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A Comparative Revisit to the Painleve' Tests for Integrability of Yang Equations, Charap Equations and Their Combinations and Some Unexpected Observations

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ABSTRACT

Painleve' test for integrability according to Weiss, Tabor and Carnevale [2] as applied by Chakraborty and Chanda [1,8,9] to the Yang-equations [3], Charap equations [4,5] and the combined Yang–Charap equations [6] have been revisited. Basically two new observations have been reported. (i) From the presentation of Chakraborty and Chanda [9] one could get the impression that for the leading order analysis of the combined equations the prominent role is played by the part that comes from the Yang equations. In the present paper it has been shown that both the Yang equations [3] and the Charap equations [4,5] contribute to the leading order analysis of combined Yang–Charap equations [6]. (ii) The Charap equations have two branches instead of one branch as was reported by Chakraborty and Chanda [8]. For the second branch of the Charap equations [4,5] other than that reported by Chakraborty and Chanda [8] the existence of requisite number of arbitrary functions in the Laurent-like expansion has been investigated. The expansion seems to allow arbitrary functions more than that is required for being a general solution. The similar situations occured to the combined Yang–Charap equations in the work of the Chakraborty and Chanda [9] which could not be completed for the involved nature of calculations.

Kew words: Painleve' analysis; integrability; *SU(2)* gauge field; Chiral invariant.

1. Introduction

Chakraborty and Chanda [1] applied the Painleve' test for integrability according to Weiss, Tabor and Carnevale [2] to Yang's self-dual equations for *SU(2)* gauge fields [3], Charap's equations for chiral invariant model of the pion dynamics and the equations [4,5] obtained by Chakraborty and Chanda [6] themselves as a combination of Yang's equations and Charap's equations. The results of the application of the test to the combined Yang–Charap equations revealed some unexpected results that are quite different from those for Yang's equations and Charap's equation. Basically two new observations have been reported here. First, the unexpected results are inherent partly in the Yang equations and fully in the Charap equations. Second, for the combined Yang–Charap equations the situation was quite complicated and the results were not conclusive whereas similar observations can be identified conclusively for Charap's equations.

2. The equations under study

(i) *The equations due to Yang* [3]: These were obtained by Yang [3] while discussing the condition of self-duality of *SU(2)* gauge fields on Euclidean four-dimensional space. The equations in terms of real variables are given by

$$\begin{aligned}\phi_{11}+\phi_{22}+\phi_{33}+\phi_{44} = &\Big[(1/\phi)\left(\phi_1^{\ 2}+\phi_2^{\ 2}+\phi_3^{\ 2}+\phi_4^{\ 2}\right)-(1/\phi)\left(\varphi_1^{\ 2}+\varphi_2^{\ 2}+\varphi_3^{\ 2}+\varphi_4^{\ 2}\right)\\ &-(1/\phi)\left(\chi_1^{\ 2}+\chi_2^{\ 2}+\chi_3^{\ 2}+\chi_4^{\ 2}\right)-(2/\phi)\left(\varphi_1\chi_2-\varphi_2\chi_1+\varphi_4\chi_3-\varphi_3\chi_4\right)\Big]\end{aligned} \tag{1}$$

$$\begin{aligned}\varphi_{11}+\varphi_{22}+\varphi_{33}+\varphi_{44} = &\left[(2/\phi)\left(\phi_1\varphi_1+\phi_2\varphi_2+\phi_3\varphi_3+\phi_4\varphi_4\right)\right.\\ &\left.+(2/\phi)\left(\phi_1\chi_2-\phi_2\chi_1+\phi_4\chi_3-\phi_3\chi_4\right)\right]\end{aligned} \tag{2}$$

$$\begin{aligned}\chi_{11}+\chi_{22}+\chi_{33}+\chi_{44} = &\left[(2/\phi)\left(\phi_1\chi_1+\phi_2\chi_2+\phi_3\chi_3+\phi_4\chi_4\right)\right.\\ &\left.+(2/\phi)\left(\phi_2\varphi_1-\phi_1\varphi_2+\phi_3\varphi_4-\phi_4\varphi_3\right)\right]\end{aligned} \tag{3}$$

where $\rho=\varphi+i\chi$, $\phi_1\equiv\partial\phi/\partial x^1$, $\phi_{11}\equiv\partial^2\phi/\partial\left(x^1\right)^2$ etc.

