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PG/IIS/MTM/203/23

M.Sc. 2nd Semester Examination, 2023

APPLIED MATHEMATICS

PAPER – MTM-203

Full Marks : 40

Time : 2 hours

The figures in the right hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

UNIT—I

(*Abstract Algebra*)

[*Marks* : 20]

1. Answer any *two* questions : 2×2

(*a*) Define solvable group with example.

(*b*) Show that every finite field extension is an algebraic extension.

(*c*) Determine the degree of

$$\left[\mathbb{Q}\left(\sqrt{3+2\sqrt{2}}\right):\mathbb{Q}\right].$$

(*d*) Show that $[\mathbb{R} : \mathbb{Q}]$ is not finite.

2. Answer any *two* questions : 4×2

(*a*) If $K \subseteq F \subseteq L$ is a tower of fields then show that

$$[L:F]\,[F:K] = [L:K]$$

where $[L:F]$ denotes the degree of L over F.

(*b*) Show that it is impossible to construct a regular heptagon by using straightedge and compass only.

(*c*) Define perfect field with example. Also let F be a field of positive characteristic p. Then show that F is perfect if and only if $F = F^p = \{a^p \mid a \in F\}$.

(*d*) Show that a pentagon is constructible by ruler and compass.

3. Answer any *one* question : 8×1

(*a*) (*i*) Let E be a field and G a finite group of automorphisms of E. Then show that E/E^G is a finite Galois extension. 5

(*ii*) Show that the Galois group of the Galois extension $\mathbb{F}_{q^n}/\mathbb{F}_q$ is a cyclic group of order n. 3

(*b*) (*i*) Show that no finite field is algebraically closed. 3

(*ii*) Define splitting field with example. Show that if F is a field then any polynomial $f(x) \in F[x]$ has a splitting field. 5

UNIT–II

(*Linear Algebra*)

[*Marks* : 20]

4. Answer any *two* questions : 2×2

(*a*) Justify the statements as *true* or *false* :

(*i*) Every linear operator has an adjoint.

(*ii*) The adjoint of a linear operator is unique.

(*b*) Let V be an n-dimensional vector space over F. What is the characteristic polynomial of the identity operator V? What is the characteristic polynomial of the zero operator ?

(*c*) Let V be vector space of dimension d and $T : V \to V$ a linear mapping with rank r and nullity n. Show that $rn \leq \frac{1}{4}d^2$..

(*d*) Determine dual basis to the basis $\{(1, 0, -1), (-1, 1, 0), (0, 1, 1)\}$ of $\mathbb{R}^3$.

5. Answer any *two* questions : 4×2

(*a*) Let T be a linear operator on a real inner product space V, and define $H : V \times V \to R$ by $H(x, y) = \langle x, T(y) \rangle$ for all $x, y \in V$.

(*i*) Show that H is a bilinear form.

(*ii*) Prove that H is symmetric if and only if T is self-adjoint.

(*iii*) Explain why H may fail to be a bilinear form if V is a complex inner product space. 4

(*b*) Define symmetric and skew-symmetric bilinear forms. Show that any bilinear form b on a vector space V is the sum of a symmetric bilinear form and a skew-symmetric bilinear form. 2 + 2

(*c*) Let V be an inner product space, and let T be a normal operator on V, and T^* be an adjoint operator of T. Then prove that following statements are true.

(*i*) $\| T(x) \| = \| T^*(x) \|$ for all $x \in V$.

(*ii*) $T - cI$ is normal for every $c \in F$.

(*iii*) If x is an eigenvector of T, then x is an eigenvector of T^*. In fact, if $T(x) = \lambda x$, then $T^*(x) = \bar{\lambda} x$.

(*iv*) If λ_1 and λ_2 are the distinct eigenvectors of T with corresponding eigenvectors x_1 and x_2, then x_1 and x_2 are orthogonal. 4

(*d*) State and prove the first isomorphism theorem. 4

6. Answer any *one* question : 8×1

(*a*) (*i*) Let A be a symmetric matrix. Prove that A is positive definite if and only if all of its eigen values are positive.

(*ii*) Let $P_2(\mathbb{R})$ denotes the collection of all polynomials of degree ≤ 2 and $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be a linear operator defined by $T(f(\mathrm{x})) = 2f(x) - f'(x)$ for all $f(x) \in P_2(\mathbb{R})$. Find the Jordan canonical form of T. 4 + 4 (*b*) (*i*) Let T be a linear operator on an inner product space V, and suppose that $\|T(x)\| = \|x\|$ for all x. Prove that T is one-to-one.

(*ii*) Let $V = C([0, 1])$ and define $\langle f, g \rangle = \int_0^{\frac{1}{2}} f(t)g(t)dt$, where $C([0,1])$ is the space of continuous real-valued functions defined on the interval $[0, 1]$. Is this an inner product on V?

(*iii*) Let $V = P(R)$ with the inner product space $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)dt$, and consider the subspace of $P_2(R)$ with the standard basis β. Using Gram Schmidt Process to compute the orthogonal vectors of standards basis β of $P_2(R)$, and then use this orthogonal basis to obtain an orthonormal basis for $P_2(R)$. 2 + 2 + 4
