Chapter 7

Application of complex fuzzy soft sets in medical diagnosis system through a similarity measure approach

7.1 Introduction

In the context of cumulative human expectancy and budge constraint, providing a proper medical care is a crucial issue in medical science. In order to help the medical consultants for accomplishing this crucial medical issue, several mathematicians have developed different types of mathematical solutions so that patients can get a fast and trustworthy diagnosis service at their hard time. Most of the existing mathematical approaches are supported to handle crisp data [120, 163]. However, in general medical practice, uncertainty is an inward rife substance due to the vagueness emerging from drawing a proper conclusion of the health status of a patient. Zadeh's [183] fuzzy set theory has been of wide use in monitoring uncertainty where, an imprecise or vague data can be defined in a conventional way by using a suitable membership function [1, 114]. Then, various approaches have been suggested to deal with disease diagnosis problems by using fuzzy set theory and its other extensions. Djatna et al. [52] provided a stroke disease diagnosis approach through intuitionistic fuzzy sets, Karaaslan [87] proposed a Gaussian single-valued neutrosophic number based multi-criteria decision-making approach for solving medical diagnosis problems, Hashmi et al. [78] used m-polar neutrosophic topology in medical science, etc. Furthermore, type-2 fuzzy set and generalized type-2 fuzzy set have also been used by the researchers [122, 123] in medical process.

Though fuzzy set theory and its several extensions are very influential for handling uncertainty, Prof. Molodtsov [118] brought out some of its difficulties in considering an appropriate membership function of a problem. Then, he appointed the notion of uncertainty from a different points of view by using parameterization and introduced soft set theory [118]. In soft set theory, construction of membership function has been replaced by the idea of parameterization for defining an element. Consequently, with a very small duration of time, soft set theory has become a herculean theoretical approach for solving uncertainty. Further, Maji et al. incorporated soft set theory with fuzzy set theory and initiated the idea of fuzzy soft set theory [104] where, all the associated parameters are in fuzzy sense. Actually, in fuzzy soft set, the issue fuzziness is handled by the human cognitive process without considering a membership function. Therefore, it is very useful in practice. In existing studies, we have observed that, researchers have been benefitted by fuzzy soft set theory in solving problems in disease diagnosis system. For instance, Basu et al. [22] mentioned a new approach, mean potentiality, for fuzzy soft set based problems. Then, Tang [155], Li et al. [99] and Wang et al. [161] developed some fuzzy soft set based algorithms by using Dempster-Shafer-Theory of evidence.

However, in many cases, all the considered parameters are not in fuzzy sense but may be in complex fuzzy sense. Complex fuzzy sense comes from complex fuzzy set. Basically, complex fuzzy set [137] is a generalization of Zadeh's fuzzy set where, range of the fuzzy membership function $[0, 1]$ is transformed into the unit circle of a complex plane. Mathematically, a complex fuzzy membership is written in the form as, $a(x)e^{ib(x)}$ where, $a(x) \in [0, 1]$ (a fuzzy value) is its amplitude part and $b(x)$ (real number) is its phase part. In many day-to-day life problems, two information need to be expressed together. For example, in periodic or recurring phenomena related problems (which are generally seems in solar activity system, financial indicator system, signal processing system, disease diagnosis system, etc.), time function is a vital part and should be considered additionally with any associated criteria. Such type of problems can be limned through complex fuzzy set. Then, Thirunavukarasu et al. [158] combined soft set theory with complex fuzzy set and introduced the idea of complex fuzzy soft set in order to consider all the parameters in complex fuzzy sense. Complex fuzzy soft set is a recent developed model of soft set theory as well as fuzzy soft set theory where, the evaluation of an alternative over a parameter is a two dimensional information in terms of amplitude part and phase part.

Moreover, similarity measure and distance measure are two conflicting representations for comparing two items; one is in favor of their closeness degree and another one is in favor of their deviation degree. In many different fields like, pattern recognition, decision-making, disease diagnosis, etc., these two representations have been used very successfully. Numerous researchers have offered different types of similarity measure and distance measure approaches for comparing fuzzy sets [26], intuitionistic fuzzy sets [151], neutrosophic sets, [27] etc. Furthermore, similarity measure and distance measure have also been well introduced in soft set theory [109, 112] and applied magnificently in many decision-making problems. But, yet for complex fuzzy soft sets, similarity measure and distance measure have not been explored.

