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Some Fixed Point Results in Non-empty Fuzzy Spaces

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ABSTRACT

We obtain a fixed point theorem involving n different fuzzy spaces and n mappings. Next we get a fixed point result in a non-empty fuzzy space with n mappings which satisfy some conditons. Lastly we obtain another fixed point result in a non-empty fuzzy space with n mappings which are pair-wise commuting. These results can generalize some existing known fixed point results in metric speces.

Keywords : *Fixed point, Commutative, Supremum, Non-negative functions, Contractive conditions.*

1. Introduction

Nung [4], Jain et al. [1] and Rao et al. [5] proved some fixed point results satifying some contractive conditions. Nung [4] and jain et al. [1] proved their results in three complete metric spaces and involving three mappings. Rao et al. [5] partially extended the results of [1] and [4]. We prove the results in more general way and prove in fuzzy spaces. We also generalize the results from three dimension to n-dimension.

2. Main Result

Theorem 3.1. Let X_i be n non-empty spaces and M_i be n non-negative functions from $X_i \times X_i \times (0, \alpha) \to [0,1]$ such that $M_i(x_i', x_i'', t) = 1$ if and only if $x_i' = x_i''$, and M_i are symmetric with first two component for $1 \le i \le n$. Further we assume that $T_i : X_i \to X_{i+1}$ $(1 \le i \le n)$ and $T_n : X_n \to X_1$ are mapping satisfying the following conditions :

$$M_1 (T_nT_{n-1}...T_3T_2x_2, T_nT_{n-1,}...T_2T_1x_{1,}t)$$
$$> \min \{M_1(x_1,T_nT_{n-1} ... T_3T_2x_2,t), M_1(x_1T_nT_{n-1} ...T_2T_1x_1,t), M_2(x_{2,} T_1x_{1,} t)\} \quad \textbf{(3.1)}$$

$M_i\,(T_{i-1}T_{i-2}\cdots T_2T_1T_nT_{n-1}\cdots T_{i+1}x_{i+1},\, T_{i-1}T_{i-2}\cdots T_2T_1T_nT_{n-1}\cdots T_{i+1}T_ix_i,t)$
$> \min\{M_i(x_{i-2}\cdots T_2T_1T_nT_{n-1}\cdots T_{i+2}T_{i+1}x_{i+1},t),$
$M_i(x_i, T_{i-1}T_{i-2}\cdots T_2T_1T_nT_{n-1}\cdots T_{i+1}T_ix_i,t), M_{i+1}(x_{i+1}, T_ix_i,t)\}(1<i<n)$ **(3.2)**

$M_n(T_{n-1}T_{n-2}\cdots T_2T_1x_1,\; T_{n-1}T_{n-2}\cdots T_2T_1T_nx_n,t)$
$> \min\{\, M_n(x_n,\, T_{n-1}T_{n-2}\ldots T_2T_1x_1,t), M_n(x_n,\, T_{n-1}T_{n-2}\ldots T_2T_1T_nx_n,t),$
$M_1(x_1, T_nx_n,t)\}$ **(3.3)**

where $x_i \in X_i$ $(1\le i \le n)$ with $x_{i+1} \ne T_ix_i$ $(1\le i < n)$ and $T_nx_n \ne,\ x_1$. Now if $x_1 \to M_1(x_1,T_nT_{n-1}\ldots T_2T_1x_1,t)$, $x_i \to M_i(x_i, T_{i-1}T_{i-2}\ldots T_2T_1T_nT_{n-1}\ldots T_{i+1}T_ix_i,t)$ $(1\le i \le n)$ attains its supremum in the respective spaces then $T_nT_{n-1}\ldots T_2T_1$, $T_{i-1}T_{i-2}\ldots T_2T_1T_nT_{n-1}\ldots T_{i+1}T_i$ $(1 \le i < n)$ has a unique fixed point in Xi for $1\le i \le n$.

Proof : Suppose the function $x_1 \to M_1(x_1,T_nT_{n-1}\ldots T_2T_1x_1,t)$ attains its supremum on X_1 , i.e. there exists a $u \in X_1$ such that $M_1(u,T_nT_{n-1}\ldots T_2T_1u,t) = \sup\{\, M_1(x,T_nT_{n-1}\ldots T_2T_1x,t) : x \in X_1\} = f(u)$ (say).

