Multiobjective Nonlinear Programming Problems Involving Second Order (b,F)-Type I Functions

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ABSTRACT

A new class of functions namely, second order (b, F)-type I functions which is the generalization of type I, F-type I and b-type I functions, is introduced. Sufficient optimality conditions for proper efficiency and second order mixed type duality theorems for multiobjective nonlinear programming problems are established under the assumptions of second order (b, F)-type I functions.

1. Introduction

Convexity plays an important role in optimization theory. Therefore, the convexity assumption on the function can be weakened to certain kind of generalized convexity assumption without destroying the results valid for the convex case. Several researchers [2,3,8,10,11,13-15,20,22-27,30] have attempted to weaken the convexity assumption and introduced some kind of generalized convexity concept and also, established optimality conditions and duality theorems for nonlinear programming problems. In this context, it is relevant to refer to works of Hanson and Mond [10] and Rueda and Hanson [27]. These authors have introduced two new classes of functions called type I and type II as generalization of invex functions introduced by Hanson [8] and also, obtained optimality conditions and various duality theorems for nonlinear programming.

The field of multiobjective programming known as vector programming has grown remarkably in different directions in the setting of optimality conditions and duality theory since the 1980's under the assumptions of various generalized convex functions [2,4-6, 13,14,25,28,29]. Kaul et al.[15] obtained optimality conditions for proper efficiency and the various Wolfe type dual theorems and also, Mond-Weir dual type theorems [19] for multiobjective programming problems under the assumptions of type I functions and its generalizations. Recently, Hachimi and Aghezzaf [12] have introduced a new class of functions, namely (F, α, ρ, d)-type I functions which unifies several concepts of generalized type I functions. Further, they have obtained optimality conditions and duality for multiobjective programming problems

A second order dual for a nonlinear programming problem was introduced by Mangasarian [17] and established duality results for nonlinear programming

problems. Mond [18] introduced the concept of second order convex functions and proved second order duality under the assumptions of second order convexity on the functions involved. Hanson [9] introduced second order invexity and proved second order duality under the assumptions of second order invexity on the functions involved. Mond and Zhang [20] established various duality results for multiobjective programming problems involving second order V-invex functions. Zhang and Mond [30] introduced second order F-convex functions as a generalization of F-convex functions [10] , and obtained various second order duality results for multiobjective nonlinear programming problems under the assumption of second order F-convexity. Srivastava and Govil [28] defined second order (F,ρ,σ)-type I functions and their generalizations and then, obtained various Mond-Weir second order duality results for multiobjective programming problems under the assumptions of the second order (F,ρ,σ)-type I functions   and   their generalizations. Recently, Ahmed and Husain [3] obtained second order Mond-Weir type dual results for multiobjective programming problems under the assumption of second order (F,α,ρ,d)-convexity on the functions involved. The study of the second order duality is significant due to the computational advantage over first order duality as it provides tighter bounds for the value of the objective functions when approximations are used [1 ,3,9,17,18,20,21,30].

        Recently, Pandian and Bharathi [24] have introduced a new class of functions namely, second order (b,F)-convex functions which is an extension of (b,F)-convex functions [22] and then, sufficient optimality and various Mond-Weir type duality theorems are established under the assumptions of second order F-convex functions.

        In this paper, we introduce a new class of functions namely, second order (b,F)-type I functions which is a generalization of second order type I functions, second order F-type I functions and second order b-type I functions. Then, we derive sufficient optimality conditions for proper efficiency and obtain second order mixed type duality theorems for multiobjective nonlinear programming problems under the assumption of second order (b,F)-type I functions. The results obtained in this paper extend many works in the literature.

