Journal of Physical Sciences, Vol. 27, 2022, 27-30
ISSN: 2350-0352 (print), www.vidyasagar.ac.in/publication/journal
Published on 30 December 2022

On the Exponential Diophantine Equation 3^x - 2.5^y = z^2

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Received 3 October 2022; accepted 20 December 2022

ABSTRACT

In this paper, we introduce the notion of the exponential Diophantine equation. We also did some characterisations of the Diophantine equation. In this article, we study and establish the theorem of the Diophantine equation $3^x - 2.5^y = z^2$ where x, y and z are non-negative integers which have no non-negative solution.

Keywords: Diophantine equation; factoring method; modular arithmetic method.

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

Many authors have been studied Diophantine equations. Thongnak, Chuayan and Kaewong [1] have recently studied exponential Diophantine equation. Thongnak, Chuayan and Kaewong has motivated us to make foundation of another exponential Diophantine equation $3^x - 2.5^y = z^2$ .The exponential diophantine equation is the diophantine equation where the unknown variables are exponents. On the other hand, many researchers studied the equations involving the exponential diophantine equation. For example, in 2007, the exponential diophantine $2^x + 5^y = z^2$ was proved by Acu [2] and which has two non-negative integer solutions such as $(x, y, z) = (3,0,3)$ and $(x, y, z) = (2,1,3)$. We refer the reader to [13] for the diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ which have no solution. Interestingly, Thongnak et al. [12] have introduced the exponential Diophantine equation $2^x - 3^y = z^2$ and found the only two non-negative integers $(x, y, z) = (1,0,1)$ and $(x, y, z) = (2,1,1)$. In 2021, Thongnak et al. [11] have proved the vital exponential Diophantine equation $7^x - 5^y = z^2$ which has only one trivial solution $(x, y, z) = (1,0,1)$. The work of Thongnak et al. [10, 4] plays an important role to proof the individual result of exponential diophantine equation.

2. Preliminary

A Diophantine equation is a polynomial equation that takes only integer values. The famous general equation $p^x + q^y = z^2$ has many forms and many authors already established the various theorem depending on this famous general equation. In this paper, we have studied the exponential Diophantine equation with the favor of the following three Lemmas which is already proved by Thongnak, Chuayan and Kaewong [1].

Lemma 2.1. If x is integer, then $x^2 \equiv 0 \bmod 4$ and $x^2 \equiv 1 \bmod 4$.

Lemma 2.2. 3 is not a common divisor of both $5^k - z$ and $5^k + z$ where k and z are non-negative integers.

Lemma 2.3. If x is an integer, then $x^2 \equiv 0,1 \pmod 3$.

3. Main result

Theorem 3.1. The exponential Diophantine equation $3^x - 2.\,5^y = z^2$ where x, y and z are non-negative integers which is not solvable.

Proof: Let us consider that $x, y, z \in Z^+ \cup \{0\}$, where Z^+ is the set of all positive integers such that

$$3^x - 2.\,5^y = z^2 \qquad (1)$$

To prove the above equation we consider the four cases:

Case 1: $x = y = 0$. In this case, it is obvious that the equation (1) has no any solution because $z^2 = -1$, which is not possible.

Cas2e 2: $x > 0$ and $y = 0$. Then the equation (1) becomes $3^x - 2 = z^2$. Thus we have $z^2 \equiv -1 \pmod 4$ or $z^2 \equiv 3 \pmod 4$. By Lemma 2.1 which is not possible.

Case 3: $x = 0$ and $y > 0$. The equation (1) becomes $1 - 2.\,5^y = z^2$. This gives the result that $z^2 < 0$ and it is clear that this case is also impossible.

Case 4: $x > 0$ and $y > 0$. Now from the equation (1) we get $z^2 \equiv (-1)^x \pmod 3$ and then by Lemma 2.3, it is obvious that x is even. If we take $x = 2k$, for all $k \in Z^+$. Then the equation (1) gives the result that:

$3^{2k} - z^2 = 2.5^y$ and this implies

$$\left(3^k\right)^2 - z^2 = 2.5^y \text{ or } \left(3^k - z\right)\left(3^k + z\right) = 2.5^y \qquad (2)$$

Now by Lemma 2.2, we can explain four possible subcases which are as follows:

Subcase 4.1: $\left(3^k + z\right) = 2.5^y$ and $\left(3^k - z\right) = 1$. Adding these two equations we get $\left(3^k - z\right) + \left(3^k + z\right) = 1 + 2.5^y$ and then we have $2.3^k = 1 + 2.5^y$, which is not possible. Again, $\left(3^k - z\right) = 1$ is implies $3^k - 1 = z$ and putting this value in $\left(3^k + z\right) = 2.5^y$ implies that $2\left(3^k - 3^y\right) = 1$ which is also impossible.

Subcase 4.2: $\left(3^k - z\right) = 2$ and $\left(3^k + z\right) = 5^y$. Adding these two equations we have $2.3^k = 2 + 5^y$ implies $2\left(3^k - 1\right) = 5^y$ which is impossible.

Subcase 4.3: $\left(3^k - z\right) = 5^y$ and $\left(3^k + z\right) = 2$. Since $\left(3^k + z\right) = 2$, we have $z = 2 - 3^k$. Obviously, $k > 0$ and it gives $z < 0$. This is impossible.

Subcase 4.4: $\left(3^k - z\right) = 2.5^y$ and $\left(3^k + z\right) = 1$ .Since $\left(3^k + z\right) = 1$, this implies $z = 1 - 3^k$. It is clear that if $k > 0$, then $z < 0$ which is impossible.

Hence $3^x - 2.\,5^y = z^2$ exponential Diophantine equation.

3. Conclusion

In this study, we have presented the basic theorems in Number theory such as the factoring method and modular method, to solve the exponential Diophantine equation $3^x - 2.\,5^y = z^2$.Finally, we have proved that the exponential Diophantine equation $3^x - 2.\,5^y = z^2$ in which x, y and z are non-negative integers which has no non-negative solution.

Acknowledgements. The author gratefully acknowledges the reviewer's valuable comments and suggestions to improve the paper.

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