(ii) *The equations due to Charap* [4]: These we reproted by Charap [4] for the chiral invariant model of pion dynamics under tangential parametrization [5]. Explicitly the Charap equation can be written as:

$$\begin{aligned}\phi_{11}+\phi_{22}+\phi_{33}-\phi_{44} = &\,2\phi[\exp(-\beta)]\left(\phi_1^{\ 2}+\phi_2^{\ 2}+\phi_3^{\ 2}-\phi_4^{\ 2}\right)+2\varphi[\exp(-\beta)]\\ &\left(\phi_1\varphi_1+\phi_2\varphi_2+\phi_3\varphi_3-\phi_4\varphi_4\right)+2\chi[\exp(-\beta)]\left(\phi_1\chi_1+\phi_2\chi_2+\phi_3\chi_3-\phi_4\chi_4\right)\end{aligned} \tag{4}$$

$$\begin{aligned}\varphi_{11}+\varphi_{22}+\varphi_{33}-\varphi_{44} = &\,2\varphi[\exp(-\beta)]\left(\varphi_1^{\ 2}+\varphi_2^{\ 2}+\varphi_3^{\ 2}-\varphi_4^{\ 2}\right)+2\phi[\exp(-\beta)]\\ &\left(\phi_1\varphi_1+\phi_2\varphi_2+\phi_3\varphi_3-\phi_4\varphi_4\right)+2\chi[\exp(-\beta)]\left(\varphi_1\chi_1+\varphi_2\chi_2+\varphi_3\chi_3-\varphi_4\chi_4\right)\end{aligned} \tag{5}$$

$$\begin{aligned}\chi_{11}+\chi_{22}+\chi_{33}-\chi_{44} = &\,2\chi[\exp(-\beta)]\left(\chi_1^{\ 2}+\chi_2^{\ 2}+\chi_3^{\ 2}-\chi_4^{\ 2}\right)+2\phi[\exp(-\beta)]\\ &\left(\phi_1\chi_1+\phi_2\chi_2+\phi_3\chi_3-\phi_4\chi_4\right)+2\varphi[\exp(-\beta)]\left(\varphi_1\chi_1+\varphi_2\chi_2+\varphi_3\chi_3-\varphi_4\chi_4\right)\end{aligned} \tag{6}$$

where $\beta=\ln\left(f_\pi^{\ 2}+\phi^2+\varphi^2+\chi^2\right)$

(iii) *The combined Yang–Charap equations* [6]: Combining the equations (1–3) and (4–6) one can write the equations in which all the terms of (1–3) and (4–6) are present as :

$$\phi_{11}+\phi_{22}+\phi_{33}+\varepsilon\phi_{44} = k'\Big[(1/\phi)\left(\phi_1^{\ 2}+\phi_2^{\ 2}+\phi_3^{\ 2}+\varepsilon\phi_4^{\ 2}\right)-(1/\phi)\Big(\varphi_1^{\ 2}+\varphi_2^{\ 2}+\varphi_3^{\ 2}$$

$$+\varepsilon\varphi^2\big)-(1/\phi)\big(\chi_1{}^2+\chi_2{}^2+\chi_3{}^2+\varepsilon\chi_4{}^2\big)-(2/\phi)(\varphi_1\chi_2-\varphi_2\chi_1+\varphi_4\chi_3-\varphi_3\chi_4)\big]$$
$$+k''\{2\phi[\exp(-\beta)](\phi_1{}^2+\phi_2{}^2+\phi_3{}^2+\varepsilon\phi_4{}^2)+2\varphi[\exp(-\beta)](\phi_1\varphi_1+\phi_2\varphi_2$$
$$+\phi_3\varphi_3+\varepsilon\phi_4\varphi_4)+2\chi[\exp(-\beta)](\phi_1\chi_1+\phi_2\chi_2+\phi_3\chi_3+\varepsilon\phi_4\chi_4)\} \tag{7}$$