Usually, in a multi-expert decision-making, different experts provide their opinions about some associated alternatives and then a common decision is taken by integrating all of their opinions. Therefore, aggregation operation is a very common aspect in solving multi-expert decision-making. In literature, various aggregation operations have been saved to deal with soft set based decision making under different uncertain environments. Roy et al. [139] and Alcantud [4] used AND aggregation operation to construct a resultant fuzzy soft set from multiple fuzzy soft sets. Then, Mao et al. [103] considered the most ancient aggregation operations, *arithmetic mean and geometric mean*, to get a resultant from a group decision-making based on intuitionistic fuzzy soft sets. Akram et al. [10] also used *arithmetic mean and geometric mean* for solving hesitant N-soft set based group decision-making. Further, Garg and Arora mentioned some aggregation operations for intuitionistic fuzzy soft sets [66, 67], Khalil et al. [90] introduced an aggregation operation for interval-valued picture fuzzy soft sets, Guleria and Bajaj [69] proposed an aggregation operation for T-spherical fuzzy soft sets, etc. But, till now, no researcher has emphasized the aggregation operation on complex fuzzy soft sets.

So, from the aforementioned discussion, complex fuzzy soft set is a very effective category of soft set by which complicated decision-making, specially disease diagnosis decision-making where, time is a significant issue corresponding to any symptom of a patient, can be handled. But, in previous literature review, there exist no work on on decision-making over complex fuzzy soft set. Therefore, to develop this recent proposed model, more research is needed. Therefore, in this chapter, we have motivated to work on complex fuzzy soft set theory. Our main goals of this chapter are as follows:

- To introduce similarity measure approach on complex fuzzy soft sets.
- To develop an aggregation operation on complex fuzzy soft sets.
- To propose a multi-expert decision-making over complex fuzzy soft sets.
- To compose decision-making problem in medical science by using our proposed decision-making approach.

This chapter has been organized as follows. In Section 7.2, we have reviewed some preliminary ideas related to our subsequent discussions. Then, in Sections 7.3, we have introduced the similarity measure approach for complex fuzzy soft sets. In Section 7.4, an aggregation operation on complex fuzzy soft sets has been presented. After that, in Section 7.5, we have proposed a complex fuzzy soft set based multi-expert decision-making approach. Section 7.6 provides a case study and its solution which is related with the medical diagnosis system. In Section 7.7, comparative analysis and sensitivity analysis have been given to examine the feasibility and effectiveness of our proposed. Finally, in Section 7.8, we have given the conclusion of this chapter.

7.2 Some basic relevant notions

(i) **Some set theoretic operations on complex fuzzy sets [185].**

Consider two complex fuzzy sets $\tilde{C}_A$ and $\tilde{C}_B$ over a universal set X as follows:
$\tilde{C}_A = \{(x_s, p_A(x_s)e^{iu_A(x_s)})|\forall x_s \in X\}$, $\tilde{C}_B = \{(x_s, p_B(x_s)e^{iu_B(x_s)})|\forall x_s \in X\}$; $\forall s = 1, 2, .., m$.

- **Complex fuzzy intersection [185]**

Complex fuzzy intersection of two complex fuzzy sets $\tilde{C}_A$ and $\tilde{C}_B$ is as follows:

$$\tilde{C}_A \cap \tilde{C}_B = \left\{\left(x_s, \mu_{\tilde{C}_A \cap \tilde{C}_B}(x_s)\right)|x_s \in X\right\}$$

where,

$$\mu_{\tilde{C}_A \cap \tilde{C}_B}(x_s) = min\left(p_A(x_s), p_B(x_s)\right) e^{imin(u_A(x_s), u_B(x_s))}; i = \sqrt{-1}, \quad \forall x_s \in X.$$

- **Complex fuzzy union [185]**

Complex fuzzy union of two complex fuzzy sets $\tilde{C}_A$ and $\tilde{C}_B$ is as follows:

$$\tilde{C}_A \cup \tilde{C}_B = \left\{\left(x_s, \mu_{\tilde{C}_A \cup \tilde{C}_B}(x_s)\right)|x_s \in X\right\}$$

where,

$$\mu_{\tilde{C}_A \cup \tilde{C}_B}(x_s) = max(p_A(x_s), p_B(x_s))e^{imax(u_A(x_s), u_B(x_s))}; i = \sqrt{-1}, \quad \forall x_s \in X.$$

- **Complex fuzzy complement [185]**

Complement of a complex fuzzy set $\tilde{C}_A$ can be derived as follows:

$$\tilde{C}_A^c = \{(x, \mu_{\tilde{C}_A^c}(x))|x \in X\}$$

where,

$$\mu_{\tilde{C}_A^c}(x) = (1 - p_A(x))e^{i(2\pi - u_A(x))}; i = \sqrt{-1}, \quad \forall x \in X.$$

