On Semiderivations in Prime Gamma Rings

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ABSTRACT

Let $M$ be a prime gamma ring. Let $d : M \rightarrow M$ be a semiderivation associated with a function $g : M \rightarrow M$. We prove that $d$ must be an ordinary derivation or of the form $d(x) = p\delta(x - g(x))$ for all $x \in M$, $\delta \in \Gamma$, where $p$ is an element of the extended centroid of $M$. We have also seen that $g$ must necessarily be an endomorphism.

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1. Introduction

Let $M$ and $\Gamma$ be additive abelian groups. $M$ is called a $\Gamma$-ring if for all $x, y, z \in M$, $\alpha, \beta \in \Gamma$ the following conditions are satisfied:

(i) $x\beta y \in M$,

(ii) $(x+y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha+\beta)y = x\alpha y + x\beta y$, $x\alpha(y+z) = x\alpha y + x\alpha z$,

(iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$.

A $\Gamma$-ring $M$ is called prime if $x\Gamma M \Gamma y = 0$ implies that $x = 0$ or $y = 0$.

The structure of semi-derivations of prime rings has been studied by C. L. Chuang [5]. He proved a structure theorem with the help of extended centroid of the classical associative rings. The same results have been obtained by M. Bresar [4].

prime rings with semiderivations and investigated the commutativity properties. J. C. Chang [5] generalized some results of prime rings with derivations to the prime rings with semi-derivations.

In this paper, we prove that \(d(x) = p\delta(x - g(x))\) for all \(x \in M\), \(\delta \in \Gamma\), where \(p\) is the element of the extended centroid of a \(\Gamma\)-ring \(M\), \(d\) is a semiderivation on \(M\) associated with a function \(g\) on \(M\).

2. Semiderivations in Prime \(\Gamma\)-rings

An additive mapping \(D\) from \(M\) to \(M\) is called a derivation if \(D(x_\alpha y) = D(x)\alpha y + x\alpha D(y)\) holds for all \(x, y \in M\), \(\alpha \in \Gamma\).

Let \(M\) be a \(\Gamma\)-ring. An additive mapping \(d: M \rightarrow M\) is called a semiderivation associated with a function \(g: M \rightarrow M\) if, for all \(x, y \in M\), \(\alpha \in \Gamma\),

\[
d(x\alpha y) = d(x)\alpha g(y) + x\alpha d(y) = d(x)\alpha y + g(x)\alpha d(y) ;
\]

(ii) \(d(g(x)) = g(d(x))\).

If \(g = I\), i.e., an identity mapping of \(M\), then all semi-derivations associated with \(g\) are merely ordinary derivations. If \(g\) is any endomorphism of \(M\), then other examples of semi-derivations are of the form \(d(x) = x - g(x)\).

Example 2.1

Let \(M_1\) be a \(\Gamma_1\)-ring and \(M_2\) be a \(\Gamma_2\)-ring. Consider \(M = M_1 \times M_2\) and \(\Gamma = \Gamma_1 \times \Gamma_2\).

Define addition and multiplication on \(M\) and \(\Gamma\) by

\[
(m_1, m_2) + (m_1', m_2') = (m_1 + m_1', m_2 + m_2')
\]

\[
(\alpha_1, \alpha_2) + (\alpha_3, \alpha_4) = (\alpha_1 + \alpha_3, \alpha_2 + \alpha_4)
\]

\[
(m_1, m_2)(\alpha_1, \alpha_2)(m_3, m_4) = (m_1\alpha_1 m_3, m_2\alpha_2 m_4),
\]

for every \((m_1, m_2)\), \((m_3, m_4)\)\(\in\)\(M\) and \((\alpha_1, \alpha_2)\), \((\alpha_3, \alpha_4)\)\(\in\)\(\Gamma\).

Under these addition and multiplication \(M\) is a \(\Gamma\)-ring. Let \(\delta:\ M_1 \rightarrow M_1\) be an additive map and \(\tau:\ M_2 \rightarrow M_2\) be a left and right \(M_2\)-\(\Gamma\)-module which is not a derivation.

