_{\in})\omega^{\in} - \frac{1}{n}\gamma_{\alpha}D_{\in}\omega^{\in}$$

Defining $\pi_{\alpha} \equiv \theta_{\alpha\in}^{\in}$ we obtain,

$$\hat{\pi}_{\alpha} = \pi_{\alpha} + (D_{\alpha}\gamma_{\delta})\omega^{\delta} - \gamma_{\alpha}\zeta$$

Finally, the supertwistor bundle can be expressed in the form,

$$\mathcal{T} \cong \mathcal{O}_{\mathcal{M}} \oplus \Pi S \oplus \Pi S^{*}$$
$$= (\zeta,\ \omega^{\alpha}, \pi_{\alpha})$$

## 3. Supertwistor Equation

**Definition 3.** Let $D$ denote a distinguished superconnection on $\mathcal{M}$, with associated scale $\in$. Also let $\omega^{\alpha} = \widehat{\omega^{\alpha}}$ be such that,

$$D^{\boldsymbol{\theta_1}\ldots\boldsymbol{\theta_{n-2}}}[\alpha_{\omega}\beta] = 0$$

where $D^{\boldsymbol{\theta_1},\ldots,\boldsymbol{\theta_{n-2}}\boldsymbol{\alpha}}\omega^{\beta} = \in^{\theta_1\ldots\theta_{n-2}}\ \alpha\mu D_{\mu}\omega^{\beta}$. We call the above *the suprtwistor equation of D.*

**Theorem 3.1.** Supertwistor equations are independent of the choice of scale.

**Proof.** Let $D$ and $\widehat{D}$ denote distinguished superconnections with associated scales $\in$ and $\widehat{\in}$ respectively. By hypothesis, $\omega^{\alpha}$ is invariant, therefore,

$$2D^{\theta_1\ldots\widehat{\theta_{n-2}}}[\alpha_{\omega}\beta] = \in^{\theta_1\ldots\theta_{n-2}\alpha\mu}\ \alpha\mu\widehat{D}_{\mu}\omega^{\beta} + \in^{\theta_1\ldots\theta_{n-2}\beta\mu}\ \widehat{D}_{\mu}\omega^{\alpha}$$
$$= \in^{\theta_1\ldots\theta_{n-2}\alpha\mu}\ D_{\mu}\omega^{\beta} + \in^{\theta_1\ldots\theta_{n-2}\beta\mu}\ D_{\mu}\omega^{\alpha}$$
$$+ \in^{\theta_1\ldots\theta_{n-2}\alpha\mu}\ \delta_{\mu}^{\beta}\gamma_{v}\omega^{v} + \in^{\theta_1\ldots\theta_{n-2}\beta\mu}\ \delta_{\mu}^{\alpha}\gamma_{v}\omega^{v}$$
$$= 2D^{\boldsymbol{\theta_1}}\ldots\boldsymbol{\theta_{n-2}}[\alpha_{\omega}\beta] \qquad \blacksquare$$

Thus, it makes sense simply to speak simply of the *supertwistor equation.* The supertwistor equation implies,

$$D^{\theta_1\ldots\theta_{n-2}\alpha}\omega^{\beta} = -D^{\theta_1\ldots\theta_{n-2}\beta}\omega^{\alpha}$$

Thus,

$$D^{\theta_1\dots\theta_{n-2}\alpha}\omega^{\beta} = \in^{\theta_1\dots\theta_{n-2}\alpha\beta}\zeta$$
$$= \in^{\theta_1\dots\theta_{n-2}\alpha\mu} D_\mu\omega^{\beta}$$

Multiplying by $\in^{\theta_1\dots\theta_{n-2}\alpha k}$ gives

$$D_k\omega^{\beta} = \delta_k^{\beta}\zeta$$

We now define $\pi_\mu = D_\mu\zeta$. Now using these expressions,

$$D_\mu D_v\omega^{\alpha} = \delta_\mu^{\alpha}\pi_\mu$$

Implying

$$[D_\mu, D_v]\omega^{\alpha} = 2\delta_{[\mu}^{\alpha}\pi_{v]}$$

Also, by definition,

$$[D_\mu, D_v]\omega^{\alpha} = D_{\mu v}\omega^{\alpha} + R_{\mu v\beta}^{\alpha}\,\omega^{\beta}$$