(ii) **Distance measure between complex fuzzy sets [9, 185].**

Distance between two complex fuzzy sets $\tilde{C}_A$ and $\tilde{C}_B$ is denoted by $d(\tilde{C}_A, \tilde{C}_B)$, which satisfies the following properties:
(i) $d(\tilde{C}_A, \tilde{C}_B) \in [0, 1]$;
(ii) $d(\tilde{C}_A, \tilde{C}_B) = 0$, if $\tilde{C}_A = \tilde{C}_B$;
(iii) $d(\tilde{C}_A, \tilde{C}_B) = \tilde{d}(\tilde{C}_B, \tilde{C}_A)$;
(iv) If, $\tilde{C}_A \subseteq \tilde{C}_B \subseteq \tilde{C}_D$ then, $d(\tilde{C}_A, \tilde{C}_D) \geq d(\tilde{C}_A, \tilde{C}_B)$ and $d(\tilde{C}_A, \tilde{C}_D) \geq d(\tilde{C}_B, \tilde{C}_D)$,

where, $\tilde{C}_D = \{(x_s, p_D(x_s)e^{iu_D(x_s)})|\forall x_s \in X\}$ is an another complex fuzzy set over X. Euclidean distance between two complex fuzzy sets $\tilde{C}_A$ and $\tilde{C}_B$ is defined as follows:

$$\tilde{d}_E(\tilde{C}_A, \tilde{C}_B) = \sqrt{\frac{1}{2m}\left[\sum_{s=1}^{m}\left\{(p_A(x_s) - p_B(x_s))^2 + \frac{1}{4\pi^2}(u_A(x_s) - u_B(x_s))^2\right\}\right]}$$

(iii) **Mathematical representation of a complex fuzzy soft set (CFSS) [158].**

A complex fuzzy soft set over X is defined as an order pair $(\tilde{\tilde{F}}, E)$, where, $\tilde{\tilde{F}}$ is a mapping defined as, $\tilde{\tilde{F}} : E \to \tilde{\rho}(X)$; E is the set of parameters which are in complex fuzzy sense and $\tilde{\rho}(X)$ is the set of all complex fuzzy subsets of the set X.

If, $X = \{x_1, x_2, .., x_m\}$ be the initial universe and $E = \{e_1, e_2, .., e_n\}$ be the set corresponding complex fuzzy parameters then, a complex fuzzy soft set $(\tilde{\tilde{F}}, E)$ over X is defined as follows:

$$\begin{aligned}(\tilde{\tilde{F}}, E) &= \{(e_1, \tilde{\tilde{F}}(e_1)), (e_2, \tilde{\tilde{F}}(e_2)), .., (e_n, \tilde{\tilde{F}}(e_n))\}\\ &= \{(e_1, ((x_1, p_{11}e^{iu_{11}}), (x_2, p_{21}e^{iu_{21}}), .., (x_m, p_{m1}e^{iu_{m1}}))), (e_2, ((x_1, p_{12}e^{iu_{12}}),\\ &(x_2, p_{22}e^{iu_{22}}), .., (x_m, p_{m2}e^{iu_{m2}}))), .., (e_n, ((x_1, p_{1n}e^{iu_{1n}}),\\ &(x_2, p_{2n}e^{iu_{2n}}), .., (x_m, p_{mn}e^{iu_{mn}})))\}\end{aligned}$$

where, $p_{sj} \in [0, 1]$ is the amplitude part and $u_{sj} \in [0, 2\pi]$ is the phase part of the evaluation of an alternative x_s; $s = 1, 2, .., m$ over a parameter e_j; $j = 1, 2, .., n$.

(iv) **Basic set theoretic operations of complex fuzzy soft sets [144].**

Consider two complex fuzzy soft sets over X as follows:

$$(\tilde{\tilde{F}}^1, E) = \{(e_j, \tilde{\tilde{F}}^1(e_j))\forall e_j \in E\} = \{(e_j, (x_s, p^1_{sj}e^{iu^1_{sj}}))|\forall e_j \in E, x_s \in X\}$$
$$(\tilde{\tilde{F}}^2, E) = \{(e_j, \tilde{\tilde{F}}^2(e_j))\forall e_j \in E\} = \{(e_j, (x_s, p^2_{sj}e^{iu^2_{sj}}))|\forall e_j \in E, x_s \in X\}$$

- **Complex fuzzy soft union [144]**

Complex fuzzy soft union of $(\tilde{\tilde{F}}^1, E)$ and $(\tilde{\tilde{F}}^2, E)$ is denoted by,
$(\tilde{\tilde{F}}^1, E) \cup (\tilde{\tilde{F}}^2, E) = (\tilde{\tilde{H}}, E)$ and is defined as follows:

$$(\tilde{\tilde{H}}, E) = \{(e_j, \tilde{\tilde{H}}(e_j))\forall e_j \in E\} = \{(e_j, (x_s, max(p^1_{sj}, p^2_{sj})e^{imax(u^1_{sj}, u^2_{sj})}))|\forall e_j \in E, x_s \in X\}$$