Define \(d:\ M \rightarrow M\) such that \(d((m_1, m_2)) = (0, \tau(m_2))\) and \(g:\ M \rightarrow M\) such that \(g((m_1, m_2)) = (\delta(m_1), 0)\), \(m_1\in M_1\), \(m_2\in M_2\). Then it is clear that \(d\) is a semi-derivation of \(M\) (with associated map \(g\)) which is not a derivation.

We refer to [8,9] for the definitions of centroid , extended centroid of \(\Gamma\)-rings.

Lemma 2.2 Let \(M\) be a prime \(\Gamma\)-ring and Let \(I \neq 0\) be an ideal of \(M\). If \(d \neq 0\) is a semiderivation on \(M\), then \(d \neq 0\) on \(I\).

Proof. Suppose \(d(I) = 0\). Then for \(r \in I\), \(x \in M\), we have

\[
0 = d(rx) = d(r)g(x) + x\alpha d(y) = rad(x).\]

Replace \(r\) by \(r\beta y\), we get \(r\beta yad(x) = 0\), for all \(r \in I\), \(x, y \in M\), \(\alpha, \beta \in \Gamma\). By the primeness of \(M\), \(r = 0\) or \(d(x) = 0\). But \(I \neq 0\), we get \(d(x) = 0\) for all \(x \in M\).
Lemma 2.3. Let M be a prime \( \Gamma \)-ring and Let \( I \neq 0 \) be an ideal of M. If \( d \neq 0 \) is a semiderivation on M and \( a \in M \) such that \( a\beta d(r) = 0 \), for all \( r \in I \) and \( \beta \in \Gamma \), then \( a = 0 \).

Proof. By Lemma 2.2 we may pick \( r \in I \) such that \( d(r) \neq 0 \). For \( s \in I \) we see that \( 0 = a\beta d(sar) = a\beta(d(s)ag(r) + sad(r)) = a\beta sad(r) \), for \( a, \beta \in \Gamma \). By the primeness of M, \( a = 0 \).

Lemma 2.4. Let M be a prime \( \Gamma \)-ring and Let \( I \neq 0 \) be an ideal of M. If \( d \neq 0 \) is a semiderivation on M, then \( d(d(I)) \neq 0 \).

Proof. Suppose \( d(d(I)) = 0 \). Then for \( r, s \in I \), we exploit the definition of \( d \) in different ways to obtain

\[
\begin{align*}
(1) \quad 0 &= d(d(ras)) = d(d(r)as + g(r)ad(s)) = d(d(r))as + g(d(r))ad(s) + d(g(r)ad(s)) \\
(2) \quad 0 &= d(d(ras)) = d(d(r)as + g(r)ad(s)) = d(d(r))ag(s) + d(r)ad(s) + d(g(r)ad(s))
\end{align*}
\]

Subtraction of (2) from (1) yields

\[
(3) \quad (g(d(r)) - d(r))ad(s) = 0, \quad r, s \in I, \quad a \in \Gamma.
\]

An application of Lemma 2.3 to (3) then says that \( d(d(r)) = d(r) \) for all \( r \in I \).

Again for \( r, s \in I, \quad a \in \Gamma \), we may also write

\[
0 = d(d(ras)) = d(d(r)as + g(r)ad(s)) = d(d(r))ag(s) + d(r)ad(s) + d(g(r)ad(s))
\]

whence we have

\[
(4) \quad d(r)ad(s) + d(g(r)ag(d(s)) = 0, \quad \text{for all} \quad r, s \in I, \quad a \in \Gamma.
\]

Since \( d(g(r)) = g(d(r)) = d(r) \) for all \( r \in I \) and characteristic \( M \neq 2 \), we conclude from (4) that \( d(r)ad(s) = 0 \) for all \( r, s \in I, \quad a \in \Gamma \). Another application of Lemma 2.3 asserts that \( d(r) = 0 \) for all \( r \in I \), which then contradicts Lemma 2.2.

Lemma 2.5 Let M be a prime \( \Gamma \)-ring and Let \( d : M \to M \) be a semiderivation with associated function \( g : M \to M \). If there exists a nonzero ideal I of M for which \( I \cap g(M) = 0 \), then there exists \( p \in C \) such that \( d(x) = p\delta(x - g(x)) \) for all \( x \in M, \delta \in \Gamma \), where \( C \) is the extended centroid of M.