$$\varphi_{11}+\varphi_{22}+\varphi_{33}+\varepsilon\varphi_{44}=k'[(2/\phi)(\phi_1\varphi_1+\phi_2\varphi_2+\phi_3\varphi_3+\varepsilon\phi_4\varphi_4)+(2/\phi)(\phi_1\chi_2-$$
$$\phi_2\chi_1+\phi_4\chi_3-\phi_3\chi_4)]+k''\{2\varphi[\exp(-\beta)](\varphi_1{}^2+\varphi_2{}^2+\varphi_3{}^2+\varepsilon\varphi_4{}^2)+2\phi[\exp(-\beta)]$$
$$(\phi_1\varphi_1+\phi_2\varphi_2+\phi_3\varphi_3+\varepsilon\phi_4\varphi_4)+2\chi[\exp(-\beta)](\varphi_1\chi_1+\varphi_2\chi_2+\varphi_3\chi_3+\varepsilon\varphi_4\chi_4)\} \tag{8}$$

$$\chi_{11}+\chi_{22}+\chi_{33}+\varepsilon\chi_{44}=k'[(2/\phi)(\phi_1\chi_1+\phi_2\chi_2+\phi_3\chi_3+\varepsilon\phi_4\chi_4)(2/\phi)(\phi_2\varphi_1-$$
$$\phi_1\varphi_2+\phi_3\varphi_4-\phi_4\varphi_3)]+k''\{2\chi[\exp(-\beta)](\chi_1{}^2+\chi_2{}^2+\chi_3{}^2+\varepsilon\chi_4{}^2)+2\phi[\exp(-\beta)]$$
$$(\phi_1\chi_1+\phi_2\chi_2+\phi_3\chi_3+\varepsilon\phi_4\chi_4)+2\varphi[\exp(-\beta)](\varphi_1\chi_1+\varphi_2\chi_2+\varphi_3\chi_3+\varepsilon\varphi_4\chi_4)\} \tag{9}$$

where $\beta=\ln\left(f_\pi{}^2+\phi^2+\varphi^2+\chi^2\right)$ and k', k'' are arbitrary coupling constants.

The equations (7–9) reduce to (1–3) when $\varepsilon=1$, $k'=1$, $k''=0$ and to (4–6) when $\varepsilon=-1$, $k'=0$, $k''=1$.For convenience we call the equations (1–3) as the 'Yang equations', the equation (4–6) as the 'Charap equations' and the equations (7–9) as the 'combined Yang–Charap (Y–C) equations'. For $\varepsilon=1$, $k'\neq 0$, $k''\neq 0$ the equations (7–9) are 'extended Yang equations'. And for $\varepsilon=-1$, $k'\neq 0$, $k''\neq 0$ the equations (7–9) are 'extended Charap equations'.

3. Painleve' test for integrability according to Weiss *et. al.* [2]

If the singularity manifold is determined by

$$u\left(x^1,x^2,x^3,x^4\right)=0 \tag{10}$$

we require ϕ, φ, χ as

$$\phi=u^\alpha\sum_{j=0}^{\infty}\phi_j u^j\ ,\qquad \varphi=u^\alpha\sum_{j=0}^{\infty}\varphi_j u^j\ ,\qquad \chi=u^\alpha\sum_{j=0}^{\infty}\chi_j u^j \tag{11}$$

where ϕ, φ, χ are a set of solutions of (1–3) , (4–6) or (7–9). ϕ_j, φ_j, χ_j are all analytic functions of $\left(x^1,x^2,x^3,x^4\right)$ in the neighbourhood of the manifold (10); $\phi_0\neq 0$, $\varphi_0\neq 0$, $\chi_0\neq 0$.

The test may be divided into four main steps after the substitution of (11) in the differential equations concerned, i.e. (1–3) , (4–6) or (7–9).

(i) Make the leading order analysis (where one gets all possible α, β, γ, ϕ_0, φ_0 and χ_0 in (11)).