We now suppose that $T_nT_{n-1}\ldots T_2T_1$ has no fixed point. If we take $P = T_nT_{n-1}\ldots T_2T_1$ then from (3.1) we have

$f(P^{n-1}u)$
$= M_1(P^{n-1}u\,,\, P^nu,\, t)$
$= M_1(T_nT_{n-1}\ldots T_2T_1P^{n-2}u\,,\, T_nT_{n-1}\ldots T_2T_1P^{n-1}u,\, t)$
$> \min\{M_1(P^{n-1}u\,,T_nT_{n-1}\ldots T_2T_1P^{n-2}u,\, t), M_1(P^{n-1}u,\, T_nT_{n-1}\ldots T_2T_1P^{n-1}u,\, t),$
$M_2(T_1P^{n-2}u,\;\; T_1P^{n-1}u,\, t)\}$
$= \min\{M_1(P^{n-1}u,\, P^{n-1}u,\, t), M_1(P^{n-1}u,\, P^nu,\, t), M_2(T_1P^{n-2}u,\, T_1P^{n-1}u,\, t)\}$
$= M_2(T_1P^{n-2}u,\, T_1P^{n-1}u,\, t)$
$= M_2(T_1T_nT_{n-1}\ldots T_2T_1P^{n-3}u,\; T_1T_nT_{n-1}\ldots T_2T_1P_{n-2}u,\; t)$
$> \min\{M_2(T_1P^{n-2}u,T_1T_nT_{n-1}\ldots T_2T_1P^{n-3}u,\, t),\; M_2(T_1P^{n-2}u,T_1T_nT_{n-1}\ldots T_2T_1P^{n-2}u,\, t),$
$M_3(T_2T_1P^{n-3}u,\; T_2T_1P^{n-2}u,\; t)\}$
$= M_3(T_2T_1P^{n-3}u,\; T_2T_1P^{n-2}u,\; t)$
$> \ldots\;\ldots\;\ldots$
$= M_n(T_{n-1}T_{n-2}\ldots T_2T_1u,\; T_{n-1}T_{n-2}\ldots T_2T_1Pu,\; t)$
$= M_n(T_{n-1}T_{n-2}\ldots T_2T_1u,\; T_{n-1}T_{n-2}\ldots T_2T_1T_nT_{n-1}\ldots T_2T_1u,\; t)$
$> \min\{\, M_n(T_{n-1}T_{n-2}\ldots T_2T_1u,\, T_{n-1}T_{n-2}\ldots T_2T_1u,\, t),$
$M_n(T_{n-1}T_{n-2}\ldots T_2T_1u,\; T_{n-1}T_{n-2}\ldots T_2T_1T_nT_{n-1}\ldots T_2T_1u,\; t)$
$M_1(u,\, T_nT_{n-1}\ldots T_2T_1u,\, t)\}$
$= M_1(u,T_nT_{n-1}\ldots T_2T_1u,\, t)$
$= f(u)$

Hence $f(P^{n-1}u) > f(u)$ which is a contradiction.