Proof. Let \( W \) be the ideal generated by

\[
\sum r_i \alpha_i (x_i - g(x_i)) \beta_i s_i, \quad \text{for all} \quad r_i, s_i, x_i \in I, \alpha_i, \beta_i \in \Gamma \]

and note that (otherwise g would be the identity mapping in contradiction to \( I \cap g(M) = 0 \).

We define a mapping \( \phi : W \to M \) according to the rule:

\[
\sum r_i \alpha_i (x_i - g(x_i)) \beta_i s_i \to \sum r_i \alpha_i d(x_i) \beta_i s_i \quad \text{where} \quad r_i, s_i \in I, x_i \in M, \alpha_i, \beta_i \in \Gamma.
\]

Now we have to show that \( \phi \) is well defined. Suppose that
\[
\sum r_i \alpha_i (x_i - g(x_i)) \beta_i s_i = 0, \text{ for all } r_i, s_i, x_i \in I, \alpha_i, \beta_i \in \Gamma.
\]
We have to prove that
\[
\sum r_i \alpha_i d(x_i) \beta_i s_i = 0, \text{ where } r_i, s_i \in I, x_i \in M, \alpha_i, \beta_i \in \Gamma.
\]
Applying the semiderivation \(d\) to \(\sum r_i \alpha_i (x_i - g(x_i)) \beta_i s_i = 0\), we see that
\[
0 = d(\sum r_i \alpha_i x_i \beta_i s_i - r_i \alpha_i g(x_i) \beta_i s_i)
\]
\[
= \sum [r_i \alpha_i d(x_i, \beta_i s_i) + d(r_i) \alpha_i g(x_i, \beta_i s_i) - d((r_i) \alpha_i g(x_i) \beta_i s_i) - r_i \alpha_i g(x_i) \beta_i d(s_i)]
\]
\[
= \sum r_i \alpha_i d(x_i) \beta_i s_i + r_i \alpha_i g(x_i) \beta_i d(s_i) + d(r_i) \alpha_i g(x_i) \beta_i s_i - d(r_i) \alpha_i g(x_i) \beta_i g(s_i)
\]
\[
- g(r_i) \alpha_i g(x_i) \beta_i d(s_i) - r_i \alpha_i g(x_i) \beta_i d(s_i)]
\]
\[
= \sum r_i \alpha_i d(x_i) \beta_i s_i - \sum r_i \alpha_i g(d(x_i)) \beta_i g(s_i)
\]
Therefore \(\sum r_i \alpha_i d(x_i) \beta_i s_i = g(\sum r_i \alpha_i d(x_i) \beta_i s_i) \in I - g(M)\) whence
\[
\sum r_i \alpha_i d_1 (x_i) \beta_i s_i = 0\]
and \(\phi\) is well defined. By the nature of the extended centroid \(C\) it follows that there exists \(p \in C\) such that \(p \delta w = \phi (w)\) for all \(w \in W, \delta \in \Gamma\). Now, regarding \(M\) as a subring of the central closure \(C(M)\), we have for all \(r, s \in I, x \in M, \alpha, \beta, \delta \in \Gamma, r_0 p \delta (x - g(x)) \beta s = p \delta r_0 (x - g(x)) \beta s = \phi (r_0 (x - g(x)) \beta v) = r_0 a \delta (x - g(x)) \beta s = 0.\)
From the primeness of \(M\) we thus see that \(d(x) = p \delta (x - g(x))\) for all \(x \in M, \delta \in \Gamma\).

**Theorem 2.6** Let \(d\) be a semiderivation of a prime \(\Gamma\)-ring \(M\) associated with the (endomorphism) mapping \(g: M \to M\). Then either one of the following two cases holds:

1. There exists an element \(p\) in the extended centroid of \(M\) such that \(d(x) = p \delta (x - g(x))\) for all \(x \in M, \delta \in \Gamma\),
2. The endomorphism \(g\) is an identity mapping and \(d\) is an ordinary derivation.