Therefore,

$$D_{\mu v}\omega^{\alpha} = 2\delta_{[\mu}^{\alpha}\pi_{v]} - R_{\mu v\beta}^{\alpha}\,\omega^{\beta}$$

Now,

$$D_\gamma D_{\mu v}\omega^{\alpha} = \delta_\mu^{\alpha}D_\gamma\pi_v + \delta_v^{\alpha}D_\gamma\pi_\mu - \left(D_\gamma R_{\mu v\beta}^{\alpha}\right)\omega^{\beta} - R_{\mu v\gamma}^{\alpha}\zeta$$

and

$$D_{\mu v}D_\gamma\omega^{\alpha} = \delta_\mu^{\alpha}D_\mu\pi_v + \delta_\gamma^{\alpha}D_v\pi_\mu$$

Giving,

$$[D_\gamma, D_{\mu v}]\omega^{\alpha} = \delta_\mu^{\alpha}D_\gamma\pi_v + \delta_v^{\alpha}D_\gamma\pi_\mu - \delta_\gamma^{\alpha}D_\mu\pi_v - \delta_\gamma^{\alpha}D_v\pi_\mu - \left(D_\gamma R_{\mu,v,\beta}^{\alpha}\right)\omega^{\beta} - R_{\mu,v,\gamma}^{\alpha}\zeta$$

Also, by definition,

$$[D_\gamma, D_{\mu v}]\omega^{\alpha} = -T_{\alpha,\mu v}^{\rho\sigma}D_{\rho\sigma}\omega^{\alpha} + R_{\gamma,\mu v,\beta}^{\alpha}\omega^{\beta}$$
$$= -T_{\alpha,\mu v}^{\rho\sigma}\left\{\delta_\rho^{\alpha}\pi_\sigma + \delta_\sigma^{\alpha}\pi_\rho - R_{\rho,\sigma,\beta}^{\alpha}\omega^{\beta}\right\} + R_{\gamma,\mu v,\beta}^{\alpha}\omega^{\beta}$$
$$= \left\{R_{\gamma,\mu v,\beta}^{\alpha} + T_{\alpha,\mu v}^{\rho\sigma}R_{\rho,\sigma,\beta}^{\alpha}\right\}\omega^{\beta} - 2T_{\gamma,\mu v}^{\alpha\beta}\pi_\beta$$

Taking the trace of the first expression for $[D_\gamma, D_{\mu v}]\omega^{\alpha}$ gives,

$$[D_\gamma, D_{\mu v}]\omega^{v} = nD_\gamma\pi_\mu - D_\mu\pi_\gamma - \left(D_\gamma R_{\mu,v,\beta}^{v}\right)\omega^{\beta} + R_{v,\gamma,\mu}^{v}\zeta$$

The trace for the second expression gives,

$$[D_\gamma, D_{\mu v}]\omega^{v} = \left\{R_{\gamma,\mu v,\beta}^{v} + T_{\gamma,\mu v}^{\rho\sigma}R_{\rho,\sigma,\beta}^{v}\right\}\omega^{\beta}$$

Thus,

$$nD_\gamma\pi_\mu - D_\mu\pi_\gamma = \left\{D_\gamma R_{\mu,v,\beta}^{v} + R_{\gamma,\mu v,\beta}^{v} + T_{\gamma,\mu v}^{\rho\sigma}R_{\rho,\sigma,\beta}^{v}\right\}\omega^{\beta} - R_{v,\gamma,\mu}^{v}\zeta$$
$$= G_{\gamma\mu}$$