Therefore $T_nT_{n-1}\ldots T_2T_1$ has a fixed point.
Now we want to prove the uniqueness.
Suppose $T_nT_{n-1}\ldots T_2T_1$ has two fixed point , say v and v´.
Now $M_1(v, v', t)$
$= M_1(T_nT_{n-1}\ldots T_2T_1v, T_nT_{n-1}\ldots T_2T_1v', t)$
$> \min\{M1(v', T_nT_{n-1}\ldots T_2T_1v,t), M_1(v', T_nT_{n-1}\ldots T_2T_1v',t), M_2(T_1v,T_1v',t)\}$
$= \min\{M1(v', v, t), M_1(v', v', t), M_2(T_1v,T_1v',t)\}$
$= M_2(T_1v,T_1v',t)$
$= M_2(T_1T_nT_{n-1}\ldots T_2T_1v, T_1T_nT_{n-1}\ldots T_2T_1v', t)$
$> \min\{M_2(T_1v', T_1T_nT_{n-1}\ldots T_2T_1v, t), M_2(T_1v', T_1T_nT_{n-1}\ldots T_2T_1v', t),$
$M_3(T_2T_1v, T_2T_1v', t)\}$
$= \min\{M_2(T_1v', T_1v, t), M_2(T_1v', T_1v', t), M_3(T_2T_1v, T_2T_1v', t)\}$
$= M_3(T_2T_1v, T_2T_1v', t)$
$> \ldots \ldots \ldots$
$= M_n(T_{n-1}T_{n-2}\ldots T_2T_1v, T_{n-1}T_{n-2}\ldots T_2T_1v', t)$
$= M_n(T_{n-1}T_{n-2}\ldots T_2T_1v, T_{n-1}T_{n-2}\ldots T_2T_1T_nT_{n-1}\ldots T_2T_1v', t)$
$> \min\{M_n(T_{n-1}T_{n-2}\ldots T_2T_1v', T_{n-1}T_{n-2}\ldots T_2T_1v, t),$
$M_n(T_{n-1}T_{n-2}\ldots T_2T_1v', T_{n-1}T_{n-2}\ldots T_2T_1T_nT_{n-1}\ldots T_2T_1v', t),$
$M_1(v, T_nT_{n-1}T_{n-2}\ldots T_2T_1v', t)\}$
$= \min\{M_n(T_{n-1}T_{n-2}\ldots T_2T_1v', T_{n-1}T_{n-2}\ldots T_2T_1v, t),$
$M_n(T_{n-1}T_{n-2}\ldots T_2T_1v', T_{n-1}T_{n-2}\ldots T_2T_1v', t),$
$M_1(v, T_nT_{n-1}T_{n-2}\ldots T_2T_1v', t)\}$
$= M_1(v, v', t)$
$\therefore M_1(v, v', t) > M_1(v, v', t)$ which is a contradiction.

Hence $T_nT_{n-1}T_{n-2}\ldots T_2T_1$ has a unique fixed point. Similarly for other cases we can prove the result.

Theorem 3.2. Let X be a nonempty set and M be a mapping from $X\times X\times(0,\alpha) \to [0,1]$such that $M(x_1,x_2,t) = 1$ if and only if $x_1 = x_2$ and M is symmetric with first two component. If $T_i : X\to X$ $(1 \le i \le n)$ be n mappings which satisfies the following conditions :

$$M(T_nT_{n-1}\ldots T_3T_2x_2,T_nT_{n-1}\ldots T_2T_1x_1,t) > \min\{M(x_1,T_nT_{n-1}\ldots T_3T_2x_2,t), M(x_1,T_nT_{n-1}\ldots T_2T_1x_1,t), M(x_2, T_1x_1,t)\} \quad \textbf{(3.4)}$$

$$M(T_{i-1}T_{i-2}\ldots T_2T_1T_nT_{n-1}\ldots T_{i+1}x_{i+1}, T_{i-1}T_{i-2}\ldots T_2T_1T_nT_{n-1}\ldots T_{i+1}T_ix_i,t) > \min\{M(x_i, T_{i-1}T_{i-2}\ldots T_2T_1T_nT_{n-1}\ldots T_{i+2}T_{i+1}x_{i+1},t), M(x_i, T_{i-1}T_{i-2}\ldots T_2T_1T_nT_{n-1}\ldots T_{i+1}T_ix_i,t), M(x_{i+1}, T_ix_i,t)\} \ (1<i<n) \quad \textbf{(3.5)}$$

$M(T_{n-1}T_{n-2}\ldots T_2T_1x_1,T_{n-1}T_{n-2}\ldots T_2T_1T_nx_n,t)$

> min { M (xn, Tn-1Tn-2…T2T1x1,t), M (xn, Tn-1Tn-2…T2T1Tnxn,t),
M $(x_1, T_n x_n,t)\}$ **(3.6)**

where $x_i \in X$ with $x_i \neq, T_{i-1}x_{i-1}$ $(1< i \leq n)$ and $T_n x_n \neq x_1$.
Now if $x_1 \to M(x_1,T_nT_{n-1} \ldots T_2T_1x_1,t)$ or $x_i \to M(x_i, T_{i-1}T_{i-2}\ldots T_2T_1T_nT_{n-1}\ldots T_{i+1}T_ix_i, t)$ $(1\leq i \leq n)$ attains its supremum in X then $T_nT_{n-1}\ldots T_2T_1$, $T_{i-1}T_{i-2}\ldots T_2T_1T_nT_{n-1}\ldots T_{i+1}T_i$ $(1 < i \leq n)$ has a unique common fixed point.