**Proof.** Set \(d_1 (x) = x - g(x)\) for \(x \in M\). Then \(d_1\) is also a semiderivation of \(M\) associated with the ring endomorphism \(g\). Let
\[
U = \{ \sum r_i \alpha_i d_1 (x_i) \beta_i s_i : r_i, s_i, x_i \in M, \alpha_i, \beta_i \in \Gamma \text{ and } \sum r_i \alpha_i d_1 (x_i) \beta_i s_i = 0 \}.
\]
Then \(U\) is obviously a two-sided ideal of \(M\). Let \(r, s, x \in M, \alpha, \beta \in \Gamma\) be such that \(\sum r_i \alpha_i d_1 (x_i) \beta_i s_i = \sum r_i \alpha_i (x_i - g(x_i)) \beta_i s_i = 0\). Applying the semiderivation \(d\) to
\[
\sum r_i \alpha_i (x_i - g(x_i)) \beta_i s_i = 0.
\]
and using the defining identities (i), (ii) for the semiderivation \(d\) to expand the resulting expression, we compute, as in Lemma 2.5.

\[
0 = d(\sum r_i \alpha_i d_i(x_i) \beta_is_i) = d(\sum (r_i \alpha_i x_i \beta_is_i - r_i \alpha_i g(x_i) \beta_is_i))
\]

\[
= \sum [r_i \alpha_i d(x_i) \beta_is_i + d(r_i) \alpha_i g(x_i) \beta_is_i - d(r_i \alpha_i g(x_i)) \beta_is_i - r_i \alpha_i g(x_i) \beta_is_i]
\]

\[
= \sum [r_i \alpha_i d(x_i) \beta_is_i - \sum g(r_i) \alpha_i g(d(x_i)) \beta_is_i] = \sum r_i \alpha_i d(x_i) \beta_is_i - g(\sum r_i \alpha_i d(x_i) \beta_is_i).
\]

Therefore \(\sum r_i \alpha_i d(x_i) \beta_is_i = g(\sum r_i \alpha_i d(x_i) \beta_is_i)\) whenever \(\sum r_i \alpha_i d(x_i) \beta_is_i = 0\).

That is, \(d_1(u) = u - g(u) = 0\) for all \(u \in U\). If the two-sided ideal \(U\) is nonzero, then by Lemma 2.3, \(d_1 = 0\) on \(M\) and hence \(g(x) = x\) for all \(x \in M\). Thus \(g\) is the identity endomorphism of \(M\) and \(d\) is merely an ordinary derivation of \(M\), as desired. Now, assume that \(U = 0\).

That is, for any \(r_i, s_i, x_i \in M, \ \alpha_i, \beta_i \in \Gamma\), \(\sum r_i \alpha_i d_i(x_i) \beta_is_i = 0\) implies

\[
\sum r_i \alpha_i d_i(x_i) \beta_is_i = 0.
\]

Let \(W\) be the two-sided ideal \(\{ \sum r_i \alpha_i d_i(x_i) \beta_is_i : r_i, s_i, x_i \in M, \ \alpha_i, \beta_i \in \Gamma \}\).

Then the mapping \(\phi\) defined on \(W\) according to the rule

\[
\phi : \sum r_i \alpha_i d_i(x_i) \beta_is_i \rightarrow \sum r_i \alpha_i d_i(x_i) \beta_is_i,
\]

where \(r_i, s_i, x_i \in M, \ \alpha_i, \beta_i \in \Gamma\), is well defined.

But \(\phi\) is obviously an \(M_\Gamma\)-bimodule map of \(W\) into \(M\). By the definition of the extended centroid of \(M\), there exists an element \(p\) in the extended centroid of \(M\) such that \(\phi(w) = p\delta w\) for all \(w \in W, \ \delta \in \Gamma\). In particular, for all \(r, s, x \in M, \ \alpha, \beta \in \Gamma\), \(d(x) = \phi(p\delta d_i(x))\beta s = \phi(p\delta d_i(x))\beta s = p\delta(p\delta d_i(x))\beta s = p\delta(x - g(x))\) for all \(x \in M, \ \alpha, \beta \in \Gamma\), as required.

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