Therefore,

$$D_\mu\pi_\gamma = \frac{1}{n^2+1}\left(nG_{\mu\gamma} + G_{\gamma\mu}\right)$$

By the above, $\nabla_\gamma\pi_\mu$ is a linear combination of $\omega^{\alpha}$ and $\zeta$, that is,

$$D_\gamma \pi_\mu = P_{\gamma\mu\delta}\,\omega^\delta + Q_{\gamma\mu}\,\zeta$$

where,

$$P_{\gamma\mu\delta} = \frac{1}{n^2-1}\{nD_\gamma R^v_{\mu,v,\delta} + nR^v_{\gamma,\mu v,\delta} + nT^{\rho\sigma}_{\gamma,\mu v} R_{\rho,\sigma,\delta^v} + D_\mu R^v_{\gamma,v,\delta} + R^v_{\mu,\gamma v,\delta} + T^{\rho\sigma}_{\mu,\gamma v} R^v_{\rho,\sigma,\delta}\}$$

$$Q_{\gamma\mu} = \frac{1}{n^2-1}\{-nR^v_{v,\gamma,\mu} - R^v_{v,\mu,\gamma}\} = -3\tfrac{n-1}{n+1} R_{\gamma\mu}$$

In the case $T^{\rho\sigma}_{\alpha,\mu v} = 0$ we can use the Bianchi identities [4] to show,

$$P_{\gamma\mu\delta} = \left(\frac{n-1}{n+1}\right)\{2D_\gamma R_{\mu\delta} - D_\mu R_{\gamma\delta}\} + \frac{1}{n-1} W^{\in}_{(\gamma\in)\mu\delta} + \frac{n-2}{n^2-1} W^{\in}_{(\gamma\mu)\in\delta}$$

Or, equivalently,

$$P_{\gamma\mu\delta} = \left(\frac{n-1}{n+1}\right)\{D_\gamma R_{\mu\delta} + D_\delta R_{\mu\gamma}\} + \frac{1}{n-1} W^{\in}_{(\gamma\in)\mu\delta} + \frac{2n-1}{n^2-1} W^{\in}_{(\gamma\mu)\in\delta}$$

Now, under a change of scale $\in \rightarrow \widehat{\in}$

$$\begin{aligned} D_\beta \widehat{\omega^\alpha} &= \widehat{D}_\beta \omega^\alpha \\ &= D_\beta \omega^\alpha + \delta^\alpha_\beta \gamma_\delta \omega^\delta \\ &= \delta^\alpha_\beta (\zeta + \gamma_\delta \omega^\delta) \\ &= \delta^\alpha_\beta \hat{\zeta} \end{aligned}$$

Therefore $\hat{\zeta} = \zeta + \gamma\beta\omega^\beta$. Given this we can now determine how $\pi_\alpha$ changes with respect to the scale,

$$\begin{aligned} \widehat{\pi_\alpha} &= \widehat{D_\alpha}\zeta \\ &= \widehat{D}_\alpha \hat{\zeta} \\ &= D_\alpha \hat{\zeta} \\ &= D_\alpha \zeta + D_\alpha \gamma_\beta \omega^\beta \\ &= \pi_\alpha + (D_\alpha \gamma_\beta)\omega^\beta - \gamma\beta(D_\alpha \omega^\beta) \\ &= \pi_\alpha + (D_\alpha \gamma_\beta)\omega^\beta - \gamma_\alpha \zeta \end{aligned}$$

In summary we have shown the following.

**Theorem 3.2.** The supertwistor equation is equivalent to the following system of equations,

$$D_\mu \omega^\alpha = \delta^\alpha_\mu \zeta$$
$$D_\mu \zeta = \pi_\mu$$
$$D_\mu \pi_\alpha = P_{\mu\alpha\beta}\,\omega^\beta + Q_{\mu\alpha}\,\zeta$$

where the $\omega^\alpha$, $\pi_\alpha$, and $\zeta$ transform in the following way,

$$\widehat{\omega}^\alpha = \omega^\alpha$$
$$\widehat{\pi_\alpha} = \pi_\alpha + (D_\alpha \gamma_\beta)\omega^\beta - \gamma_\alpha \zeta$$

$$\hat{\zeta} = \zeta + \gamma_\beta \omega^\beta$$

**4. Supertwistor connection**

We will now define a connection on the supertwistor bundle $\mathcal{T}$, which we denote by $\nabla_\mu$, in the following way,