(ii) Define the recursion relations for ϕ_m, φ_m, χ_m for leading orders obtained in Step -I. If one directly substitutes (11) with the values of α, β, γ, ϕ_0, φ_0 and

χ_0 obtained from (i) above and then equates the coefficients of powers of u in the various terms and thereby observes the behaviour of the expansion coefficients, then the recursion relations for ϕ_m, φ_m, χ_m are

$$[T]=\begin{bmatrix}\phi_m\\ \varphi_m\\ \chi_m\end{bmatrix}=[\text{other terms with } \phi_j,\ \varphi_j,\ \chi_j \text{ and their derivatives where } j<m] \tag{12}$$

where $[T]$ is the system matrix and it is written as

$$[T]=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix} \tag{13}$$

(iii) Determine the resonance positions (those values of m for which the system matrix $[T]$ vanishes).

(iv) Check whether the expansions allow requisite number of arbitrary functions at the resonance positions. $m=-1$ is always a resonance point [7]. It indicates that the singularity manifold defined by (10) is arbitrary. [1,6,8,9].

4. First observation: *The unexpected results are inherent partly in the Yang equations and fully in the Charap equations.*

At first we report the previous results:
Chakraborty and Chanda assume that

$$\phi\sim\phi_0 u^{\alpha},\qquad \varphi\sim\varphi_0 u^{\alpha},\qquad \chi\sim\chi_0 u^{\alpha} \tag{14}$$

They substituted (14) in (1–3), (4–6) and (7–9) and equated the coefficients of the negative powers of u (considering that all of α, β and γ are negative) in each case. The results were as follows:

For the Yang equations [3,1]:
From (1–3) one gets

$$\alpha=\beta=\gamma \tag{15}$$

$$-\alpha\phi_0{}^2+\alpha^2\varphi_0{}^2+\alpha^2\chi_0{}^2=0 \tag{16}$$

$$\alpha(\alpha-1-2\alpha)\phi_0\varphi_0=0 \tag{17}$$

$$\alpha(\alpha-1-2\alpha)\phi_0\chi_0=0 \tag{18}$$

which lead to the results reported by Chakraborty and Chanda [1]:

$$\alpha=-1,\quad \beta=-1,\quad \gamma=-1 \tag{19}$$

and $$\phi_0^{\ 2} + \varphi_0^{\ 2} + \chi_0^{\ 2} = 0 \tag{20}$$

One may note that (20) is allowed, because all of ϕ_0, φ_0 and χ_0 are complex functions.

For the Charap equations [4,5]:
From (4–6) one gets

$$\alpha = \beta = \gamma \tag{21}$$
$$\alpha(\alpha - 1 - 2\alpha)\left(\phi_0^{\ 2} + \varphi_0^{\ 2} + \chi_0^{\ 2}\right)\phi_0 = 0 \tag{22}$$
$$\alpha(\alpha - 1 - 2\alpha)\left(\phi_0^{\ 2} + \varphi_0^{\ 2} + \chi_0^{\ 2}\right)\varphi_0 = 0 \tag{23}$$
$$\alpha(\alpha - 1 - 2\alpha)\left(\phi_0^{\ 2} + \varphi_0^{\ 2} + \chi_0^{\ 2}\right)\chi_0 = 0 \tag{24}$$

The choice of Chakraborty and Chanda [8] was $(\alpha - 1 - 2\alpha) = 0$ i.e. $\alpha = -1$. Or, in other words, the branch reported by them is given by

$$\alpha = -1, \quad \beta = -1, \quad \gamma = -1 \tag{25}$$

and all of ϕ_0, φ_0 and χ_0 are arbitrary . (26)

For the combined Yang–-Charap equations [6] (*with* $k' = 1$, $k'' = 1$):
From (7–9), with $k' = 1$ and $k'' = 1$, One gets

$$\alpha = \beta = \gamma \tag{27}$$
$$\left[-\alpha(2\alpha + 1)\phi_0^{\ 2} + \alpha^2\varphi_0^{\ 2} + \alpha^2\chi_0^{\ 2}\right]\left(\phi_0^{\ 2} + \varphi_0^{\ 2} + \chi_0^{\ 2}\right) = 0 \tag{28}$$
$$\left[\alpha(\alpha - 1) - 4\alpha^2\right]\left(\phi_0^{\ 2} + \varphi_0^{\ 2} + \chi_0^{\ 2}\right)\phi_0\varphi_0 = 0 \tag{29}$$
$$\left[\alpha(\alpha - 1) - 4\alpha^2\right]\left(\phi_0^{\ 2} + \varphi_0^{\ 2} + \chi_0^{\ 2}\right)\phi_0\chi_0 = 0 \tag{30}$$

The first possibility appears to be

$$\phi_0^{\ 2} + \varphi_0^{\ 2} + \chi_0^{\ 2} = 0 \tag{31}$$

with $\alpha = \beta = \gamma$ (with no other information about α , β , γ).