Proof : Suppose the function $x_1 \to M(x_1,T_nT_{n-1}\ldots T_2T_1x_1,t)$ attains its supremum on X , i.e., there exists a $u \in X$ such that $M(u,T_nT_{n-1}\ldots T_2T_1u,t) = \sup \{ M_1(x,T_nT_{n-1}\ldots T_2T_1x,t) : (x \in X) = f(u)$ (say). We claim that u is the common fixed point of $T_nT_{n-1}\ldots T_2T_1$, $T_{i-1}T_{i-2}\ldots T_2T_1T_nT_{n-1}\ldots T_{i+1}T_i$ $(1 < i \leq n)$. If otherwise let u is not a fixed point of $T_nT_{n-1}\ldots T_2T_1 = P$ (say) , then from (3.4) we have

$f(P^{n-1}u)$
$= M(P^{n-1}u , P^nu, t)$

$= M(T_nT_{n-1}\ldots T_2T_1P^{n-2}u , T_nT^{n-1}\ldots T_2T_1P^{n-1}u, t)$
$> \min \{M(P^{n-1}u,T_nT_{n-1}\ldots T_2T_1P^{n-2}u, t), M(P^{n-1}u, T_nT_{n-1}\ldots T_2T_1P^{n-1}u, t),$
$M(T_1P^{n-2}u, T_1P^{n-1}u, t)\}$
$= \min \{M(P^{n-1}u, P^{n-1}u, t), M(P^{n-1}u, P^nu, t), M(T_1P^{n-2}u, T_1P^{n-1}u, t)\}$
$= M(T_1P^{n-2}u, T_1P^{n-1}u, t)$
$= M(T_1T_nT_{n-1}\ldots T_2T_1P^{n-3}u, T_1T_nT_{n-1}\ldots T_2T_1P^{n-2}u, t)$
$> \min \{M(T_1P^{n-2}u,T_1T_nT_{n-1}\ldots T_2T_1P^{n-3}u, t), M(T_1P^{n-2}u,T_1T_nT_{n-1}\ldots T_2T_1P^{n-2}u, t),$
$M(T_2T_1P^{n-3}u, T_2T_1P^{n-2}u, t)\}$
$= M(T_2T_1P^{n-3}u, T_2T_1P^{n-2}u, t)$

$> \ldots \ldots \ldots$
$= M(T_{n-1}T_{n-2}\ldots T_2T_1u, T_{n-1}T_{n-2}\ldots\ldots T_2T_1Pu, t)$
$= M(T_{n-1}T_{n-2}\ldots T_2T_1u, T_{n-1}T_{n-2}\ldots T_2T_1T_nT_{n-1}\ldots T_2T_1u, t)$
$> \min \{M(T_{n-1}T_{n-2}\ldots T_2T_1u, T_{n-1}T_{n-2}\ldots T_2T_1u, t),$
$M(T_{n-1}T_{n-2}\ldots T_2T_1u, T_{n-1}T_{n-2}\ldots T_2T_1T_nT_{n-1}\ldots T_2T_1u, t)$
$M(u,T_nT_{n-1}\ldots T_2T_1u, t)\}$
$= M(u,T_nT_{n-1}\ldots T_2T_1u, t)$
$= f(u)$

Hence $f(P^{n-1}u) > f(u)$ which is a contradiction.

Therefore $T_nT_{n-1}\ldots T_2T_1$has a fixed point. Similarly we can prove that u is a fixed point of $T_{i-1}T_{i-2}\ldots T_2T_1T_nT_{n-1}\ldots T_{i+1}T_i$ $(1 < i \leq n)$.

Now we want to prove the uniqueness.

Suppose $T_nT_{n-1}\ldots T_2T_1$ has two fixed point , say v and v´.

Now M(v, v´, t)

$= M(T_nT_{n-1}\ldots T_2T_1v,\ T_nT_{n-1}\ldots T_2T_1v',\ t)$

$> \min\{M(v', T_nT_{n-1}\ldots T_2T_1v, t), M(v', T_nT_{n-1}\ldots T_2T_1v',t), M(T_1v,T_1v',t)\}$

$= \min\{ M(v', v, t), M(v', v', t), M(T_1v,T_1v',t)\}$

$= M(T_1v,T_1v',t)$

$= M(T_1T_nT_{n-1}\ldots T_2T_1v,\ T_1T_nT_{n-1}\ldots T_2T_1v',\ t)$

$> \min\{M(T1v', T_1T_nT_{n-1}\ldots T_2T_1v, t), M(T_1v', T_1T_nT_{n-1}\ldots T_2T_1v', t),$
$M(T_2T_1v, T_2T_1v', t)\}$