2. Preliminaries

        Throughout this paper, the following conventions for vectors in R^n will be followed. For any $x=(x_1,x_2,\cdots,x_n)^T$ and $y=(y_1,y_2,\cdots,y_n)^T$, we follow the notations of Mangasarian [16]

  $x < y$ if and only if $x_i < y_i$, i = 1,2,…,n ;
  $x \leqq y$ if and only if $x_i \leq y_i$, i =1,2,…,n ;
  $x \leq y$ if and only if $x_i \leq y_i$, i =1,2,…,n and $x_r < y_r$ for some r $\in \{1,2,…,n\}$;
  $x = y$ if and only if $x_i = y_i$, i = 1,2,…,n and $x \nleq y$ is the negation of $x \leq y$.

Let X be an open convex subset of R^n and R_+ denote the set of all positive real numbers and $e = (1,1,...,1) \in R^k$.

Let us assume that $g_\circ : X \to R$, $h : X \to R$ $f : X \to R^k$ and $g : R^n \to R^m$ where $f = (f_1,...,f_k)$ and $g = (g_1,...,g_m)$ are twice differentiable functions on X. The functions $f_i's$ and $g_j's$ are defined as follows,

$$f_i : X \to R, \ i = 1,2,...,k \ \text{ and } \ g_j : X \to R, j = 1,2,...,m.$$

Let the vectors $p = (p_1, p_2,..., p_n)^T \in R^n$, $\lambda = (\lambda_1, \lambda_2,..., \lambda_k) \in R^k$ and $y = (y_1, y_2,..., y_m) \in R^m$.

Let F be a function defined by $F : X \times X \times R^n \to R$ and the functions $c_\circ(x,u)$, $b_\circ(x,u)$, $c_j(x,u), j = 1,2,...,m$ and $b_i(x,u), i = 1,2,...,k$ be defined as follows,

$$b_\circ(x,u) : X \times X \to R_+, \quad c_\circ(x,u) : X \times X \to R_+,$$
$$b_i(x,u) : X \times X \to R_+ \ \text{ and } \ c_j(x,u) : X \times X \to R_+.$$

Consider the following multiobjective nonlinear programming problem (MOP) Minimize $f(x) = (f_1(x), f_2(x),..., f_k(x))$

subject to $g(x) \leq 0$, $x \in X$,

where $f_i : X \to R, i = 1,2,...,k$ and $g : X \to R^m$ where $g = (g_1,...,g_m)$ are twice differentiable functions on X.

We need the following definition which can be found in [10].

Definition 1: A   function $F : X \times X \times R^n \to R$ is said to be sublinear in its third argument if for each $x, u \in X$,

$$F(x,u;a+b) \leq F(x,u;a) + F(x,u;b), \text{ for all } a,b \in R^n \text{ and}$$
$$F(x,u;\alpha a) = \alpha F(x,u;a), \text{ for all } \alpha \geq 0 \text{ in } R \text{ and } a \in R^n.$$

Note: $F(x,u;0) = 0$, for all $x,u \in X$.

Let $P = \{x \in X : g_j(x) \leq 0, j = 1,2,...m\}$. That is, P is the set of all feasible solutions for the problem (MOP).

We need the following definitions which can be found in [6,7,23].

Definition 2: A feasible point x° is said to be efficient for (MOP) if there exists no other feasible point x in (MOP) such that

$$f_i(x) \le f_i(x^\circ), \ i = 1,2,...,k \ \text{ and}$$

$$f_r(x) < f_r(x^\circ), \ \text{for some } r \in \{1,2,...,k\}.$$

Definition 3: A feasible point x° is said to be a properly efficient solution of (MOP), if it is   efficient and if there exists a scalar $M > 0$ such that, for each $i \in \{1,2,...,k\}$ and for all feasible x of (MOP) satisfying $f_i(x) < f_i(x^\circ)$, we have

$$f_i(x^\circ) - f_i(x) \le M(f_r(x) - f_r(x^\circ))$$

for some r such that $f_r(x) > f_r(x^\circ)$.

     We need the following theorem for proving sufficient optimality conditions for proper efficiency and duality theorems which can be found in [6,14,23].