$$\nabla_\mu \begin{pmatrix} \zeta \\ \omega^\alpha \\ \pi_\alpha \end{pmatrix} = D_\mu \begin{pmatrix} \zeta \\ \omega^\alpha \\ \pi_\alpha \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\delta_\mu^\beta \\ \delta_\mu^\alpha & 0 & 0 \\ Q_{\mu\alpha} & P_{\mu\alpha\beta} & 0 \end{pmatrix} \begin{pmatrix} \zeta \\ \omega^\beta \\ \pi_\beta \end{pmatrix}$$

so that covariantly constant sections of $\nabla_\mu$ are solutions to the twistor equation. In accordance with Super Yang-Mills theory we set,

$$[\nabla_\mu, \nabla_v]Z = \nabla_{\mu v} Z$$

where
$$Z = \begin{pmatrix} \zeta \\ \omega^\alpha \\ \pi_\alpha \end{pmatrix}$$

Given this, we can show that,

$$\nabla_{\mu v} \begin{pmatrix} \zeta \\ \omega^\alpha \\ \pi_\alpha \end{pmatrix}$$
$$= D_{\mu v} \begin{pmatrix} \zeta \\ \omega^\alpha \\ \pi_\alpha \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & -2P_{[\mu v]\beta} & 0 \\ 0 & R^\alpha_{\mu,v,\beta} & -2\delta^\alpha_{[\mu}\delta^\beta_{v]} \\ -2D_{[\mu}Q_{v]\alpha} + 2P_{[\mu|\alpha|\alpha]} & -2D_{[\mu}P_{v]\alpha\beta} & 2\delta^\beta_{[\mu}Q_{v]\alpha} - R^\beta_{\mu,v,\alpha} \end{pmatrix} \begin{pmatrix} \zeta \\ \omega^\beta \\ \pi_\beta \end{pmatrix}$$

Notice that the bottom right $2 \times 2$ block of the matrix above gives the results of the usual twistor connection on an almost Lagrangian manifold plus extra terms. We now consider the commutator$[\nabla_\gamma, \nabla_{\mu v}]$, and setting $T^{\rho\sigma}_{\alpha,\mu v} = 0$ we obtain

$$[\nabla_\gamma, \nabla_{\mu v}] \begin{pmatrix} \zeta \\ \omega^\alpha \\ \pi_\alpha \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \zeta \\ \omega^\beta \\ \pi_\beta \end{pmatrix}$$

where
$$a_{11} = 2D_{[\mu}Q_{v]\gamma} + 2P_{[\mu v]\gamma} - 2P_{[\mu|\gamma|v]}$$
$$a_{12} = 2D_{[\mu}P_{v]\gamma\beta} - D_\gamma 2P_{[\mu v]\beta}$$
$$a_{13} = R^\beta_{\mu v\gamma} + 2\delta^\beta_{[\mu}Q_{v]\gamma}$$
$$a_{21} = 2Q_{\gamma[\mu}\delta^\alpha_{v]} - R^\alpha_{\mu v\gamma}$$
$$a_{22} = 2\delta^\alpha_{[\mu}P_{|\gamma|v]\beta} - \delta^\alpha_\gamma 2P_{[\mu v]\beta} - D_\gamma R^\alpha_{\mu v\beta} - R^\alpha_{\gamma,\mu v\beta}$$
$$a_{23} = 0$$
$$a_{31} = D_\gamma(2D_{[\mu}Q_{v]\alpha}) + 2D_{[\mu}P_{v]\alpha\gamma} + 2\delta^\delta_{[\mu}Q_{v]\alpha}Q_{\gamma\delta} + R^\delta_{\mu v\alpha}Q_{\gamma\delta} - D_\gamma 2P_{[\mu|\alpha|v]} - D_{\mu v}Q_{\gamma\alpha}$$
$$a_{32} = D_\gamma(2D_{[\mu}P_{v]\alpha\beta}) + P_{\gamma\alpha\delta}R^\delta_{\mu v\beta} + P_{\gamma\delta\beta}R^\delta_{\mu v\alpha} + P_{\gamma\delta\beta}2\delta^\delta_{[\mu}Q_{v]\alpha} - 2P_{[\mu v]\beta}Q_{\gamma\alpha} - D_{\mu v}P_{\gamma\alpha\beta}$$