$= \min\{M(T_1v', T_1v, t), M(T_1v', T_1v', t), M(T_2T_1v, T_2T_1v', t)\}$

$= M(T_2T_1v, T_2T_1v', t)$

$> \ldots\ \ldots\ \ldots$

$= M(T_{n-1}T_{n-2}\ldots T_2T_1v,\ T_{n-1}T_{n-2}\ldots T_2T_1v',\ t)$

$= M(T_{n-1}T_{n-2}\ldots T_2T_1v,\ T_{n-1}T_{n-2}\ldots T_2T_1T_nT_{n-1}\ldots T_2T_1v',\ t)$

$> \min\{M(T_{n-1}T_{n-2}\ldots T_2T_1v', T_{n-1}T_{n-2}\ldots T_2T_1v, t),$
$M(T_{n-1}T_{n-2}\ldots T_2T_1v',\ T_{n-1}T_{n-2}\ldots T_2T_1T_nT_{n-1}\ldots T_2T_1v',\ t),$
$M(v,\ T_nT_{n-1}T_{n-2}\ldots T_2T_1v',\ t)\}$

$= \min\{M(T_{n-1}T_{n-2}\ldots T_2T_1v', T_{n-1}Tn\text{-}2\ldots T_2T_1v, t),$
$M(T_{n-1}T_{n-2}\ldots T_2T_1v',\ T_{n-1}T_{n-2}\ldots T_2T_1v',\ t),$
$M(v,\ T_nT_{n-1}T_{n-2}\ldots T_2T_1v',\ t)\}$

$= M(v, v', t)$

$\therefore\ M(v, v', t) > M(v, v', t)$ which is a contradiction.

Hence $T_nT_{n-1}T_{n-2}\ldots T_2T_1$ has a unique fixed point. Similarly for other cases we can prove the result. Therefore u is the common fixed point. This proves the result.

Theorem 3.3. Let X be a nonempty set and M be a mapping from $X\times X\times(0,\alpha) \to [0,1]$with M(x , y , t) = 1 if and only if x = y and M is symmetric with first two component. Let $T_i : X \to X$ $(1\le i \le n)$ be n mappings which satisfies the following conditions :

$$M(T_2T_3T_4\ldots T_{n-1}T_nx, T_3T_4T_5\ldots T_nT_1y, t) > \min\{M(T_3T_4T_5\ldots T_{n-1}T_nx, T_4T_5\ldots T_nT_1y, t), M(T_2T_3T_4\ldots T_{n-1}T_nx,T_3T_4T_5\ldots T_nx, t), M(T_3T_4\ldots T_{n-1}T_nT_1y, T_4T_5\ldots T_nT_1y, t), M(T_3T_4\ldots T_{n-1}T_nx, T_3T_4T_5\ldots T_nT_1y, t)\} \quad \textbf{(3.7)}$$

$$M(T_iT_{i+1}T_{i+2}\ldots T_{n-1}T_nT_1T_2\ldots T_{i-3}T_{i-2}x,\ T_{i+1}T_{i+2}\ldots T_{n-1}T_nT_1T_2\ldots T_{i-2}T_{i-1}y,\ t)$$
$$> \min\{M(T_{i+1}T_{i+2}\ldots T_{n-1}T_nT_1T_2\ldots T_{i-3}T_{i-2}x,\ T_{i+2}\ T_{i+3}\ .\ .\ .T_{n-1}T_nT_1T_2\ldots T_{i-2}T_{i-1}y,\ t),$$

$M(T_iT_{i+1}T_{i+2}\ldots T_{n-1}T_nT_1T_2\ldots T_{i-3}T_{i-2}x,\ T_{i+1}T_{i+2}\ldots T_{n-1}T_nT_1T_2\ldots T_{i-3}T_{i-2}x,\ t),$
$M(T_{i+1}T_{i+2}\ldots T_{n-1}T_nT_1T_2\ldots T_{i-2}T_{i-1}y,\ T_{i+2}\ T_{i+3}\ \ldots T_{n-1}T_nT_1T_2\ldots T_{i-2}T_{i-1}y,\ t),$
$M(T_{i+1}T_{i+2}\ldots T_{n-1}T_nT_1T_2\ldots T_{i-3}T_{i-2}x,\ T_{i+1}T_{i+2}\ldots T_{n-1}T_nT_1T_2\ldots T_{i-2}T_{i-1}y,\ t)\}$ **(3.8)**