Theorem 1: Let $\lambda^\circ > 0$ in R^k be fixed with $\lambda^{\circ T} e = 1$. If x° is an optimal solution of the scalar programming problem (MOP_{λ°) where

$$(MOP_{\lambda^\circ}) \quad \text{Minimize } \lambda^{\circ T} f(x), \ \ x \in P,$$

then x° is a properly efficient solution for (MOP).

     We need the following necessary optimality conditions for proving strong duality theorem which can be found in Pandian [23].

Theorem 2: (Necessary Optimality Conditions): Assume that x° is an efficient solution for (MOP) at which a constraint qualification [16] is satisfied for each $(MOP_r(x^\circ))$, $r \in \{1,2,...,k\}$ where

$$(MOP_r(x^\circ)) \ \text{Minimize } f_r(x)$$

subject to

$$f_i(x) \le f_i(x^\circ), \ \text{for all } i \ne r,$$
$$x \in P.$$

Then, there exists $\lambda^\circ > 0$ in R^k with $\lambda^{\circ T} e = 1$ and $y^\circ \geqq 0$ in R^m such that $(x^\circ, \lambda^\circ, y^\circ)$ satisfies

$$\sum_{i=1}^{k} \lambda_i^\circ \nabla f_i(x^\circ) + \sum_{j=1}^{m} y_j^\circ \nabla g_j(x^\circ) = 0 \tag{1}$$

$$y_j^\circ g_j(x^\circ) = 0, \ j = 1,2,...,m. \tag{2}$$

3. Second order (b,F)-type I functions

We, now, define a new class of functions namely, second order (b,F)-type I functions which is a generalization of second order type I, second order F-type I and second order b-type I functions.

Definition 4: The function $(h, g_\circ)$ is said to be second order (b,F)-type I at $u \in X$ with respect to $b_\circ(x,u)$ and $c_\circ(x,u)$ if for all $x \in X$ and $p \in R^n$,

$$b_\circ(x,u)[h(x) - h(u) + \frac{1}{2} p^T \nabla^2 h(u) p] \geq F(x,u; \nabla h(u) + \nabla^2 h(u) p)$$

and

$$c_\circ(x,u)(-g_\circ(u) + \frac{1}{2} p^T \nabla^2 g_\circ(u) p) \geq F(x,u; \nabla g_\circ(u) + \nabla^2 g_\circ(u) p).$$

Remark 1: If $b_\circ(x,u) = 1, c_\circ(x,u) = 1$ and $F(x,u;z) = \eta^T(x,u)z$, then the definition 3. becomes the definition of second order type I functions.

Remark 2: If $b_\circ(x,u) = 1$ and $c_\circ(x,u) = 1$, then the definition 3. becomes the definition of second order F-type I functions.

Remark 3: If $F(x,u;z) = \eta^T(x,u)z$, then the definition 3. becomes the definition of second order b-type I functions.

4. Sufficient optimality conditions

We, now, prove the sufficient optimality conditions for a feasible point of the problem (MOP) to be a properly efficient solution under the assumption of second order (b,F)- type I functions.

Theorem 3: (Sufficient Optimality Conditions) Let x° be feasible for (MOP) and there exists $\lambda^\circ > 0$ in R^k with $\lambda^{\circ T} e = 1$, $y_j^\circ \geq 0$ in R, $j \in I(x^\circ)$ and $p^\circ \in R^n$ such that

$$\sum_{i=1}^{k} \lambda_i^\circ \nabla f_i(x^\circ) + \sum_{j \in I(x^\circ)} y_j^\circ \nabla g_j(x^\circ) +$$

$$[\sum_{i=1}^{k} (\lambda_i^\circ \nabla^2 f_i(x^\circ)) + \sum_{j \in I(x^\circ)} (y_j^\circ \nabla^2 g_j(x^\circ))] p^\circ = 0, \tag{3}$$

where $I(x^\circ) = \{j : g_j(x^\circ) = 0\}$.