And the second possibility appears to be the combination of three equations given by

$$-\alpha(2\alpha + 1)\phi_0^{\ 2} + \alpha^2\varphi_0^{\ 2} + \alpha^2\chi_0^{\ 2} = 0 \tag{32}$$
$$\alpha(\alpha - 1) - 4\alpha^2 = 0 \tag{33}$$
$$\alpha(\alpha - 1) - 4\alpha^2 = 0 \tag{34}$$

with $\alpha = \beta = \gamma$.

From (33) & (34) one arrives at $\alpha = -\frac{1}{3}$. It is interesting to note that if one puts $\alpha = -\frac{1}{3}$ in (32) one again arrives at (31), i.e. ${\phi_0}^2 + {\varphi_0}^2 + {\chi_0}^2 = 0$. That is, for this possibility the branch is given by

$$\alpha = -\tfrac{1}{3}, \quad \beta = -\tfrac{1}{3}, \quad \gamma = -\tfrac{1}{3} \tag{35}$$

$${\phi_0}^2 + {\varphi_0}^2 + {\chi_0}^2 = 0 \tag{36}$$

Now we report our new observations:

Our observations (new) start with (21–24). Chakraborty and Chanda [8] ignored the option ${\phi_0}^2 + {\varphi_0}^2 + {\chi_0}^2 = 0$ in (21–24). Once we accept that option we get another branch given by

$$\alpha = \beta = \gamma \text{ (with no other information about } \alpha,\ \beta,\ \gamma\text{)} \tag{37}$$

$${\phi_0}^2 + {\varphi_0}^2 + {\chi_0}^2 = 0 \tag{38}$$

It is interesting to note that only when we accept (37) & (38) one can visualize the effect of superimposition of the Yang equation [3] and the Charap equation [4,5] in the combined Yang–Charap equation [6] (with $k' = 1$, $k'' = 1$) in the stage of leading order analysis itself. This is as stated below.

For the combined equations, along with $\alpha = \beta = \gamma$ we get ${\phi_0}^2 + {\varphi_0}^2 + {\chi_0}^2 = 0$ which appears via two different pathways:

First — directly, from (27–30), just as in (21–24) and (37–38) for the Charap equations (4–6).

Second — via $\alpha = -\frac{1}{3}$ in (32–34), just as (via $\alpha = -1$) in (15–18) for the Yang equations (1–3).

The branch given by '$\alpha = -1, \beta = -1, \gamma = -1$ with all of ϕ_0, φ_0 and χ_0 are arbitrary' that appears in (21–24) from the Charap equations (4–6) does not occur to the Yang equation and the same is not observable in the combined Yang–Charap equation (7–9). The branch given by (35–36), $\alpha = -\frac{1}{3}$ for the combined equations is definitely an interesting study. However, the task is quite involved. The work is in progress. The results, when available, will be reported separately. Here we have tried to complete the Painleve' analysis for the Charap equations. One can summarize the leading order branches for the Yang equations (1–3), Charap equations (4–6) and the Combined Yang–Charap equations (7–9) follows.

One can note that the *branch(i)* given by (31) of the leading order analysis of the combined Yang–Charap equations (7–9) is the same as the *branch(ii)* given by (37–38) of the leading order analysis of the Charap equations (4–6). On the other hand, the *branch(ii)* given by (35–36) of the leading order analysis of the combined Yang–Charap equations (7–9) is almost same (α being changed from -1 to $-\frac{1}{3}$) as the single branch given by (19–20) of the Yang equations (1–3). From the presentation of Chakraborty and Chanda [8,9] one could get the impression that the Yang equations [3] had the prominent role in the leading order analysis of the combined equations (7–9). Thus, here we see that both the Yang equations and Charap equations contribute to the leading order analysis of the combined Yang–Charap equations (7–9).

