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The Shift Map and the Symbolic Dynamics and Application of Topological Conjugacy

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ABSTRACT

The aim of this paper is to prove some chaotic properties of the symbol space Σ_2 and the shift map σ and on the other hand apply the topological conjugacy property of the shift map on the logistic map. We have proved that the shift map is generically δ-chaotic on Σ_2. It is also proved that Σ_2 is a Cantor set and the shift map has sensitive dependence on initial conditions in an alternative way. In two other theorems we have directly proved that the dynamical system (Σ_2, σ) has modified weakly chaotic dependence on initial conditions as well as chaotic dependence on initial conditions. Hence by topological conjugacy the dynamical system (I, F_μ), for $\mu > 4$, has those properties.

Keywords: Shift Map, Symbolic Dynamics, Generically δ-chaotic, Cantor Set, Lyapunov ε-unstable, Topological Conjugacy, Li –Yorke Pair, Modified Weakly Chaotic Dependence on Initial Conditions, Chaotic Dependence on Initial Conditions.

AMS Subject Classifications: 37B10, 37D45.

1. Introduction

We all know that symbolic dynamical system is a very interesting example of topological dynamical system. A topological dynamical system is a pair (X, T), where X is a compact metric space and $T: X \to X$ is a continuous mapping. Dynamical systems given by the iteration of a continuous map on an interval are broadly studied because although they are simple, they nevertheless exhibit complex behaviors. Moreover they allow numerical simulations using a computer or a mere pocket calculator, which enable us to discover some chaotic phenomena. However continuous maps on intervals have many properties that are not found in other spaces. As a consequence, the study of one-dimensional dynamics is very rich but not necessarily representative of other systems. Almost in every introduction to chaos and dynamical systems, such as [3, 5, 8, 10 and 11], we get the examples of the logistic map $F_\mu(x) = \mu x(1-x)$. The logistic map F_μ has all of its interesting dynamics in the unit interval $I = [0,1]$. For low values of μ, the dynamics of F_μ is not too complicated, but as μ increases the dynamics of F_μ become more and more complicated. We now describe some aspects of the dynamical system (I, F_μ).

1. F_μ has an attracting fixed point at $(\mu-1)/\mu$ and a repelling fixed point at 0, if $1 < \mu < 3$.
2. Let, $A_n = \{x \in I; F_\mu^i(x) \in I$, for $i \le n$ but $F_\mu^{n+1}(x) \notin I\}$ and $\Lambda = I - (\bigcup_{n=0}^{\infty} A_n)$, then for $\mu > 4$ the set Λ is a Cantor set.
3. F_μ is chaotic on I for $\mu \ge 4$.
4. F_4 is topologically conjugate to the tent map $T_2(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \le x \le 1. \end{cases}$

By the symbolic dynamical system we mean here the sequence space $\Sigma_2 = \{\alpha : \alpha = (\alpha_0\alpha_1 \ldots\ldots\ldots); \alpha_i = 0 \text{ or } 1\}$ and the shift map $\sigma : \Sigma_2 \to \Sigma_2$. Also Σ_2 is a compact metric space by the metric $d(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^{i+1}}$, where $s = (s_0 s_1 \ldots\ldots\ldots)$ and $t = (t_0 t_1 \ldots\ldots\ldots)$ are any two points of Σ_2. The maximum distance between any two points of Σ_2 is 1 by our chosen metric. Also the shift map σ is a continuous map on Σ_2. There are many references of the shift map and the symbolic dynamical system in many papers and books such as [3, 4, 5, 6, 9, 11 and 14]. It is also well known that the dynamical system (Σ_2, σ) is an example of chaotic dynamical system. The shift map obeys all the conditions of Devaney's definition [3] of chaos such as sensitive dependence on initial conditions, topological transitivity and dense periodic points. Sensitive dependence on initial conditions is an important property for any chaotic map. There are some interesting research works on this particular property in [13, 18 and 19]. Recently Bau – Sen Du [6] gave a new strong definition of chaos by using shift map in the symbol space Σ_2 and by taking a dense uncountable invariant scrambled set in Σ_2. The term scrambled set was first introduced by Li and Yorke in their paper 'Period Three Implies Chaos' [15]. After that, this term became popular [1, 5, 6, 7, 11 and 12] day