If each $i = 1,2,....,k$ and $j \in I(x^\circ)$, (f_i, g_j) is second order (b,F)-type I at x° with respect to $b_i(x,x^\circ)$ and $c_j(x,x^\circ)$ with $b_i(x,x^0) > 0$, then x° is a properly efficient solution of the problem (MOP) with $p^{\circ T}(\nabla^2 f_i(x^\circ)) p^\circ \leq 0$, for all i and $p^{\circ T}(\nabla^2 g_j(x^\circ)) p^\circ \leq 0$, for all $j \in I(x^\circ)$.

Proof. Let z be a feasible solution of (MOP).

Since (f_i, g_j) is second order (b, F)-type I at x° with respect to $b_i(x, x^\circ)$ and

$c_j(x, x^\circ)$ with $b_i(x, x^0) > 0$ for all $i = 1, 2, \ldots, k$ and $j \in I(x^\circ)$, we have

$$b_i(z, x^\circ)[f_i(z) - f_i(x^\circ) + \frac{1}{2} p^{\circ T} \nabla^2 f_i(x^\circ) p^\circ]$$

$$\geq F(z, x^\circ; \nabla f_i(x^\circ) + \nabla^2 f_i(x^\circ) p^\circ), \text{ for all } i \quad (4)$$

and

$$c_j(z, x^\circ)(-g_j(x^\circ) + p^{\circ T} \nabla^2 g_j(x^\circ) p^\circ)$$

$$\geq F(z, x^\circ; \nabla g_j(x^\circ) + \nabla^2 g_j(x^\circ) p^\circ), \text{ for all } j \in I(x^\circ). \quad (5)$$

Now, since z is feasible and $p^{\circ T} \nabla^2 g_j(x^\circ) p^\circ \leq 0$, for all $j \in I(x^\circ)$, we have,

$$c_j(z, x^\circ)(-g_j(x^\circ) + \frac{1}{2} p^{\circ T} \nabla^2 g_j(x^\circ) p^\circ) \leq 0, \text{ for all } j \in I(x^\circ). \quad (6)$$

Suppose that x° is not an efficient solution for (MOP).
Then, there exists a feasible x for (MOP) such that

$$f(x) \leq f(x^\circ)$$

Since $b_i(x, x^\circ) > 0$ and $p^{\circ T} \nabla^2 f_i(x^\circ) p^\circ \leq 0$, for all i, it follows that

$$b_i(x, x^\circ)[f_i(x) - f_i(x^\circ) + \frac{1}{2} p^{\circ T} \nabla^2 f_i(x^\circ) p^\circ] \leq 0, \text{ for all } i \quad \text{and}$$

$$b_r(x, x^\circ)[f_r(x) - f_r(x^\circ) + \frac{1}{2} p^{\circ T} \nabla^2 f_r(x^\circ) p^\circ] < 0,$$

$$\text{for some } r \in \{1, 2, \ldots, k\}. \quad (7)$$

Now, from (4), (5), (6) and (7) and since $y_j^\circ \geq 0$, for all $j \in I(x^\circ)$, $\lambda_i > 0$, for all i
and F is sublinear, we have

$$F(x, x^\circ; \sum_{i=1}^{k} \lambda_i^\circ \nabla f_i(x^\circ) + \sum_{i=1}^{k} \lambda_i^\circ (\nabla^2 f_i(x^\circ)) p^\circ) < 0 \quad \text{and}$$

$$F(x, x^\circ; \nabla(\sum_{j \in I(x^\circ)} y_j^\circ g_j(x^\circ)) + \nabla^2(\sum_{j \in I(x^\circ)} y_j^\circ g_j(x^\circ)) p^\circ)) \leq 0.$$

Since F is sublinear, we can conclude that

$$F(x, x^\circ; \sum_{i=1}^{k} \lambda_i^\circ \nabla f_i(x^\circ) + \sum_{j\in I(x^\circ)} y_j^\circ \nabla g_j(x^\circ) +$$

$$[\sum_{i=1}^{k} (\lambda_i^\circ \nabla^2 f_i(x^\circ)) + \sum_{j\in I(x^\circ)} (y_j^\circ \nabla^2 g_j(x^\circ))]p^\circ) < 0,$$

which contradicts the fact that $F(x, x^\circ; 0) = 0$. Thus, x° is an efficient solution for (MOP).