- **Complex fuzzy soft intersection [144]**

Complex fuzzy soft intersection of $(\tilde{\tilde{F}}^1, E)$ and $(\tilde{\tilde{F}}^2, E)$ is denoted by,
$(\tilde{\tilde{F}}^1, E) \cap (\tilde{\tilde{F}}^2, E) = (\tilde{\tilde{H}}, E)$ and is defined as follows:

$$(\tilde{\tilde{H}}, E) = \{(e_j, \tilde{\tilde{H}}(e_j))\forall e_j \in E\} = \{(e_j, (x_s, min(p^1_{sj}, p^2_{sj})e^{imin(u^1_{sj}, u^2_{sj})}))|\forall e_j \in E, x_s \in X\}$$

- **Complex fuzzy soft subset and complex fuzzy soft equal [144]**

The complex fuzzy soft set $(\tilde{\tilde{F}}^1, E)$ is said to be complex fuzzy soft subset of $(\tilde{\tilde{F}}^2, E)$

$$if,\ \forall e_j \in E\ and\ \forall x_s \in X, p^1_{sj} \leq p^2_{sj}\ and\ u^1_{sj} \leq u^2_{sj}$$

If, $\forall e_j \in E$ and $x_s \in X$, $p^1_{sj} = p^2_{sj}$ and $u^1_{sj} = u^2_{sj}$ then, $(\tilde{\tilde{F}}^1, E)$ is said to be the complex fuzzy soft equal to $(\tilde{\tilde{F}}^2, E)$.

- **Absolute complex fuzzy soft set and null complex fuzzy soft set [144]**

A complex fuzzy soft set $(\tilde{\tilde{F}}, E)$ over X is said to be an absolute complex fuzzy soft set, if $\forall e_j \in E$ and $x_s \in X, p_{sj} = 1$ and $u_{sj} = 2\pi$. Mathematically, it is denoted by, $(\tilde{\tilde{F}}, E)_{\bar{1}}$.
Similarly, a complex fuzzy soft set $(\tilde{\tilde{F}}, E)$ is called a null complex fuzzy soft set over X, if $\forall e_j \in E$ and $x_s \in X, p_{sj} = 0$ and $u_{sj} = 0\pi$. Mathematically, it is denoted by, $(\tilde{\tilde{F}}, E)_{\bar{0}}$.

7.3 Similarity measure approach to complex fuzzy soft sets

Similarity measure of two objects determines the degree of closeness or degree of sameness between them. In many different fields like, pattern recognition, cluster analysis, decision-making, etc. it plays an effective role. Now, we have introduced a new similarity measure approach, named as ratio similarity measure, for complex fuzzy soft sets. Firstly, we have introduced this new approach for complex fuzzy sets (as given in Subsection 7.2.1) and then, we have extended it to complex fuzzy soft sets (as given in Subsection 7.2.2).

7.3.1 Ratio similarity measure to complex fuzzy sets

Axiomatic definition of similarity measure of complex fuzzy sets.

Definition 7.1. Let, $X = \{x_1, x_2, .., x_m\}$ and C_A, C_B be two complex fuzzy sets with respect to X where, $\tilde{C}_A = \left\{(x_s, p_A(x_s)e^{iu_A(x_s)})|s = 1, 2, .., m\right\}$ with each $p_A(x_s) \in [0, 1]$ and each $u_A(x_s) \in [0, 2\pi]$ and $\tilde{C}_B = \left\{(x_s, p_B(x_s)e^{iu_B(x_s)})|s = 1, 2, .., m\right\}$ with each $p_B(x_s) \in [0, 1]$ and each $u_B(x_s) \in [0, 2\pi]$.

Now consider a function $\tilde{S}$ as, $\tilde{S} : \wp(X) \times \wp(X) \to [0, 1]$, where, $\wp(X)$ is the set of all