by day. Another interesting definition of chaos is generic chaos. In 2000, Murinova [16] introduced generic chaos in metric spaces.
It is well known that mappings which are topologically conjugate are completely equivalent in terms of their dynamics. In particular, if $\mu > 4$ then F_μ is topologically conjugate to the shift map σ. Hence the shift map is an exact model for the quadratic map F_μ when $\mu > 4$. Also topological conjugacy [3, 5 and 8] is a very important property for any dynamical system.
In section three firstly we have proved that the shift map σ is generically δ-chaotic on Σ_2 with $\delta = 1$ by proving that σ is topologically mixing and hence it is weak mixing [2]. Also our proof of topologically mixing property is different from the other proofs [5]. Theorem –3.2 shows independently that Σ_2 is a Cantor set. In Theorem –3.3 we have proved alternatively that the shift map σ has sensitive dependence on initial conditions.
In section four we have proved directly that the dynamical system (Σ_2, σ) has modified weakly chaotic dependence on initial conditions (Theorem – 4.1) and also chaotic dependence on initial conditions (Theorem – 4.2). Since the shift map σ is topologically conjugate to the logistic map F_μ when $\mu > 4$, then the dynamical system (I, F_μ) also has modified weakly chaotic dependence on initial conditions as well as chaotic dependence on initial conditions for $\mu > 4$.
We also require some notations which we have used in this paper. If A is a set, we denote the diameter of A by $diam(A)$. For any point $x \in X$, $f^n(x) = f \circ f \circ ....\circ f(x)$ [f composed n- times] and similarly we can define $f^n(U)$ for any subset U of X. If β_i be any binary numeral, then we denote the complement of β_i by β_i', that is if $\beta_i = 0$ or 1, then $\beta_i' = 1$ or 0.

2. Preliminaries

In this section we are giving some definitions and lemmas needed for the main theorems. We start with some elementary definitions.

Definition 2.1 (Shift Map [3]): The shift map $\sigma : \Sigma_2 \to \Sigma_2$ is defined by $\sigma(\alpha_0\alpha_1..........)$
$= (\alpha_1\alpha_2.......)$, where $\alpha = (\alpha_0\alpha_1..............)$ is any point of Σ_2.

Definition 2.2 (Topologically Transitive [3]): A continuous map $f : S \to S$ is called topologically transitive if for any pair of non empty open sets $U, V \subset S$ there exists $k \geq 0$ such that $f^k(U) \cap V \neq \phi$, where S is a compact metric space.

Definition 2.3 (Topologically Mixing [5]): Let X be a compact metric space and $f : X \to X$ a continuous map. The map f is called topologically mixing if for any two non empty open sets $U, V \subset X$ there exists $m \geq 0$ such that for all $n \geq m$, $f^n(U) \cap V \neq \phi$.

Definition 2.4 (Weak Mixing [1]): Let X be a compact metric space and $f : X \to X$ a continuous map. The map f is called (topologically) weak mixing if $f \times f$ is transitive on $X \times X$.

Definition 2.5 (Li -Yorke Pair [7]): A pair $(x, y) \in X^2$ is called Li –Yorke (with modulus $\delta > 0$) if $\underset{n\to\infty}{LtSup}\, d(f^n(x), f^n(y)) \geq \delta$ and $\underset{n\to\infty}{LtInf}\, d(f^n(x), f^n(y)) = 0$, where X is a compact metric space with the metric d and f is a continuous mapping on X. The set of all Li -Yorke pairs of modulus δ is denoted by $LY(f, \delta)$.

Definition 2.6 (Generically $\delta -$chaotic [1]): Let $f : X \to X$ is a continuous map on a compact metric space X and $\delta > 0$. Then f is called generically $\delta -$chaotic if $LY(f, \delta)$ is residual in X^2.

Definition 2.7 (Cantor Set [3]): A set S is called a Cantor set provided it is (i) perfect, (ii) totally disconnected and (iii) compact. We recall that a set S is perfect provided it is closed and every point of S is a limit point of S and the set S is totally disconnected if connected components are single points.