Suppose that x° is not a properly efficient solution for (MOP). Then, for every M > 0, there exists a feasible solution x of (MOP) and an index i such that

$$f_i(x^\circ) - f_i(x) > M[f_r(x) - f_r(x^\circ)]$$

for all r satisfying $f_r(x) - f_r(x^\circ) > 0$ whenever $f_i(x^\circ) - f_i(x) > 0$.

This means that $f_i(x^\circ) - f_i(x)$ can be made arbitrarily large. Since $b_i(x, x^\circ) > 0$ and $p^{\circ T} \nabla^2 f_i(x^\circ) p^\circ \leq 0$, for all i and also, from (7) and (5), it follows that $-F(x, x^\circ; \nabla f_i(x^\circ) + (\nabla^2 f_i(x^\circ))p^\circ)$ can be made arbitrarily large and hence for $\lambda_i^\circ > 0$, for all i and since F is sublinear, we have

$$F(x, x^\circ; \sum_{i=1}^{k} \lambda_i^\circ \nabla f_i(x^\circ) + \sum_{i=1}^{k} \lambda_i^\circ (\nabla^2 f_i(x^\circ))p^\circ) < 0.$$

From (3) and since F is sublinear, it follows that

$$F(x, x^\circ; \nabla(\sum_{j\in I(x^\circ)} y_j^\circ g_j(x^\circ)) + \nabla^2(\sum_{j\in I(x^\circ)} y_j^\circ g_j(x^\circ))p^\circ)) > 0. \quad (8)$$

Now, since x is feasible and from (5) and (6), we have

$$F(x, x^\circ; \nabla g_j(x^\circ) + \nabla^2 g_j(x^\circ)p^\circ) \leq 0, \text{ for all } j\in I(x^\circ).$$

Since $y_j^\circ \geq 0$, for all $j\in I(x^\circ)$ and F is sublinear, it follows that

$$F(x, x^\circ; \nabla(\sum_{j\in I(x^\circ)} y_j^\circ g_j(x^\circ)) + \nabla^2(\sum_{j\in I(x^\circ)} y_j^\circ g_j(x^\circ))p^\circ)) \leq 0,$$

which contradicts (8). Thus, x° is a properly efficient solution for (MOP). Hence the theorem.

5. Duality theorems

Let J_1 be a subset of $M = \{1, 2, ..., m\}$ and $J_2 = M \setminus J_1$. We consider the following second order mixed type dual [1] for (MOP).

(XMOP) Maximize $f(u) + y_{J_1}g_{J_1}(u)e - \dfrac{1}{2}p^T\nabla^2[f(u) + y_{J_1}g_{J_1}(u)e]p$

$$= (f_1(u) + y_{J_1}g_{J_1}(u) - \frac{1}{2}p^T\nabla^2[f_1(u) + y_{J_1}g_{J_1}(u)]p ,...,$$

$$f_k(u) + y_{J_1}g_{J_1}(u) - \frac{1}{2}p^T\nabla^2[f_k(u) + y_{J_1}g_{J_1}(u)]p)$$

subject to

$$\nabla\lambda^T f(u) + (\nabla^2\lambda^T f(u))p + \nabla y^T g(u) + (\nabla^2 y^T g(u))p = 0$$

$$y_{J_2}g_{J_2}(u) - \frac{1}{2}p^T(\nabla^2 y_{J_2}g_{J_2}(u))p \geq 0$$

$$y \geqq 0, \ \lambda > 0, \lambda^T e = 1,$$

where $y_{J_1}g_{J_1} = \sum\limits_{j\in J_1} y_j g_j(u)$ and $y_{J_2}g_{J_2} = \sum\limits_{j\in J_2} y_j g_j(u)$.