5. Second observation: *For the combined Yang–Charap equations the situation was quite complicated and the results in relation to the existence of requisite number of arbitrary functions were not conclusive whereas similar observations can be identified conclusively for Charap equations.*

System matrix and determination of resonance points:

With (37) & (38) the system matrix $[T]$ will be of the form as stated below

$$[T]=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

where

$a_{11} = 2\alpha\{(\alpha-1)-(m+2\alpha)\}\phi_0^{\,2}$, $a_{12} = 2\alpha\{(\alpha-1)-(m+2\alpha)\}\phi_0\varphi_0$,

$a_{13} = 2\alpha\{(\alpha-1)-(m+2\alpha)\}\phi_0\chi_0$, $a_{21} = 2\alpha\{(\alpha-1)-(m+2\alpha)\}\phi_0\varphi_0$,

$a_{22} = 2\alpha\{(\alpha-1)-(m+2\alpha)\}\varphi_0^{\,2}$, $a_{23} = 2\alpha\{(\alpha-1)-(m+2\alpha)\}\varphi_0\chi_0$

$a_{31} = 2\alpha\{(\alpha-1)-(m+2\alpha)\}\phi_0\chi_0$, $a_{32} = 2\alpha\{(\alpha-1)-(m+2\alpha)\}\varphi_0\chi_0$,

$a_{33} = 2\alpha\{(\alpha-1)-(m+2\alpha)\}\chi_0^{\,2}$

and $|T|=0$ leads to

$$[2\alpha\{(\alpha-1)-(m+2\alpha)\}]^3\phi_0\varphi_0\chi_0\begin{vmatrix} \phi_0 & \varphi_0 & \chi_0 \\ \phi_0 & \varphi_0 & \chi_0 \\ \phi_0 & \varphi_0 & \chi_0 \end{vmatrix}=0$$

so that we get no further information about α and the resonance points. This is just the same situation, which occurred to the combined Yang-Charap equations with $k'=1$, $k''=1$ [9].

In order to get specific results we take $\alpha=-1$ in an ad-hoc fashion.

For m = 1 (in(12):

We get three equations with ϕ_1, φ_1, χ_1.

With the help of (38) and its derivatives all the three equations reduce to a single equation given by

$$\phi_0\phi_1+\varphi_0\varphi_1+\chi_0\chi_1=0 \tag{39}$$

(since $\phi_0 \neq 0$, $\varphi_0 \neq 0$, $\chi_0 \neq 0$)

Thus, from (39) we can say that there is a possibility of getting resonance point at $m =$ (1 , 1). Definitely, at this stage we can say that the resonance point $m =$ (0 , 0) is confirmed.

For m = 2 (in(12)):

We get three equations with ϕ_2, φ_2, χ_2.

Here again we take the help of (38) , (39) and its derivatives. We arrive at

$$\phi_0\phi_2 + \varphi_0\varphi_2 + \chi_0\chi_2 = -\tfrac{1}{2}\left(\phi_1^{\,2} + \varphi_1^{\,2} + \chi_1^{\,2} + f_\pi^{\,2}\right) \tag{40}$$

Here, we have the possibility of getting the resonance point at $m =$(2 , 2). However, the existence of resonance point at $m =$(1 , 1) gets confirmed.

For m = 3 (in(12)):

We get three equations with ϕ_3, φ_3, χ_3.

With the help of (38), (39) & (40) and its derivatives the three equations reduce to a single equation given by

$$\phi_0\phi_3 + \varphi_0\varphi_3 + \chi_0\chi_3 = -\left(\phi_1\phi_2 + \varphi_1\varphi_2 + \chi_1\chi_2\right) \tag{41}$$

which introduces the possibility of getting the resonance point at $m =$(3, 3) and the existence of resonance point at $m =$ (2 , 2) gets confirmed.

Now we can identify the unexpected nature of observations mentioned in the introduction. The arbitrary function whose existence in the Laurent-like expansion (11) have been confirmed are:

u (singularity manifold defined in (10)), any two of ϕ_0, φ_0, χ_0, any two of ϕ_1, φ_1, χ_1 and any two of ϕ_2, φ_2, χ_2 i.e. seven arbitrary functions where as for being (11) a set of general solution one is required to have just six arbitrary function.