Definition 2.8 (Sensitive Dependence on Initial Conditions [3]): A continuous map $f : X \to X$ has sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any $x \in X$ and any neighborhood $N(x)$ of x, there exist $y \in N(x)$ and $n \geq 0$ such that $d(f^n(x), f^n(y)) > \delta$, where (X, d) is a compact metric space.

Definition 2.9 (Lyapunov ε-unstable [7]): Let $f : X \to X$ is a continuous map on a compact metric space (X, d). Given $\varepsilon > 0$, the map f is called Lyapunov ε-unstable at a point $x \in X$ if for every neighborhood $N(x)$ of x, there is a $y \in N(x)$ and $n \geq 0$ with $d(f^n(x), f^n(y)) > \varepsilon$. If a dynamical system is point wise Lyapunov ε-unstable and also transitive then it has sensitive dependence on initial conditions.

Definition 2.10 (Weakly Chaotic Dependence on Initial Conditions [7]): A dynamical system (X, f) is called weakly chaotic dependence on initial conditions if for any $x \in X$ and every neighborhood $N(x)$ of x there are $y, z \in N(x)$, such that the pair $(y, z) \in X^2$ is Li – Yorke.

We now give the definition of modified weakly chaotic dependence on initial conditions.

Definition 2.11 (Modified Weakly Chaotic Dependence on Initial Conditions): A dynamical system (X, f) is called modified weakly chaotic dependence on initial conditions if for any $x \in X$ and every neighborhood $N(x)$ of x there are $y, z \in N(x)$, $y \neq x, z \neq x$, such that the pair $(y, z) \in X^2$ is Li – Yorke.

Definition 2.12 (Chaotic Dependence on Initial Conditions [7]): A dynamical system (X, f) is called chaotic dependence on initial conditions if for any $x \in X$ and every neighborhood $N(x)$ of x there is a $y \in N(x)$ such that the pair $(x, y) \in X^2$ is Li -Yorke.

Definition 2.13 (Topological Conjugacy [3]): Let, $f : A \to A$ and $g : B \to B$ be two continuous mappings. Then f and g are said to be topologically conjugate if there exists a homeomorphism $h : A \to B$ such that $h \circ f = g \circ h$. The homeomorphism h is called a topological conjugacy between f and g.

Definition 2.14 (Transitive Point [3]): Any point on a compact metric space (X, d) is called transitive point if it has dense orbit.

We also need the following lemmas.

Lemma-2.1 [3]: Let $s, t \in \Sigma_2$ and $s_i = t_i$, for $i = 0,1,.....,m$. Then, $d(s,t) < \frac{1}{2^m}$ and conversely if $d(s,t) < \frac{1}{2^m}$ then $s_i = t_i$, for $i = 0,1,.....,m.$

Lemma-2.2 [2]: Let X be a compact metric space and $T : X \to X$ is a continuous topologically mixing map then it is also (topologically) weak mixing map.

Lemma-2.3 [1]: Let $T : X \to X$ be a continuous map on a compact metric space X. If T is (topologically) weak mixing then it is generically δ-chaotic on X with $\delta = diam(X)$.

3. Some Properties of the Shift Map

Theorem 3.1: The shift map $\sigma : \Sigma_2 \to \Sigma_2$ is generically δ-chaotic on Σ_2 with $\delta = 1$.

Proof: First we prove that the shift map is topologically mixing. Let U and V be two arbitrary non empty open set of Σ_2. Let, $u = (u_0 u_1 ....) \in U$ be any point such that $\min\{d(u, \beta_1)\} = \varepsilon_1$, for any β_1 belongs to the boundary of the set U and $v = (v_0 v_1 ....) \in V$ be any point such that $\min\{d(v, \beta_2)\} = \varepsilon_2$, for any β_2 belongs to the boundary of the set V, where, $\varepsilon_1, \varepsilon_2 > 0$ be arbitrary. We now choose two positive integers k_1 and k_2 such that $\frac{1}{2^{k_1 - 1}} < \varepsilon_1$ and $\frac{1}{2^{k_2 - 1}} < \varepsilon_2$.