for $(3 \le i \le n)$
$M(T_1T_2T_3\ldots T_{n-2}T_{n-1}x,\ T_2T_3T_4\ldots T_{n-1}T_ny,\ t) > \min\{M(T_2T_3T_4\ldots T_{n-1}x, T_3T_4T_5\ldots T_{n-1}T_ny, t),$
$M(T_1T_2T_3\ldots T_{n-1}x, T_2T_3T_4\ldots T_{n-2}T_{n-1}x,\ t),\ M(T_2T_3T_4\ldots T_{n-1}T_ny,\ T_3T_4T_5\ldots T_{n-1}T_ny,\ t),$
$M(T_2T_3T_4\ldots T_{n-2}T_{n-1}x,\ T_2T_3T_4\ldots T_{n-1}T_ny,\ t)\}$ **(3.9)**

for all x , $y \in X$ and $T_3T_4T_5\ldots T_{n-1}T_nx \neq T_4T_5\ldots T_nT_1y$, $T_{i+1}T_{i+2}\ldots T_{n-1}T_nT_1T_2\ldots T_{i-3}T_{i-2}x \neq T_{i+2}\ldots T_{n-1}T_nT_1T_2\ldots T_{i-2}T_{i-1}y$ $(3 \le i \le n)$, $T_2T_3T_4\ldots T_{n-2}T_{n-1}x \neq T_3T_4T_5\ldots T_{n-1}T_ny$ respectively. If the functions $x \to M(T_2T_3T_4\ldots T_{n-1}T_nx, T_3T_4T_5\ldots T_{n-1}T_nx,\ t)$ or $x \to M(T_iT_{i+1}T_{i+2}\ldots T_{n-1}T_nT_1T_2\ldots T_{i-3}T_{i-2}x,\ T_{i+1}T_{i+2}\ldots T_{n-1}T_nT_1T_2\ldots T_{i-3}T_{i-2}x,\ t)$ $(3 \le i \le n)$or $x \to M(T_1T_2T_3\ldots T_{n-2}T_{n-1}x,\ T_2T_3T_4\ldots T_{n-2}T_{n-1}x,\ t)$ attains its supremum on X then $T_3T_4\ldots T_{n-1}T_n$ or $T_{i+1}T_{i+2}\ldots T_{n-1}T_nT_1T_2\ldots T_{i-3}T_{i-2}$ $(1\le i \le n-1)$ has a fixed point .

Proof : Assume that $x \to M(T_2T_3T_4\ldots T_{n-1}T_nx,\ T_3T_4T_5\ldots T_{n-1}T_nx,\ t)$ attains its supremum on X. Then there exists $z \in X$ such that $M(T_2T_3T_4\ldots T_{n-1}T_nz,\ T_3T_4T_5\ldots T_{n-1}T_nz,t) = \sup\{M(T_2T_3T_4\ldots T_{n-1}T_nx,T_3T_4T_5\ldots T_{n-1}T_nx,\ t) : x \in X\} = f(z)$ (say). Also we suppose that $T_3T_4\ldots T_{n-1}T_n$ or $T_{i+1}T_{i+2}\ldots T_{n-1}T_nT_1T_2\ldots T_{i-3}T_{i-2}$ $(1\le i \le n-1)$ have no fixed point.