complex fuzzy sets of the set X. Then, $\tilde{S}$ is said to be a similarity measure for complex fuzzy sets if, $\tilde{S}$ satisfies the following properties: $\forall \tilde{C}_A, \tilde{C}_B, \tilde{C}_D \in \wp(X)$,
(i) $\tilde{S}(\tilde{C}_A, \tilde{C}_B) \in [0,1]$;
(ii) $\tilde{S}(\tilde{C}_A, \tilde{C}_B) = 1$ if and only if $\tilde{C}_A = \tilde{C}_B$;
(iii) $\tilde{S}(\tilde{C}_A, \tilde{C}_B) = \tilde{S}(\tilde{C}_B, \tilde{C}_A)$;
(iv) If, $\tilde{C}_A \subseteq \tilde{C}_B \subseteq \tilde{C}_D$, then, $\tilde{S}(\tilde{C}_A, \tilde{C}_B) \geq \tilde{S}(\tilde{C}_A, \tilde{C}_D)$ and $\tilde{S}(\tilde{C}_B, \tilde{C}_D) \geq \tilde{S}(\tilde{C}_A, \tilde{C}_D)$, where, $\tilde{C}_D = \{(x_s, p_D(x_s)e^{iu_D(x_s)}) | s = 1, 2, .., m\}$ is the another complex fuzzy set over X.

Ratio similarity measure of complex fuzzy sets.

Since, in every complex fuzzy evaluation there exist two decision information where, one is expressed through *amplitude part* and another one is expressed through *phase part* therefore, to measure the similarity degree of two complex fuzzy sets, we have measured their similarity for amplitude part and phase part individually and then added them for deriving the total similarity. In the following, mathematically it has been illustrated.

Definition 7.2. The *ratio similarity measure* between two complex fuzzy sets $\tilde{C}_A$ and $\tilde{C}_B$ is denoted by, $\tilde{S}_R(\tilde{C}_A, \tilde{C}_B)$ and is defined by the following equation,

$$\tilde{S}_R(\tilde{C}_A, \tilde{C}_B) = \frac{\sum_{s=1}^{m} min\big(p_A(x_s), p_B(x_s)\big)}{\sum_{s=1}^{m} \big(p_A(x_s) + p_B(x_s)\big)} + \frac{\sum_{s=1}^{m} min\big(u_A(x_s), u_B(x_s)\big)}{\sum_{s=1}^{m} \big(u_A(x_s) + u_B(x_s)\big)} \tag{7.1}$$

Example 7.1. Let, $X = \{x_1, x_2, x_3\}$ be a universal set. Now, assume two complex fuzzy sets over X as, $\tilde{C}_A = \{(x_1, 1e^{i2\pi}), (x_2, 1e^{i\pi}), (x_3, 0.9e^{i2\pi})\}$;
$\tilde{C}_B = \{(x_1, .9e^{i3\pi/2}), (x_2, 1e^{i2\pi/3}), (x_3, 0.8e^{i2\pi})\}$.
Then, the ratio similarity between them is as follows:
$\tilde{S}_R(\tilde{C}_A, \tilde{C}_B) = \frac{0.9+1+0.8}{1.9+2+1.7} + \frac{\frac{3\pi}{2}+\frac{2\pi}{3}+2\pi}{\frac{7\pi}{2}+\frac{5\pi}{3}+4\pi} = \frac{2.7}{5.6} + \frac{25}{55} = 0.93.$

Theorem 7.1. Ratio similarity measure of two complex fuzzy sets (Equation 7.1) satisfies all the properties of Definition 7.1.

Proof. (i) Since, each $p_A(x_s),\ p_B(x_s) \in [0,1]$ and each $u_A(x_s), u_B(x_s) \in [0, 2\pi]$, then, $min(p_A(x_s), p_B(x_s)) \geq 0$ and $min(u_A(x_s), u_B(x_s)) \geq 0$.
Moreover, $min(p_A(x_s), p_B(x_s)) \leq \frac{1}{2}(p_A(x_s) + p_B(x_s))$ and $min(u_A(x_s), u_B(x_s)) \leq \frac{1}{2}(u_A(x_s) + u_B(x_s))$.

Then, $\frac{\sum_{s=1}^{m} min(p_A(x_s),p_B(x_s))}{\sum_{s=1}^{m} (p_A(x_s)+p_B(x_s))} \leq \frac{1}{2}$ and $\frac{\sum_{s=1}^{m} min(u_A(x_s),u_B(x_s))}{\sum_{s=1}^{m} (u_A(x_s)+u_B(x_s))} \leq \frac{1}{2}$.

Hence, from Equation 7.1 it is obvious that, $\tilde{S}_R(\tilde{C}_A, \tilde{C}_B) \in [0,1]$.

(ii) $\tilde{C}_A = \tilde{C}_B \Leftrightarrow p_A(x_s) = p_B(x_s)$ and $u_A(x_s) = u_B(x_s)$.
Then from Equation 7.1 we get, $\tilde{S}_R(\tilde{C}_A, \tilde{C}_B) = 1$.