Next, we consider the sequence of points given by,

$\alpha_i = (u_0 u_1 .... u_{k_1 - 1} (1)^{i-1} v_0 v_1 .... v_{k_2 - 1} .....)$, for $i \geq 2$ and

$\alpha_1 = (u_0 u_1 .... u_{k_1 - 1} v_0 v_1 ...... v_{k_2 - 1} .....)$

We now prove the theorem with the help of Lemma-2.1.

Now, $d(u, \alpha_i) < \frac{1}{2^{k_1 - 1}} < \varepsilon_1$, for all $i \geq 1$, (by Lemma-2.1).

Hence, $\alpha_i \in U$, for all $i \geq 1$, that is $\sigma^k(\alpha_i) \in \sigma^k(U)$, for any $k \geq 0$. (3.1)

On the other hand, $\sigma^{k_1}(\alpha_1) = (v_0 v_1 .... v_{k_2 - 1} ....)$. Hence, $d(\sigma^{k_1}(\alpha_1), v) < \frac{1}{2^{k_2 - 1}} < \varepsilon_2$, by applying Lemma-2.1 again. This gives $\sigma^{k_1}(\alpha_1) \in V$ also. (3.2)

By virtue of (3.1) and (3.2) we can say that $\sigma^{k_1}(U) \cap V \neq \phi$.

Next consider the point α_2. Then $\sigma^{k_1 + 1}(\alpha_2) = (v_0 v_1 .... v_{k_2 - 1} ....)$, which again belongs to V. Hence, $\sigma^{k_1 + 1}(U) \cap V \neq \phi$. Continuing this process by taking all α_i's we can easily prove that $\sigma^k(U) \cap V \neq \phi$, for all $k \geq k_1$.

Hence σ is topologically mixing on Σ_2. Since Σ_2 is a compact metric space and the shift map $\sigma : \Sigma_2 \to \Sigma_2$ is a continuous map, by Lemma-2.2 it is also weak mixing. Again applying Lemma-2.3 we get our desired result that is, $\sigma : \Sigma_2 \to \Sigma_2$ is generically δ-chaotic on Σ_2 with $\delta = 1$.

Theorem 3.2: The set $\Sigma_2 = \{\alpha : \alpha = (\alpha_0 \alpha_1 ....), \alpha_i = 0 \text{ or } 1\}$ is a Cantor set.

Proof: We only have to prove that the set Σ_2 is perfect and totally disconnected. Let, $x = (x_0 x_1 ....)$ be any point of Σ_2 and $N(x)$ be any neighborhood of x. Then there exists an open ball $S_r(x)$, with centre at x and radius $r > 0$ such that, $S_r(x) \subset N(x)$. We choose n sufficiently large such that $\frac{1}{2^n} < r$. We now consider the sequence of points $y_1 = (x_0 x_1 .....x_n 100.....)$, $y_2 = (x_0 x_1 .....x_n 110.......)$, $y_3 = (x_0 x_1 ......x_n 1110.......)$ and so on.

In general $y_i = (x_0 x_1 ......x_n \overbrace{11..............1}^{i-times}000.........)$, for all $i \geq 1$.

Then, $d(x, y_i) = d((x_0 x_1 ....x_n ....), (x_0 x_1 ....x_n 11.......100......))$

$$< \frac{1}{2^n} < r \text{, by Lemma-2.1 and our construction above.}$$

Hence we can say that $y_i \in S_r(x)$, for all $i \geq 1$. So, $N(x)$ contains infinitely many points of Σ_2, that is x is a limit point of Σ_2. Since x is an arbitrary point, we can say that every point of Σ_2 is a limit point of Σ_2. Again by construction of Σ_2 it is easy to see that if α is a limit point of Σ_2 then α must be an infinite sequence of 0's and (or) 1's, that is, $\alpha = (\alpha_0 \alpha_1 ....), \alpha_i = 0 \text{ or } 1$. This gives $\alpha \in \Sigma_2$, proves that Σ_2 is a closed set. Which concludes that Σ_2 is a perfect set.

Lastly, Σ_2 is a set whose points are infinite sequence of 0's and 1's. So, we can say that connected components of Σ_2 are single points. Hence, Σ_2 is a totally disconnected set.

By the above two arguments we can say that Σ_2 is a perfect and totally disconnected set. Since Σ_2 is also a compact set so, we conclude that Σ_2 is a Cantor set.