$$a_{33} = \delta^{\beta}_{\gamma} 2P_{[\mu|\alpha|v]} - 2P_{\gamma\alpha[\mu}\delta^{\beta}_{v]} + D_{\gamma} R^{\beta}_{\mu v\alpha} + D_{\gamma}\left(2\delta^{\beta}_{[\mu} Q_{v]\alpha}\right) - 2D_{[\mu} Q_{v]\alpha}\delta^{\beta}_{\gamma} + R^{\beta}_{\gamma,\mu v,\alpha}$$

and

$$X_{[\alpha\beta]} = \frac{1}{2} X_{\alpha\beta} - (-1)^{\hat{\alpha}\hat{\beta}} \frac{1}{2} X_{\beta\alpha}$$

$$Q_{\mu v} = \frac{n-1}{n+1} R_{\mu v}$$

$$P_{\gamma\alpha\beta} = \frac{1}{n^2-1}\Big\{ nD_{\gamma} R^{\delta}_{\delta\alpha\beta} + nR^{\delta}_{\gamma,\delta\alpha,\beta} + nT^{\rho\sigma}_{\gamma,\alpha\delta} R^{\delta}_{\rho\sigma\ \beta} + D_{\alpha} R^{\delta}_{\delta\gamma\beta} + R^{\delta}_{\alpha,\delta\gamma,\beta} + T^{\rho\sigma}_{\alpha,\gamma\delta} R^{\delta}_{\rho\sigma\beta}\Big\}$$

By using the solutions of the super-Bianchi identities we can show that the matrix above reduces to the following,

$$\left[\nabla_{\gamma}, \nabla_{\mu v}\right]\begin{pmatrix}\zeta\\ \omega^{\alpha}\\ \pi_{\alpha}\end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{n} D_{\delta} Y^{\delta}_{\gamma[\mu v]\beta} & 0\\ 0 & -Y^{\alpha}_{\gamma[\mu v]\beta} & 0\\ -\frac{1}{n} D_{\delta} Y^{\delta}_{\gamma[\mu v]\alpha} & -B_{\gamma,\mu v,\alpha,\beta} & +Y^{\beta}_{\gamma[\mu v]\alpha}\end{pmatrix}\begin{pmatrix}\zeta\\ \omega^{\beta}\\ \pi_{\beta}\end{pmatrix}$$

where $Y^{\alpha}_{\gamma[\mu v]\beta}$ is the trace free scale invariant quantity $W^{\alpha^0}_{(\gamma\mu)v\beta} + W^{\alpha^0}_{(\gamma v)\mu\beta}$.

Now given,

$$Z = \begin{pmatrix}\zeta\\ \omega^{\alpha}\\ \pi_{\alpha}\end{pmatrix} \text{ and } Z' = \begin{pmatrix}\zeta'\\ \omega'^{\alpha}\\ \pi'_{\alpha}\end{pmatrix}$$

We define $b \in Osp(1|2n)$ in the following way,

$$b(Z, Z') = \zeta\zeta' - \omega^{\alpha}\pi'_{\alpha} + \pi_{\alpha}\omega'^{\alpha} + 3\left(\frac{n-1}{n+1}\right) R_{\alpha\beta}\,\omega^{\alpha}\omega'^{\beta}$$

Or, equivalently, in supermatrix notation,

$$b(Z, Z') = Z^{st} B Z'$$

$$= \begin{pmatrix}\zeta\\ \omega^{\alpha}\\ \pi_{\alpha}\end{pmatrix}^{st} \begin{pmatrix} 1 & 0 & 0\\ 0 & 3\left(\frac{n-1}{n+1}\right) R_{\alpha\beta} & -\delta^{\beta}_{\alpha}\\ 0 & \delta^{\beta}_{\alpha} & 0\end{pmatrix}\begin{pmatrix}\zeta'\\ \omega'^{\beta}\\ \pi'_{\beta}\end{pmatrix}$$

where $Z^{st}$ denotes the supertranspose. We now prove two important properties of $b$.