(iii) $\tilde{S}_R(\tilde{C}_A, \tilde{C}_B) = \frac{\sum_{s=1}^{m} min\left(p_A(x_s), p_B(x_s)\right)}{\sum_{s=1}^{m}\left(p_A(x_s)+p_B(x_s)\right)} + \frac{\sum_{s=1}^{m} min\left(u_A(x_s), u_B(x_s)\right)}{\sum_{s=1}^{m}\left(u_A(x_s)+u_B(x_s)\right)} =$
$\frac{\sum_{s=1}^{m} min\left(p_B(x_s), p_A(x_s)\right)}{\sum_{s=1}^{m}\left(p_B(x_s)+p_A(x_s)\right)} + \frac{\sum_{s=1}^{m} min\left(u_B(x_s), u_A(x_s)\right)}{\sum_{s=1}^{m}\left(u_B(x_s)+u_A(x_s)\right)} = \tilde{S}_R(\tilde{C}_B, \tilde{C}_A).$

(iv) $\tilde{C}_A \subseteq \tilde{C}_B \subseteq \tilde{C}_D \Leftrightarrow p_A(x_s) \leq p_B(x_s) \leq p_D(x_s); u_A(x_s) \leq u_B(x_s) \leq u_D(x_s).$
Then, $\sum_{s=1}^{m} min(p_A(x_s), p_B(x_s)) = \sum_{s=1}^{m} min(p_A(x_s), p_D(x_s));$
$\sum_{s=1}^{m} min(u_A(x_s), u_B(x_s)) = \sum_{s=1}^{m} min(u_A(x_s), u_D(x_s))$ and
$\sum_{s=1}^{m}(p_A(x_s) + p_B(x_s)) \leq \sum_{s=1}^{m}(p_A(x_s) + p_D(x_s));$
$\sum_{s=1}^{m}(u_A(x_s) + u_B(x_s)) \leq \sum_{s=1}^{m}(u_A(x_s) + u_D(x_s)).$
These results imply that, $\frac{\sum_{s=1}^{m} min(p_A(x_s), p_B(x_s))}{\sum_{s=1}^{m}(p_A(x_s)+p_B(x_s))} \geq \frac{\sum_{s=1}^{m} min(p_A(x_s), p_D(x_s))}{\sum_{s=1}^{m}(p_A(x_s)+p_D(x_s))}$ and
$\frac{\sum_{s=1}^{m} min(u_A(x_s), u_B(x_s))}{\sum_{s=1}^{m}(u_A(x_s)+u_B(x_s))} \geq \frac{\sum_{s=1}^{m} min(u_A(x_s), u_D(x_s))}{\sum_{s=1}^{m}(u_A(x_s)+u_D(x_s))}.$
Then, from Equation 7.1 it is obtained that,
$\tilde{S}_R(\tilde{C}_A, \tilde{C}_B) = \frac{\sum_{s=1}^{m} min\left(p_A(x_s), p_B(x_s)\right)}{\sum_{s=1}^{m}\left(p_A(x_s)+p_B(x_s)\right)} + \frac{\sum_{s=1}^{m} min\left(u_A(x_s), u_B(x_s)\right)}{\sum_{s=1}^{m}\left(u_A(x_s)+u_B(x_s)\right)} \geq$
$\frac{\sum_{s=1}^{m} min\left(p_A(x_s), p_D(x_s)\right)}{\sum_{s=1}^{m}\left(p_A(x_s)+p_D(x_s)\right)} + \frac{\sum_{s=1}^{m} min\left(u_A(x_s), u_D(x_s)\right)}{\sum_{s=1}^{m}\left(u_A(x_s)+u_D(x_s)\right)} = \tilde{S}_R(\tilde{C}_A, \tilde{C}_D).$
In the similar way, it can also be proved that, $\tilde{S}_R(\tilde{C}_B, \tilde{C}_D) \geq \tilde{S}_R(\tilde{C}_A, \tilde{C}_D).$

7.3.2 Ratio similarity measure to complex fuzzy soft sets

Axiomatic definition of similarity measure of complex fuzzy soft sets.

Definition 7.3. Let, $X = \{x_1, x_2, .., x_m\}$ and $E = \{e_1, e_2, .., e_n\}$. Now, consider a mapping $\hat{S} : \wp_{CFSS}(X) \times \wp_{CFSS}(X) \to [0, 1]$ where, $\wp_{CFSS}(X)$ is the set of all complex fuzzy soft sets over X. Then, the mapping $\hat{S}$ is said to be a similarity measure for complex fuzzy soft sets if and only if it follows the following conditions: $\forall(\tilde{\tilde{F}}, E), (\tilde{\tilde{G}}, E), (\tilde{\tilde{H}}, E) \in \wp_{CFSS}(X)$,

(S1) $\hat{S}((\tilde{\tilde{F}}, E), (\tilde{\tilde{G}}, E)) \in [0, 1]$;