Theorem 3.3: The shift map $\sigma : \Sigma_2 \to \Sigma_2$ has sensitive dependence on initial conditions.

Proof: We first prove that the shift map $\sigma : \Sigma_2 \to \Sigma_2$ is point wise Lyapunov ε-unstable on Σ_2. Let, $x = (x_0 x_1 ....)$ be any point of Σ_2 and U be any open neighborhood of x. Since U is open we can always get an $\varepsilon > 0$ such that $\min\{d(x, \alpha)\} = \varepsilon$, for all α belonging to the boundary of the set U. Now the maximum distance between any two points of Σ_2 is 1, by our chosen metric, so we can not take $\varepsilon > 1$. Hence, $\varepsilon \leq 1$ always. We take $n > 0$ such that $\frac{1}{2^n} < \varepsilon$. We now consider the point $y = (x_0 x_1 .....x_n x'_{n+1} x'_{n+2} x'_{n+3} ......)$ of Σ_2. Hence y is a point of Σ_2 which agrees with x up to x_n, but after the term x_n all the terms of y are complementary terms of that term of x.

Now, $d(x,y) < \frac{1}{2^n} < \varepsilon$, by Lemma-2.1 and our construction above. Then obviously $y \in U$. Also,

$$d(\sigma^{n+1}(x),\sigma^{n+1}(y)) = d((x_{n+1}x_{n+2}.............),(x'_{n+1}x'_{n+2}.............))$$
$$= \frac{1}{2}+\frac{1}{2^2}+..................$$
$$=1.$$

So we can say that $d(\sigma^{n+1}(x),\sigma^{n+1}(y)) \geq \varepsilon$. That is the shift map $\sigma:\Sigma_2 \to \Sigma_2$ is Lyapunov ε-unstable at x. Hence the shift map $\sigma:\Sigma_2 \to \Sigma_2$ is point wise Lyapunov ε-unstable on Σ_2. Again it is well known that the shift map is also topologically transitive. Since the shift map is both point wise Lyapunov ε-unstable and topologically transitive, it has sensitive dependence on initial conditions.

4. Applications of Topological Conjugacy

We now apply the topological conjugacy property between the shift map and logistic map to prove two properties of the logistic map.

Theorem 4.1: The dynamical system (Σ_2,σ) has modified weakly chaotic dependence on initial conditions.

Proof: Let, $x = (x_0x_1.............)$ be any point of Σ_2. Also assume that V be any open neighborhood of x. Since V is open, we can always choose an $\varepsilon_1 > 0$ such that, $\min\{d(x,\beta)\} = \varepsilon_1$, for any β belongs to the boundary of the set V. Again choose n sufficiently large such that $\frac{1}{2^n} < \varepsilon_1$. We now give some notations which help us to prove this theorem.

1. Let, $S = s_0s_1................s_i$ and $P = p_0p_1................p_m$ are two finite sequences of 0's and
1's, then $SP = s_0s_1..............s_ip_0p_1.............p_m$. Further, if we suppose that $T_1,T_2,.......\ ....,T_p$ are p finite sequences of 0's and 1's, then $T_1T_2.............T_p$ can be defined in a similar manner as above.

2.Let, $A(x,2n+2) = (x'_{3n+1}x'_{3n+2}....................x'_{4n+1}x_{4n+2}x_{4n+3}....................x_{5n+2})$,

$A(x,2n+4) = (x'_{5n+3}x'_{5n+4}....................x'_{6n+4}x_{6n+5}x_{6n+6}...................x_{7n+6})$,

$A(x, 2n+6) = (x'_{7n+7}x'_{7n+8}.....................x'_{8n+9}x_{8n+10}x_{8n+11}................x_{9n+12})$

and so on. That is for any even integer $k > 0$, $A(x,\ 2n+k)$ is a finite string of length $2n+k$.