**Proposition 4.1.** The Bilinear form $b$ is a scale invariant.

**Proof.** Using the transformation rules for $\omega^{\alpha}, \pi_{\alpha}$ and $\zeta$ we find,

$$b(\widehat{Z, Z'}) = \widehat{\zeta\zeta'} - \widehat{\omega^{\alpha}\pi'_{\alpha}} + \widehat{\pi_{\alpha}\omega'^{\alpha}} + 3\left(\frac{n-1}{n+1}\right) R_{\alpha\beta}\,\widehat{\omega^{\alpha}\omega'^{\beta}}$$

$$= \zeta\zeta' + \zeta\gamma_{\epsilon}\omega'^{\epsilon} + \zeta'\gamma_{\epsilon}\omega^{\epsilon} + (\gamma_{\epsilon}\omega^{\epsilon})(\gamma_{\beta}\omega^{\beta}) - \omega^{\alpha}\pi'_{\alpha}$$

$$-\omega^{\alpha}(D_{\alpha}\gamma_{\epsilon})\omega'^{\epsilon} + \omega^{\alpha}\gamma_{\alpha}\zeta' + \pi_{\alpha}\omega'^{\alpha} + (D_{\alpha}\gamma_{\epsilon})\omega^{\epsilon}\omega'^{\alpha}$$

$$-\gamma_\alpha \zeta \omega'^{\alpha} + 3\left(\frac{n-1}{n+1}\right) R_{\alpha\beta}\, \omega^\alpha \omega'^\beta + \left(D_\alpha \gamma_\beta\right)\omega^\alpha \omega'^\beta$$
$$-\left(D_\beta \gamma_\alpha\right)\omega^\alpha \omega'^\beta + \left(\gamma_\alpha \gamma_\beta\right)\omega^\alpha \omega'^\beta .$$
$$= \zeta\zeta' - \omega^\alpha \pi'_\alpha + \pi_\alpha \omega'^\alpha + 3\left(\tfrac{n-1}{n+1}\right) R_{\alpha\beta}\, \omega^\alpha \omega'^\beta$$
$$= b(Z, Z') \qquad \blacksquare$$

**Proposition 4. 2.** The bilinear form $b$ is consistent with the supertwistor connection, that is,

$$\nabla_\mu b(Z, Z') = b(\nabla_\mu Z, Z') + b(Z, \nabla_\mu Z')$$

**Proof.** Using the first form of b be find that**,**

$$\nabla_\mu b(Z, Z') = \nabla_\mu \left\{ \zeta\zeta' - \omega^\alpha \pi'_\alpha + \pi_\alpha \omega'^\alpha + 3\left(\frac{n-1}{n+1}\right) R_{\alpha\beta}\, \omega^\alpha \omega'^\beta \right\}$$

$$= (\nabla_\mu \zeta)\zeta' - (\nabla_\mu \omega^\alpha)\pi'_\alpha + (\nabla_\mu \pi_\alpha)\omega'^\alpha + 3\left(\tfrac{n-1}{n+1}\right) R_{\alpha\beta} (\nabla_\mu \omega^\alpha)\omega'^\beta +$$
$$\zeta(\nabla_\mu \zeta') + \omega^\alpha(\nabla_\mu \pi'_\alpha) - \pi_\alpha(\nabla_\mu \omega'^\alpha) -$$
$$3\left(\tfrac{n-1}{n+1}\right) R_{\alpha\beta}\, \omega^\alpha (\nabla_\mu \omega'^\beta)$$

We now calculate $b(\nabla_\mu Z, Z') + b(Z, \nabla_\mu Z')$ using the supermatrix form of $b$. First, by the rules of supermatrices,

$$\nabla_\mu Z = \nabla_\mu \begin{pmatrix} \zeta \\ \omega^\alpha \\ \pi_\alpha \end{pmatrix} = \begin{pmatrix} \nabla_\mu \zeta \\ -\nabla_\mu \omega^\alpha \\ -\nabla_\mu \pi_\alpha \end{pmatrix}$$

Now, as this vector is odd,

$$(\nabla_\mu Z)^{st} = \begin{pmatrix} \nabla_\mu \zeta \\ -\nabla_\mu \omega^\alpha \\ -\nabla_\mu \pi_\alpha \end{pmatrix}^{st} = (\nabla_\mu \zeta,\ \nabla_\mu \omega^\alpha,\ \nabla_\mu \pi_\alpha)$$