We, now, prove the following weak duality theorems between the problems (MOP) and (XMOP) under the assumptions of second order (b,F)-type I functions.

Theorem 4:(Weak Duality Theorem) Let x be feasible for (MOP) and (u,λ,y,p) be feasible for (XMOP). If each $i=1,2,...,k$, $(f_i + y_{J_1}g_{J_1}, y_{J_2}g_{J_2})$ is second order (b,F)-type I at u with respect to $b_i(x,u)$ and $c_\circ(x,u)$ with $b_i(x,u) > 0$, then

$$f(x) \nleq f(u) + y_{J_1}g_{J_1}(u)e - \frac{1}{2}p^T(\nabla^2[f(u) + y_{J_1}g_{J_1}(u)e])p .$$

Proof. Now, since $(f_i + y_{J_1}g_{J_1}, y_{J_2}g_{J_2})$ is second order (b,F)-type I at u with respect to $b_i(x,u)$ and $c_\circ(x,u)$, we have for $i=1,2,...,k$,

$$b_i(x,u)[f_i(x) + y_{J_1}g_{J_1}(x) - f_i(u) - y_{J_1}g_{J_1}(u) + \frac{1}{2}p^T(\nabla^2(f_i(u) + y_{J_1}g_{J_1}(u)))p]$$

$$\geq F(x,u;\nabla(f_i(u) + y_{J_1}g_{J_1}(u)) + \nabla^2(f_i(u) + y_{J_1}g_{J_1}(u))p) \quad (9)$$

and

$$c_\circ(x,u)(-y_{J_2}g_{J_2}(u) + \frac{1}{2}\nabla^2(y_{J_2}g_{J_2}(u))p))$$

$$\geq F(x,u;\nabla(y_{J_2}g_{J_2}(u)) + \nabla^2(y_{J_2}g_{J_2}(u))p)). \quad (10)$$

Suppose that $f(x) \leq f(u) + y_{J_1} g_{J_1}(u)e - \frac{1}{2} p^T (\nabla^2 [f(u) + y_{J_1} g_{J_1}(u)e])p.$

Since $y_{J_1} g_{J_1}(x) \leq 0$, $b_i(x,u) > 0$, i = 1,2,...,k and $\lambda_i > 0$, i = 1,2,...,k and from (9), we have.

$F(x,u; \lambda_i \nabla(f_i(u) + y_{J_1} g_{J_1}(u)) + \lambda_i \nabla^2 (f_i(u) + y_{J_1} g_{J_1}(u))p) \leq 0,$ for all i

and

$F(x,u; \lambda_r \nabla(f_r(u) + y_{J_1} g_{J_1}(u)) + \lambda_r \nabla^2 (f_r(u) + y_{J_1} g_{J_1}(u))p) < 0,$

for some $r \in \{1,2,...,k\}$.

By the sublinearity of F, it follows that

$$F(x,u; \sum_{i=1}^{k} \lambda_i \nabla(f_i(u) + y_{J_1} g_{J_1}(u)) + \sum_{i=1}^{k} \lambda_i \nabla^2 (f_i(u) + y_{J_1} g_{J_1}(u))p) < 0. \qquad (11)$$

Now, since x is feasible for (MOP) and (u, λ, y, p) is feasible for (XMOP) and from (10), we have

$$F(x,u; \nabla(y_{J_2} g_{J_2}(u)) + \nabla^2 (y_{J_2} g_{J_2}(u))p)) \leq 0. \qquad (12)$$

Now, from (11) and (12) and since F is sublinear, we have

$$F(x,u; \nabla \lambda^T f(u) + (\nabla^2 \lambda^T f(u))p + \nabla y^T g(u) + (\nabla^2 y^T g(u))p) < 0. \qquad (13)$$

Now, by the sublinearity of F and the feasibility of (u, λ, y, p) for (XMOP),   we have

$$F(x,u; \nabla \lambda^T f(u) + (\nabla^2 \lambda^T f(u))p + \nabla y^T g(u) + (\nabla^2 y^T g(u))p) = 0,$$

which contradicts (13).