3. Lastly we take $t_1, t_2 \in \Sigma_2$, such that,

$t_1 = (x_0x_1..................x_nx'_{n+1}x'_{n+2}....................x'_{3n}x_{3n+1}x_{3n+2}x_{3n+3}...................)$

and

$t_2 =$

$(x_0x_1................x_n(1)^n(0)^n A(x,2n+2)\,A(x,2n+4)\,A(x,2n+6).......................)$

We now prove Theorem – 4.1, with the help of those notations and Lemma –2.1 above. Since, both t_1 and t_2 agree with x up to the $(n+1)$-th term, then by Lemma –2.1 we get, $d(x,t_i) < \frac{1}{2^n} < \varepsilon_1, i = 1,2$. Hence, both t_1 and t_2 belong to V. Here also note that t_2 contains infinitely many finite sequences of the type $A(x,\ 2n+k)$, when $k > 0$ is an even integer.

Also, $\sigma^{5n+3}(t_1) = (x_{5n+3}x_{5n+4}..........x_{6n+4}........)$ and $\sigma^{5n+3}(t_2) = (x'_{5n+3}x'_{5n+4}..........x'_{6n+4}........)$.

Hence we get,

$$\underset{n\to\infty}{Lt\,Sup}\, d(\sigma^n(t_1),\sigma^n(t_2)) \geq \underset{n\to\infty}{Lt}\, d((x_{5n+3}x_{5n+4}.......x_{6n+4}.........),(x'_{5n+3}x'_{5n+4}.......x'_{6n+4}.........))$$

$$\geq \underset{n\to\infty}{Lt}\left(\frac{1}{2}+\frac{1}{2^2}+..........+\frac{1}{2^{n+2}}\right)$$

$$=1.$$

So, $$\underset{n\to\infty}{Lt\,Sup}\, d(\sigma^n(t_1),\sigma^n(t_2)) = 1. \qquad (4.1)$$

Similarly,

$\sigma^{6n+5}(t_1) = (x_{6n+5}x_{6n+6}............x_{7n+6}........)$ and

$\sigma^{6n+5}(t_2) = (x_{6n+5}x_{6n+6}............x_{7n+6}........)$

Hence,

$$\underset{n\to\infty}{Lt\,Inf}\, d(\sigma^n(t_1),\sigma^n(t_2)) \leq \underset{n\to\infty}{Lt}((x_{6n+5}x_{6n+6}...........x_{7n+6}.......),(x_{6n+5}x_{6n+6}...........x_{7n+6}.......))$$

$$\leq \underset{n\to\infty}{Lt}\left(\frac{0}{2}+\frac{0}{2^2}+..........+\frac{0}{2^{n+2}}\right)$$

$$= 0.$$

Hence $$\underset{n\to\infty}{Lt\,Inf}\, d(\sigma^n(t_1),\sigma^n(t_2)) = 0. \qquad (4.2)$$

The equations (4.1) and (4.2) prove that the pair (t_1, t_2) is Li -Yorke. Hence, the dynamical system (Σ_2, σ) has modified weakly chaotic dependence on initial conditions.

Theorem 4.2: The dynamical system (Σ_2, σ) has chaotic dependence on initial conditions.

Proof: Let, $s = (s_0 s_1 ....)$ be any point of Σ_2. Also assume that U be any open neighborhood of s. Since, U is open we similarly take an $\varepsilon_2 > 0$, such that $\min\{d(s,\alpha)\} = \varepsilon_2$, for any α belongs to the boundary of the set U. Choose n sufficintly large such that $\frac{1}{2^n} < \varepsilon_2$. We now require some similar notations given in the Theorem - 4.1 as above.

1. Let, $S = s_0 s_1 .... s_i$ and $P = p_0 p_1 .... p_m$ are two finite sequences of 0's and 1's, then $SP = s_0 s_1 .... s_i p_0 p_1 .... p_m$. Further, if we suppose that $T_1, T_2, ... \quad ...., T_p$ are p finite sequences of 0's and 1's, then $T_1 T_2 .... T_p$ can be defined in a similar manner as above.

2. $A(s,\ 2n+k)$ be the same as in the proof of Theorem – 4.1.

3. Lastly we take $t \in \Sigma_2$, such that,

$t = (s_0 s_1 .... s_n (0)^n (1)^n A(s, 2n+2)\, A(s, 2n+4)\, A(s, 2n+6) ....)$,

where $(\alpha)^n = \alpha\alpha .... \alpha \quad n-$ times.

With those three notations and the Lemma-2.1 above we now prove Theorem –4.2 .