Now $f(T_1T_2T_3\ldots T_{n-1}T_nz)$

$= M(T_2T_3T_4\ldots T_{n-1}T_n\ T_1T_2T_3\ldots T_{n-1}T_nz,\ T_3T_4T_5\ldots T_n\ T_1T_2T_3\ldots T_{n-1}T_nz,\ t)$
$> \min\{M(T_3T_4T_5\ldots T_{n-1}T_nT_1T_2T_3\ldots T_{n-1}T_n\ z,\ T_4T_5\ldots T_nT_1T_2T_3\ldots T_{n-1}T_n\ z,\ t),$
$M(T_2T_3T_4\ldots T_{n-1}T_nT_1T_2T_3\ldots T_{n-1}T_n\ z\ ,\ T_3T_4T_5\ldots T_nT_1T_2T_3\ldots T_{n-1}T_n\ z\ ,\ t),$
$M(T_3T_4\ldots T_{n-1}T_nT_1T_2T_3\ldots T_{n-1}T_n\ z,\ T_4T_5\ldots T_nT_1T_2T_3\ldots T_{n-1}T_n\ z,\ t),$
$M(T_3T_4\ldots T_{n-1}T_nT_1T_2T_3\ldots T_{n-1}T_n\ z,\ T_3T_4T_5\ldots T_nT_1T_2T_3\ldots T_{n-1}T_nz,\ t)\}$(by (3.7))
$= M(T_3T_4\ldots T_{n-1}T_nT_1T_2T_3\ldots T_{n-1}T_n\ z,\ T_4T_5\ldots T_nT_1T_2T_3\ldots T_{n-1}T_nz,\ t)$
$> \min\{M(T_4T_5\ldots T_{n-1}T_nT_1T_2T_3\ldots T_{n-1}T_n\ z,\ T_5T_6\ldots T_nT_1T_2T_3\ldots T_{n-1}T_nz,\ t),$
$M(T_3T_4\ldots T_{n-1}T_nT_1T_2T_3\ldots T_{n-1}T_nz,\ T_4T_5\ldots T_nT_1T_2T_3\ldots T_{n-1}T_n\ z,\ t),$
$M(T_4T_5\ldots T_{n-1}T_nT_1T_2T_3\ldots T_{n-1}T_n\ z,\ T_5T_6\ldots T_nT_1T_2T_3\ldots T_{n-1}T_n\ z,\ t),$
$M(T_4T_5\ldots T_{n-1}T_nT_1T_2T_3\ldots T_{n-1}T_n\ z,\ T_5T_6\ldots T_nT_1T_2T_3\ldots T_{n-1}T_nz,\ t)\}$
$= M(T_4T_5\ldots T_{n-1}T_nT_1T_2T_3\ldots T_{n-1}T_n\ z,\ T_5T_6\ldots T_nT_1T_2T_3\ldots T_{n-1}T_n\ z,\ t)$
… … … … … …
$= M(T_1T_2T_3\ldots T_{n-1}T_n\ z,\ T_2T_3\ldots T_{n-1}T_n\ z,\ t)$
$> \min\{M(T_2T_3\ldots T_{n-1}T_nz,\ T_3T_4\ldots T_{n-1}T_n\ z,\ t),\ M(T_1T_2T_3\ldots T_{n-1}T_n\ z,\ T_2T_3\ldots T_{n-1}T_n\ z,\ t),$
$M(T_2T_3\ldots T_{n-1}T_n\ z,\ T_3T_4\ldots T_{n-1}T_n\ z,\ t),\ M(T_2T_3\ldots T_{n-1}T_nz,\ T_3T_4\ldots T_{n-1}T_nz,\ t)\}$
$= M(T_2T_3\ldots T_{n-1}T_nz,\ T_3T_4\ldots T_{n-1}T_n\ z,\ t)$
$= f(z)$

$\therefore$ $f(T_1T_2T_3\ldots T_{n-1}T_nz) > f(z)$ which contradicts the fact that f(z) is supremum. Hence $T_3T_4\ldots T_{n-1}T_n$ or $T_{i+1}T_{i+2}\ldots T_{n-1}T_nT_1T_2\ldots T_{i-3}T_{i-2}$ $(1\leq i \leq n-1)$ has a fixed point.

Corollary 3.4. Let X be a nonempty set and M be a mapping from $X\times X\times(0,\alpha) \to [0,1]$ with $M(x , y , t) = 1$ if and only if $x = y$ and M is symmetric with first two component. Let T_1,T_2 , $T_3 : X \to X$ be 3 mappings which satisfies the following conditions :

$$M(T_2T_3x, T_3T_1y, t) > \min\{M(T_3x, T_1y, t), M(T_2T_3x, T_1x, t), M(T_3T_1y, T_1y, t), M(T_3x, T_3T_1y, t)\} \quad \textbf{(3.10)}$$

$$M(T_1T_2x, T_2T_3y, t) > \min \{ M(T_2x, T_3y, t), M(T_1T_2x, T_3x, t), M(T_2T_3y, T_3y, t), M(T_2x, T_2T_3y, t)\} \quad \textbf{(3.11)}$$

$$M(T_3T_1x, T_1T_2y, t) > \min\{M(T_1x, T_2y, t), M(T_3T_1x,T_2x, t), M(T_3T_1y, T_1y, t), M(T_1x, T_1T_2y, t)\} \quad \textbf{(3.12)}$$

for all $x , y \in X$ and $T_3x \neq T_1y$, $T_2x \neq T_3y$, $T_11x \neq T_2y$ respectively. If the functions

$x \to M(T_2T_3x,T_3x, t)$ or $x \to M(T_1T_2x, T_3x, t)$ or $x \to M(T_3T_1x, T_2x, t)$ attains its supremum on X then T_2T_3 or T_1T_2, T_3T_1 has a fixed point.

