PG 1st Semester Examination, 2023

MATHEMATICS

(*Real Analysis*)

PAPER – MTM-101

Full Marks : 50

Time : 2 hours

The figures in the right hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

1. Answer any *four* questions : 2×4

(*a*) Show that difference of two measurable sets is a measurable set.

(*b*) Let X be a measurable space and $\chi_E : X \to \mathbb{R}$ be a measurable function, where

$$\chi_E(x) = \begin{cases} 1 \text{ if } x \in E \\ 0 \text{ if } x \notin E \end{cases}.$$

Is E a measurable set in X?

(c) Define Borel set.

(d) Show that the set of all natural numbers is a null subset of $\mathbb{R}$.

(e) If α is continuous and β is of bounded variation on $[a,b]$, show that $\alpha \in R(\beta)$ on $[a,b]$.

(f) State Lusin's theorem.

2. Answer any *four* questions : 4 × 4

(a) Establish a necessary and sufficient condition for a function $f : [a,b] \to \mathbb{R}$ to be a function of bounded variation on $[a,b]$.

(b) Show that the function $f(x)$ defined on $[4,7]$ by

$$f(x) = \begin{cases} 5, \text{ for all rotationals } x \text{ in } [4,7] \\ 6, \text{ for all irrationals } x \text{ in } [4,7] \end{cases}$$

is not a function of bounded variation on $[4,7]$.

(*c*) Suppose f is continous on $[a,b]$ and α is monotonically increasing on $[a,b]$. Show that $f \in \mathcal{R}(\alpha)$ on $[a,b]$.

(*d*) Show that every finite sum of real numbers can be expressed as the R-S integral over some interval.

(*e*) Let $f_n : X \to \mathbb{R}^*$ be measurable for n=1,2, 3,... Then show that $\liminf\limits_{n\to\infty} f_n$ and $\inf\limits_{n\to\infty} f_n$ are measurable functions on X.

(*f*) If $f_n : X \to [0,\infty]$ is measurable for n=1,2, 3,..., and $f(x) = \sum_{n=1}^{\infty} f_n(x), x \in X$, then show that $\int f\, d\mu = \sum_{n=1}^{\infty} \int f_n d\mu$.

3. Answer any *two* questions : 8×2

(*a*) (*i*) State and prove the Fatou's lemma. Give an example to show that strict inequality can occur in Fatou's lemma. 4

(*ii*) Let μ be a measure on a σ-algebra of subsets of X. Show that the outer measure μ^* induced by μ is countably subadditive. 4

(*b*) (*i*) Let $f(x)$ be defined as $f(x) = \frac{1}{x^{\frac{4}{5}}}$ if $0 < x \leq 1$ and $f(0) = 0$. Show that f is Lebesgue integrable on $[0,1]$. Also compute the integral. 6

(*ii*) Evaluate the following :

$$\int_{-3}^{4} (x^2 + 7) d([x] - x).$$ 2

(*c*) (*i*) Construct a non-measurable subsets of $\mathbb{R}$. 5

(*ii*) Let $\{E_k\}$ be a sequence of measurable sets in X such that $\sum_{k=1}^{\infty} u(E_k) < \infty$. Then prove that almost all $x \in X$ lie in at most finitely many of the sets E_k. 3

(*d*) (*i*) Let $\{f_n\}_{n \geq 1}$ and $\{g_n\}_{n \geq 1}$ be sequence of measurable functions such that $|f_n| \leq g_n$ for all n. Let f and g be measurable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. $x(\mu)$ and $\lim_{n\to\infty} g_n(x) = g(x)$ for a.e. $x(\mu)$. If $\lim_{n\to\infty} \int g_n \, d\mu = \int g \, du < +\infty$, show that $\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$. 5

(*ii*) Show that the Cantor set is an uncountable set. 3

[Internal Assessment – 10 Marks]