(S2) $\hat{S}((\tilde{\tilde{F}}, E), (\tilde{\tilde{G}}, E)) = \hat{S}((\tilde{\tilde{G}}, E), (\tilde{\tilde{F}}, E))$;

(S3) $\hat{S}((\tilde{\tilde{F}}, E), (\tilde{\tilde{G}}, E)) = 1 \Leftrightarrow (\tilde{\tilde{F}}, E) = (\tilde{\tilde{G}}, E)$;

(S4) If, $(\tilde{\tilde{F}}, E) \subseteq (\tilde{\tilde{G}}, E) \subseteq (\tilde{\tilde{H}}, E)$, then, $\hat{S}((\tilde{\tilde{F}}, E), (\tilde{\tilde{G}}, E)) \geq \hat{S}((\tilde{\tilde{F}}, E), (\tilde{\tilde{H}}, E))$ and $\hat{S}((\tilde{\tilde{G}}, E), (\tilde{\tilde{H}}, E)) \geq \hat{S}((\tilde{\tilde{F}}, E), (\tilde{\tilde{H}}, E))$.

Ratio similarity measure of complex fuzzy soft sets.

Definition 7.4. Consider two complex fuzzy soft sets over X as follows:

$$\begin{aligned}(\tilde{\tilde{F}}, E) \quad &= \{(e_1, ((x_1, p_{11}^F e^{iu_{11}^F}), (x_2, p_{21}^F e^{iu_{21}^F}), .., (x_m, p_{m1}^F e^{iu_{m1}^F}))),\\ &(e_2, ((x_1, p_{12}^F e^{iu_{12}^F}), (x_2, p_{22}^F e^{iu_{22}^F}), .., (x_m, p_{m2}^F e^{iu_{m2}^F}))), ..,\\ &(e_n, ((x_1, p_{1n}^F e^{iu_{1n}^F}), (x_2, p_{2n}^F e^{iu_{2n}^F}), .., (x_m, p_{mn}^F e^{iu_{mn}^F})))\}\end{aligned}$$

$$\begin{aligned}(\tilde{\tilde{G}}, E) \quad &= \{(e_1, ((x_1, p_{11}^G e^{iu_{11}^G}), (x_2, p_{21}^G e^{iu_{21}^G}), .., (x_m, p_{m1}^G e^{iu_{m1}^G}))),\\ &(e_2, ((x_1, p_{12}^G e^{iu_{12}^G}), (x_2, p_{22}^G e^{iu_{22}^G}), .., (x_m, p_{m2}^G e^{iu_{m2}^G}))), ..,\\ &(e_n, ((x_1, p_{1n}^G e^{iu_{1n}^G}), (x_2, p_{2n}^G e^{iu_{2n}^G}), .., (x_m, p_{mn}^G e^{iu_{mn}^G})))\}\end{aligned}$$

where, $p_{sj}^F \in [0, 1]$ is the amplitude part and $u_{sj}^F \in [0, 2\pi]$ is the phase part of the evaluation of an alternative x_s over the parameter e_j with respect to the complex fuzzy soft set $(\tilde{\tilde{F}}, E)$ and similarly, $p_{sj}^G \in [0, 1]$ is the amplitude part and $u_{sj}^G \in [0, 2\pi]$ is the phase part of the evaluation of an alternative x_s over the parameter e_j with respect to the complex fuzzy soft set $(\tilde{\tilde{G}}, E)$.

Then, the ratio similarity of $(\tilde{\tilde{F}}, E)$ and $(\tilde{\tilde{G}}, E)$ is denoted by, $\hat{S}_R((\tilde{\tilde{F}}, E), (\tilde{\tilde{G}}, E))$ and is defined by the following equation,

$$\hat{S}_R((\tilde{\tilde{F}}, E), (\tilde{\tilde{G}}, E)) = \frac{\sum_{j=1}^{n} w_j \hat{S}_R(\tilde{\tilde{F}}(e_j), \tilde{\tilde{G}}(e_j))}{\sum_{j=1}^{n} w_j} \tag{7.2}$$

where,

$$\hat{S}_R(\tilde{\tilde{F}}(e_j), \tilde{\tilde{G}}(e_j)) = \frac{\sum\limits_{s=1}^{m} min\big(p_{sj}^F, p_{sj}^G\big)}{\sum\limits_{s=1}^{m} \big(p_{sj}^F + p_{sj}^G\big)} + \frac{\sum\limits_{s=1}^{m} min\big(u_{sj}^F, u_{sj}^G\big)}{\sum\limits_{s=1}^{m} \big(u_{sj}^F + u_{sj}^G\big)}$$

and $\{w_1, w_2, .., w_n\}$ are the weights of the parameters with each $w_j \in [0, 1]$.
If, $\sum_{j=1}^{n} w_j = 1$, then the above equation takes the form as follows:

$$\hat{S}_R((\tilde{\tilde{F}}, E), (\tilde{\tilde{G}}, E)) = \sum_{j=1}^{n} w_j \hat{S}_R(\tilde{\tilde{F}}(e_j), \tilde{\tilde{G}}(e_j)) \tag{7.3}$$