Thus, $f(x) \nleq f(u) + y_{J_1} g_{J_1}(u)e - \frac{1}{2} p^T (\nabla^2 [f(u) + y_{J_1} g_{J_1}(u)e])p.$

Hence the theorem.

We, now, prove the strong duality theorem between the problems (MOP) and (XMOP) under the assumption of second order (b,F)-type I functions.

Theorem 5:(Strong Duality Theorem) Assume that x° is an efficient solution for (MOP) at which a constraint qualification [16] is satisfied for each $(MOP_r(x^\circ))$, $r \in \{1,2,...,k\}$. Then,   there exists scalars $\lambda^\circ \in R^k$, $y^\circ \in R^m$ such that $(x^\circ, \lambda^\circ, y^\circ, p^\circ = 0)$ is a feasible solution for (XMOP) and the corresponding objective function values of (MOP) and (XMOP) are equal. If the conditions of weak duality (Theorem 4) holds, then $(x^\circ, \lambda^\circ, y^\circ, p^\circ = 0)$ is a properly efficient solution for (XMOP).

Proof. By the Theorem 2,   there exists $\lambda° > 0$ in R^k with $\lambda°^T e = 1$ and $y° \geqq 0$ in R^m such that $(x°, \lambda°, y°)$ satisfies (1) and (2). Therefore, $(x°, \lambda°, y°, p° = 0)$ is feasible for (XMOP) and the objective value of the problem (MOP) at $x°$ and the objective value of (XMOP) at $(x°, \lambda°, y°, p° = 0)$ are the same.

Suppose that $(x°, \lambda°, y°, p° = 0)$ is not efficient for (XMOP). Then, there exists a feasible (u, λ, y, p) for (XMOP) such that

$$f(x°) \leq f(u) + y_{J_1} g_{J_1}(u)e - \frac{1}{2} p^T \nabla^2 [f(u) + y_{J_1} g_{J_1}(u)e]p.$$

Since $\lambda > 0$ in R^k with $\lambda^T e = 1$, it follows that

$$\lambda^T f(x°) < \lambda^T f(u) + y_{J_1} g_{J_1}(u) - \frac{1}{2} p^T \nabla^2 [\lambda^T f(u) + y_{J_1} g_{J_1}(u)]p,$$

which contradicts weak duality (Theorem 4). Thus, $(x°, \lambda°, y°, p° = 0)$ is an efficient solution for (XMOP).

Now, since $(\lambda^T f + y_{J_1} g_{J_1}, y_{J_2} g_{J_2})$ is second order $(b°, F)$-type I at u with respect to $b°(x,u)$ and $c°(x,u)$, we have

$$b°(x,u)[\lambda^T f(x) + y_{J_1} g_{J_1}(x) - \lambda^T f(u) - y_{J_1} g_{J_1}(u)$$

$$+ \frac{1}{2} p^T \nabla^2 (\lambda^T f(u) + y_{J_1} g_{J_1}(u))p]$$

$$\geq F(x,u; \nabla(\lambda^T f(u) + y_{J_1} g_{J_1}(u)) + \nabla^2(\lambda^T f(u) + y_{J_1} g_{J_1}(u))p) \quad (14)$$

and

$$c°(x,u)(-y_{J_2} g_{J_2}(u) + \frac{1}{2} \nabla^2 (y_{J_2} g_{J_2}(u))p))$$

$$\geq F(x,u; \nabla(y_{J_2} g_{J_2}(u)) + \nabla^2(y_{J_2} g_{J_2}(u))p)), \quad (15)$$

for all feasible x.