Now, by using the supermatrix form of $b$ we have,

$$b(\nabla_\mu Z, Z') = (\nabla_\mu \zeta)\zeta' - (\nabla_\mu \omega^\alpha)\pi'_\alpha + (\nabla_\mu \pi_\alpha)\omega'^\alpha + 3\left(\tfrac{n-1}{n+1}\right) R_{\alpha\beta} (\nabla_\mu \omega^\alpha)\omega'^\beta$$

Similarly, we can show,

$$b(Z, \nabla_\mu Z') = \zeta(\nabla_\mu \zeta') + \omega^\alpha(\nabla_\mu \pi'_\alpha) - \pi_\alpha(\nabla_\mu \omega'^\alpha) - 3\left(\frac{n-1}{n+1}\right) R_{\alpha\beta}\, \omega^\alpha (\nabla_\mu \omega'^\beta)$$

upon summing these expressions we find that

$$\nabla_\mu b(Z, Z') = b(\nabla_\mu Z, Z') + b(Z, \nabla_\mu Z') \qquad \blacksquare$$

We now come to the main theorem.

**Theorem 4.3.** Let $\mathcal{M}$ be an almost Lagrangian supermanifold. Then $\mathcal{M}$ is locally isomorphic to the isotropic Grassmanian *GI* **,** if and only if the entries of the twistor matrix vanish.

**Proof.** Suppose the remaining entries in the twistor matrix are zero, then the corresponding parallel transport is integrable. The superspace of globally horizontal sections of the twistor bundle $\mathcal{T}$ over a neighborhood $\mathcal{U}$ of some $\widetilde{\in}\ \mathcal{M}$ can be identified with the fibre of $\mathcal{T}$ at $p$, which we will denote by $\mathbb{T}$. Since the bilinear form $b$ defined previously, is consistent with the twistor connection, there is a natural symmetric bilinear form defined on $\mathbb{T}$. By definition $\prod S^*$ is a sub-bundle of $\mathcal{T}$ and thus we may relate to each point $p \in \mathcal{U}$ a $(0|n)$ dimensional plane $S_q^{0|n} \subset \mathbb{T}$ as the set of those horizontal sections of $\mathcal{T}|_{\mathcal{U}}$ whose values at $q$ lie in $\prod S^*$, that is in some trivialization $(\zeta, \omega^\alpha, \pi_\alpha)$ of $\mathcal{T}$ the following holds, $\omega^\alpha(q) = \zeta(q) = 0$. By the definition of $b$ we have that for any $q \in \mathcal{U}$ each plane $S_q^{0|n}$ is $b$ isotropic [1]. Thus we have defined a mapping, $\gamma: q \to S_q^{0|n}$ from $\mathcal{U}$ to some superdomain $\mathbb{U} \subset GI$. Further, we are free to choose $\mathcal{U}$ so small that the map $\gamma$ is injective. ■

## REFERENCES


1. Ali, M. S., Moduli spaces of compact complex isotropic submanifilds in complex contact manifold, Phd thesis, Department of Mathematics, University of Glasgow, UK, 2003.
2. Ali, M. S., A New Approach on the Torsion and Curvature Tensors in Supermanifolds, BRAC University Journal, II (2) (2005) 57-64.
3. Ali, M. S., Superconnections on an almost Lagrangian Supermanifolds, The CDR journal, I(I) (2006) 21-32.
4. Ali, M. S., Talukder, M. G. M., Khan, M. R. and Sarkar, Md. Abu Hanif, Bianchi Identities on Distinguished Superconnections, Dhaka University Journal of Science, 59(2), (2011) 235-239.
5. Bernstein, J., Lectures on Supersymmetry, Quantum Field Theory Program at IAS, http://www.math.ias.edu/QFT/fall/bernstein2.ps (1996).
6. Boothby, W. M., An Introduction to Differentiable Manifolds and Riemannian Geometry, 2003.
7. Isham, J. Cherish, Modern Differential Geometry for Physicists, World Scientific Publishing Co. Pte. Ltd., 1989.