PG 1st Semester Examination, 2023

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

(*ODE and Special Functions*)

PAPER – MTM-103

Full Marks : 50

Time : 2 hours

The figures in the right hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

1. Answer any *four* questions : 2 × 4

(*a*) Find all the singularities of the following differential equation and then classify them : $(z-z^2)\omega''+(1-5z)\omega'-4\omega=0$.

(*b*) Show that $J_n(z)$ is an odd function of z if n is odd.

(Turn Over)

(*c*) Define fundamental set of solutions for a system of ordinary differential equation.

(*d*) Define orthogonal functions associated with Sturm-Liouville problem.

(*e*) Prove that : $F(-n, b, b; -z) = (1+z)^n$ where $F(a,b,c;z)$ denotes the hypergeometric function.

(*f*) Under suitable transofrmation to be considered by you, prove that Legendre differential equation can be reduced to hypergeometric equation.

2. Answer any *four* questions : 4×4

(*a*) Show that $J_0^2(z) + 2\sum_{n=1}^{\infty} J_n^2(z) = 1$ and prove that for real z, $|J_0(z)| < 1$, and $|J_n(z)| < \frac{1}{\sqrt{2}}$, for all $n \geq 1$.

(*b*) Using Green's function method, solve the following differential equation

$$y''(x) - y(x) = -2e^x,$$

subject to the boundary conditions $y(0) = y'(0), y'(l) = -y(1)$.

(*c*) Establish the generating function for the Bessel's function $J_n(z)$.

(*d*) Show that

$$1 + 3P_1(z) + 5P_2(z) + 7P_3(z) + \ldots + (2n+1)\, P_n(z) = \frac{d}{dz}\left[P_{n+1}(z) + P_n(z)\right],$$

where $P_n(z)$ denotes the Legendre's Polynomial of degree n.

(*e*) If the vector functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ defined as follows :

$$\varphi_1 = \begin{bmatrix} \varphi_{11} \\ \varphi_{21} \\ \cdot \\ \varphi_{n1} \end{bmatrix}, \varphi_2 = \begin{bmatrix} \varphi_{12} \\ \varphi_{22} \\ \cdot \\ \varphi_{n2} \end{bmatrix}, \ldots \varphi_n = \begin{bmatrix} \varphi_{1n} \\ \varphi_{2n} \\ \cdot \\ \varphi_{nn} \end{bmatrix}$$

be n solutions of the homogeneous linear differential equation $\frac{dx}{dt} = A(t)x(t)$ in the interval $a \le t \le b$, then these n solutions are linearly independent in $a \le t \le b$ iff Wronskian

$$W[\varphi 1, \varphi 2, \ldots, \varphi n] \ne 0 \forall t, \text{ on } a \le t \le b.$$

(f) Consider the boundary value problem

$$\frac{d^2 y}{dx^2} + \lambda y = 0, 0 \le x \le \pi$$

subject to $y(0) = 0$, $y'(\pi) = 0$. Find the eigen values and eigen functions of the problem.

3. Answer any *two* questions : 8×2

(*a*) (*i*) All the eigen values of regular SL problem with $r(x) > 0$, are real. 3

(*ii*) Find the general solution of the homogeneous system

$$\frac{dX}{dt} = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} X \quad \text{where } X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$ 5

(*b*) (*i*) Establish generating function for Legendre polynomial. Use it to prove that

$$(2n+1)zP_n(z) = (z+1)P_{n+1}(z) + nP_{n-1}(z).$$

4 + 2

(*ii*) Deduce the confluent hypergeometric differential equation from hypergeometric differential equation. 2

(*c*) (*i*) Prove that if $f(z)$ is continuous and has continuous derivatives in $[-1, 1]$

then $f(z)$ has unique Legendre series expansion is given by

$$f(z) = \sum_{n=0}^{\infty} C_n P_n(z)$$

where P_n's are Legendre Polynomials

$$C_n = \frac{2n+1}{2} \int_{-1}^{1} f(z) P_n(z),\ n = 1, 2, 3 \ldots$$ 6

(*ii*) Prove that $\frac{d}{dz}\left[J_0(z)\right] = -J_1(z)$. 2

(*d*) (*i*) Find the general solution of the ODE $2zw''(z) + (1+z)w'(z) - kw = 0$, (where k is a real constant) in series form for which values of k, is there a polynomial solution ? 5

(*ii*) Deduce the integral formula for hypergeometric function. 3

[Internal Assessment – 10 Marks]