Suppose that $(x°, \lambda°, y°, p° = 0)$ is not properly efficient for (XMOP). Then, for every $M > 0$, there exists a feasible solution (u, λ, y, p) of (XMOP) and an index i such that

$$f_i(u) + y_{J_1} g_{J_1}(u) - f_i(x°) > M[f_r(x°) - f_r(u) - y_{J_1} g_{J_1}(u)]$$

for   all   r   satisfying $f_r(x°) - f_r(u) - y_{J_1} g_{J_1}(u) > 0$ whenever

$$f_i(u) + y_{J_1} g_{J_1}(u) - f_i(x°) > 0.$$

This means that $f_i(u) + y_{J_1} g_{J_1}(u) - f_i(x^\circ)$ can be made arbitrarily large. Since

$\lambda > 0$ in R^k with $\lambda^T e = 1$ and from (14), we can conclude that

$$F(x^\circ, u; \nabla(\lambda^T f(u) + y_{J_1} g_{J_1}(u)) + \nabla^2(\lambda^T f(u) + y_{J_1} g_{J_1}(u))p) < 0.$$

Since (u, λ, y, p) is feasible for (XMOP) and F is sublinear, it follows that

$$F(x^\circ, u; \nabla(y_{J_2} g_{J_2}(u)) + \nabla^2(y_{J_2} g_{J_2}(u))p)) > 0. \tag{16}$$

Now, since x° is feasible for (MOP) and (u, λ, y, p) is feasible for (XMOP) and from (15), we have

$$F(x^\circ, u; \nabla(y_{J_2} g_{J_2}(u)) + \nabla^2(y_{J_2} g_{J_2}(u))p) \leq 0,$$

which contradicts (16). Thus, $(x^\circ, \lambda^\circ, y^\circ, p^\circ = 0)$ is a properly efficient solution for (XMOP).

Hence the theorem.

6. Generalization of Second order (b,F)-type I functions

The generalization of second order (b, F) - type I functions and semistrictly second order (b, F) -type I functions are given as follows.

Let ρ_1 and ρ_2 be real numbers and let $d(\cdot, \cdot)$ .be a pseudometric on R^n.

Definition 5: The function $(h, g_\circ)$ is said to be second order (b, F, ρ) -type I at $u \in X$ with respect to $b_\circ(x, u)$, $c_\circ(x, u)$ and $\rho = (\rho_1, \rho_2)$ if for all $x \in X$ and $p \in R^n$,

$$b_\circ(x, u)[h(x) - h(u) + \frac{1}{2} p^T \nabla^2 h(u)p]$$

$$\geq F(x, u; \nabla h(u) + \nabla^2 h(u)p) + \rho_1 d^2(x, u)$$

and

$$c_\circ(x, u)(-g_\circ(u) + \frac{1}{2} p^T \nabla^2 g_\circ(u)p)$$

$$\geq F(x, u; \nabla g_\circ(u) + \nabla^2 g_\circ(u)p) + \rho_2 d^2(x, u).$$

Remark 4: If $b_\circ(x, u) = 1, c_\circ(x, u) = 1$ and $F(x, u; z) = \eta^T(x, u)z$, then the definition 6. becomes the definition of second order $\rho -$ type I functions

Remark 5: If $b_\circ(x, u) = 1$ and $c_\circ(x, u) = 1$, then the definition 6. becomes the definition of second order $(F, \rho) -$ type I functions

Remark 6: If $F(x, u; z) = \eta^T(x, u)z$, then the definition 6. becomes the definition of second order $(b, \rho) -$ type I functions.

On the assumption of the second order (b, F, ρ) - type I functions, sufficient optimality conditions and various duality theorems can be proved on the same lines as in Section 4 and 5.

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