Theorem 3.5. Let X be a nonempty set and M be a mapping from $X\times X\times(0,\alpha) \to [0,1]$ with $M(x , y , t) = 1$ if and only if $x = y$. Let $T_i : X \to X$ $(1 \leq i \leq n)$ be pair wise commuting mappings which satisfies the following conditions :

$$M(T_1T_2T_3\ldots T_{n-2}T_{n-1}\, x, T_1T_2T_3\ldots T_{n-1}T_n\, y, t) > M(x, T_n\, y, t) \quad \textbf{(3.13)}$$
for $x \neq T_n\, y$, $T_1T_2T_3\ldots T_{n-2}T_{n-1}\, x \neq T_1T_2T_3\ldots T_{n-1}T_n\, y$

$$M(T_iT_{i+1}T_{i+2}\ldots T_{n-1}T_nT_1T_2\ldots T_{i-3}T_{i-2}\, x,\ T_iT_{i+1}T_{i+2}\ldots T_{n-1}T_nT_1T_2\ldots T_{i-2}T_{i-1}\, y,\ t) > M(x, T_{i-1}\, y, t) \quad \textbf{(3.14)}$$

for $x \neq T_{i-1}\, y$, $T_iT_{i+1}T_{i+2}\ldots T_{n-1}T_nT_1T_2\ldots T_{i-3}T_{i-2}\, x \neq T_iT_{i+1}Ti_{+2}\ldots T_{n-1}T_nT_1T_2\ldots T_{i-2}T_{i-1}\, y$ $(1< i <n)$

$$M(T_1T_2\ldots T_{n-3}T_{n-2}\, x, T_nT_1T_2\ldots T_{n-2}T_{n-1}\, y, t) > M(x, T_{i-1}\, y, t) \quad \textbf{(3.15)}$$
for $x \neq T_{n-1}\, y$, $T_nT_1T_2\ldots T_{n-3}T_{n-2}\, x \neq T_nT_1T_2\ldots T_{n-2}T_{n-1}\, y$

then T_i $(1\leq i \leq n)$ has a fixed point if the function $x \to M(x, T_ix, t)$ attains its supremum on X.

Proof : Assume that the function $x \to M(x, T_1x, t)$ attains its supremum at $z \to X$. Hence $M(z, T_1z, t) = \sup \{ M(x , T_1x, t) : x \in X \} = f(z)$ (say) . Let $P = T_1T_2T_3\ldots T_{n-1}T_n$ Suppose T_1 has no fixed point .

Now $f(P_{n-1}z) = M(P_{n-1}z, T_1P_{n-1}z, t)$

$= M(P^{n-1}z, P^{n-1}T_1z, t)$ (since Ti 's are pair wise commuting)

$= M(T_1T_2T_3\ldots T_{n-1}T_n P^{n-2}z, T_1T_2T_3\ldots T_{n-1}T_n P^{n-2}T_1z, t)$

$> M(T_n P^{n-2}z, P^{n-2}T_1z, t)$ (by (3.13))

$= M(T_n T_1T_2T_3\ldots T_{n-1}T_n P^{n-3}z, T_nT_1T_2T_3\ldots T_{n-1}T_n P^{n-3}T_1z, t)$

$> M(T_{n-1}T_n P^{n-3}z, T_{n-1}T_n P^{n-3}T_1z, t)$

… … … … … …

$> M(T_2T_3\ldots T_{n-1}T_n z, T_2T_3\ldots T_{n-1}T_n T_1z, t)$

$> M(z, T_1z, t)$

$= f(z)$, which contradicts the fact that $f(z)$ is supremum.

Hence T_1 has a fixed point.

Similarly , we can prove for other T_i's.

This completes the proof of the theorem.

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