There is every chance that one can confirm the existence of more arbitrary functions in (11). However, we have not proceeded further. Because, those cases indicate the same observation — existence of arbitrary functions which is more in number than that is required for being (11) a set of general solutions.

Next we check the case given by $\alpha = -2$ (*again chosen in ad-hoc fashion*):

Here, m = 1 (in(12)):

We get three equations in ϕ_1, φ_1, χ_1.

With the help of (38) and its derivatives all the three equations are identically satisfied.

Then it is confirmed that any two of ϕ_0, φ_0 and χ_0 can be kept arbitrary. Thus, two more resonance, are there at $m =$(0,0). Also, there is a possibility that all of the three functions ϕ_1, φ_1 and χ_1 are arbitrary.

It may be checked that the equations arising for $m =$ 2, 3 etc. are quite involved. In order to make life simpler we made a choice (based on our experience from investigation with $\alpha = -1$). The choice is given by

$$\phi_0\phi_1 + \varphi_0\varphi_1 + \chi_0\chi_1 = 0 \tag{42}$$

However, with the (42) one gets that any two out of ϕ_1, φ_1, χ_1 may be arbitrary.

For m = 2 (in(12)):

With the help of (38), (42) and its derivatives we get

$$\phi_0\phi_2 + \varphi_0\varphi_2 + \chi_0\chi_2 = -\tfrac{1}{2}\left(\phi_1^{\,2} + \varphi_1^{\,2} + \chi_1^{\,2}\right) \tag{43}$$

Thus, from (43) one can confirm that two out of the three functions ϕ_1, φ_1 and χ_1 are arbitrary. At the same time, one can say that there is a possibility that two out of the functions ϕ_2, φ_2 and χ_2 arbitrary.

For m = 3 (in(12)):

Similarly with the help of (38), (42) & (43) and its derivatives the three equations reduce to a single equation given by

$$\phi_0\phi_3 + \varphi_0\varphi_3 + \chi_0\chi_3 = -\left(\phi_1\phi_2 + \varphi_1\varphi_2 + \chi_1\chi_2\right) \tag{44}$$

Thus, from (44) one can confirm that two out of the three functions ϕ_2, φ_2 and χ_2 are arbitrary. At the same time, one can say that there is a possibility that two out of the functions ϕ_3, φ_3 and χ_3 are arbitrary.

Here also, there is every chance that one can confirm the existence of more arbitrary function in (11). As before, we have not proceeded further. Because those cases indicate the same observation — existence of arbitrary functions which is more in number than that is required for being (11) a set of general solutions.

6. Summary

6.1. The present work is in relation to the Painleve' test for integrability according to Weiss, Tabor and Carnevale [2] of Yang's self-dual equations (1–3) for *SU(2)* gauge fields [3], Charap's equations (4–6) for chiral invariant [4,5] model of the pion dynamics and the equations (7–9) obtained by Chakraborty and Chanda [6] as a combination of Yang's equations and Charap's equations These tests were performed Chakraborty and Chanda [1,8,9]. In this paper we have extended their work.

6.2. Yang's equations (1–3) allow only one branch. Whereas Charap's equations (4–6) allow only two branches. From the presentation of Chakraborty and Chanda [8,9] one could get the impression that for the combined equations (7–9) in the leading order analysis Yang equations had the prominent role. However, here we have shown that both the Yang equations (1–3) and the Charap equations (4–6) contribute to the leading order analysis of the combined Yang–Charap equations (7–9).

6.3. The existence of requisite number of arbitrary function (six) in the Laurent like expansion (11) for the first branch (given by (21–24) and (25–26) of the Charap equations (4–6)) was confirmed by Chakraborty and Chanda. The present authors have investigated the existence of the requisite number of arbitrary functions for the second branch (given by (37–38) of the Charap equations (4–6)). It has been observed that the expansion (11) allow arbitrary functions which are more, in number, than that is required for being (11) a general solution. That is here the number allowed of arbitrary function is more than six. Such an observation is not known in the literature.

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