By construction s and t agree up to s_n. Hence, $d(s,t) < \frac{1}{2^n} < \varepsilon_2$, by Lemma -2.1.

So, $t \in U$. Now, $\sigma^{3n+1}(s) = (s_{3n+1} s_{3n+2} .... s_{4n+1} ....)$ and $\sigma^{3n+1}(t) = (s'_{3n+1} s'_{3n+2} .... s'_{4n+1} .....)$. Note that t consists of infinitely many finite sequences of the type $A(s, 2n+k)$.

So we get,

$$\underset{n\to\infty}{Lt\,Sup}\, d(\sigma^n(s), \sigma^n(t)) \geq \underset{n\to\infty}{Lt}\, d((s_{3n+1} s_{3n+2} .... s_{4n+1} ....), (s'_{3n+1} s'_{3n+2} .... s'_{4n+1} ....))$$

$$\geq \underset{n\to\infty}{Lt} \left(\frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^{n+1}}\right)$$

$$= 1 .$$

Hence, $\underset{n\to\infty}{Lt\,Sup}\, d(\sigma^n(s), \sigma^n(t)) = 1.$ (4.3)

Similarly, $\sigma^{4n+2}(s) = (s_{4n+2}s_{4n+3}.........s_{5n+2}.......)$ and $\sigma^{4n+2}(t) = (s_{4n+2}s_{4n+3}.........s_{5n+2}....)$. So again we get,

$$\underset{n\to\infty}{Lt\,Inf}\, d(\sigma^n(s), \sigma^n(t)) \le \underset{n\to\infty}{Lt}((s_{4n+2}s_{4n+3}.........s_{5n+2}.......), (s_{4n+2}s_{4n+3}.........s_{5n+2}.......))$$

$$\le \underset{n\to\infty}{Lt}\left(\frac{0}{2} + \frac{0}{2^2} + ... + \frac{0}{2^{n+1}}\right)$$

$$= 0.$$

Hence $\quad , \underset{n\to\infty}{Lt\,Inf}\, d(\sigma^n(s), \sigma^n(t)) = 0. \qquad (4.4)$

From (4.3) and (4.4) it is proved that the pair (s,t) is Li -Yorke. Hence, the dynamical system (Σ_2, σ) has chaotic dependence on initial conditions.
Hence, from Theorem – 4.1 and Theorem – 4.2 and by topological conjugacy discussed above we get our desired results.

Theorem 4.3: The dynamical system (I, F_μ) has modified weakly chaotic dependence on initial conditions.

Theorem 4.4: The dynamical system (I, F_μ) has chaotic dependence on initial conditions.

Remarks: We have proved Theorem – 4.1 and Theorem – 4.2 by taking open neighborhood only. If we take any neighborhood, the proof is also similar. We discuss it below.
Let, $x \in \Sigma_2$ be any point and F be any neighborhood of x. Then there exists an open sphere U centered at x and contained in F. Then the Theorem – 4.1 and Theorem – 4.2 are also proved by taking U similarly. Since, $U \subset F$ the conditions of the Definition -2.11 and Definition -2.12 hold for F also.

5. Conclusions

In this paper we have proved some properties of shift map and symbol space as well as applied the topological conjugacy property.
It is well known that the shift map is chaotic in the sense of Devaney and Li -Yorke, but in Theorem –3.1 we have proved that it is also generically δ-chaotic on Σ_2 with $\delta = 1$. In Theorem –3.2 we have proved directly that Σ_2 is a Cantor set. In Theorem –3.3 we have proved that shift map is point wise Lyapunov ε-unstable on Σ_2 and then by transitivity of the shift map we can say that it also has sensitive dependence on initial conditions. So, this is an alternative proof of the sensitive dependence on initial conditions.
Also the two properties in Definition -2.11 and Definition -2.12 are very important for any dynamical system, because these two properties are mainly based on Li –

Yorke pair but have some common features with sensitive dependence on initial conditions. Hence, we can say that the logistic map has properties which are very similar to Li –Yorke pair but also have some common features with sensitivity. Also our proof's (Theorem – 4.1 and Theorem – 4.2) are independent and straightforward and we have not applied any result from any other paper.

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