Table 7.1: CFSS $(\tilde{\tilde{F}}, E)$ (Example 7.2)

	e_1	e_2	e_3
x_1	$0.9e^{i3\pi/2}$	$0.1e^{i\pi}$	$0.5e^{i\pi/2}$
x_2	$1e^{i\pi}$	$0.6e^{i3\pi/2}$	$0.1e^{i\pi/2}$
x_3	$0.6e^{i\pi/2}$	$0.4e^{i3\pi/2}$	$0.3e^{i\pi}$

Table 7.2: CFSS $(\tilde{\tilde{G}}, E)$ (Example 7.2)

	e_1	e_2	e_3
x_1	$0.8e^{i2\pi}$	$0.5e^{i\pi/2}$	$0.6e^{i3\pi/2}$
x_2	$0.2e^{i\pi/2}$	$0.6e^{i\pi}$	$0.2e^{i\pi/2}$
x_3	$0.6e^{i\pi}$	$0.6e^{i3\pi/2}$	$0.5e^{i\pi}$

Example 7.2. Now, consider three elements as the universal set, $X = \{x_1, x_2, x_3\}$. $E = \{e_1, e_2, e_3\}$ be the set of three corresponding parameters which are in complex fuzzy sense. Let, $(\tilde{\tilde{F}}, E)$ and $(\tilde{\tilde{G}}, E)$ be two complex fuzzy soft sets as follows:
$(\tilde{\tilde{F}}, E) = \{\big(e_1, \big((x_1, 0.9e^{i3\pi/2}), (x_2, 1e^{i\pi}), (x_3, 0.6e^{i\pi/2})\big)\big),$
$\big(e_2, \big((x_1, 0.1e^{i\pi}), (x_2, 0.6e^{i3\pi/2}), (x_3, 0.4e^{i3\pi/2})\big)\big),$
$\big(e_3, \big((x_1, 0.5e^{i\pi/2}), (x_2, 0.1e^{i\pi/2}), (x_3, 0.3e^{i\pi})\big)\big)\};$
$(\tilde{\tilde{G}}, E) = \{\big(e_1, \big((x_1, 0.8e^{i2\pi}), (x_2, 0.2e^{i\pi/2}), (x_3, 0.6e^{i\pi})\big)\big),$
$\big(e_2, \big((x_1, 0.5e^{i\pi/2}), (x_2, 0.6e^{i\pi}), (x_3, 0.6e^{i3\pi/2})\big)\big),$
$\big(e_3, \big((x_1, 0.6e^{i3\pi/2}), (x_2, 0.2e^{i\pi/2}), (x_3, 0.5e^{i\pi})\big)\big)\}.$
Tabular forms of $(\tilde{\tilde{F}}, E)$ and $(\tilde{\tilde{G}}, E)$ have been given in Table 7.1 and Table 7.2.
Now, assume that, $w_1 = 0.8$; $w_2 = 0.9$; $w_3 = 0.7$.
Then,

$$\begin{aligned}
\hat{S}_R(\tilde{\tilde{F}}(e_1), \tilde{\tilde{G}}(e_1)) \quad &= \left\{\frac{(0.8+0.2+0.6)}{(1.7+1.2+1.2)} + \frac{(\frac{3\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2})}{(\frac{7\pi}{2}+\frac{3\pi}{2}+\frac{3\pi}{2})}\right\} \\
&= \frac{1.6}{4.1} + \frac{5}{13} \approx 0.77
\end{aligned}$$

$$\begin{aligned}
\hat{S}_R(\tilde{\tilde{F}}(e_2), \tilde{\tilde{G}}(e_2)) \quad &= \left\{\frac{(0.1+0.6+0.4)}{(0.6+1.2+1)} + \frac{(\frac{\pi}{2}+\pi+\frac{3\pi}{2})}{(\frac{3\pi}{2}+\frac{5\pi}{2}+\frac{3\pi}{)})}\right\} \\
&= \frac{1.1}{2.8} + \frac{3}{7} \approx 0.82
\end{aligned}$$

$$\begin{aligned}
\hat{S}_R(\tilde{\tilde{F}}(e_3), \tilde{\tilde{G}}(e_3)) \quad &= \left\{\frac{(0.5+0.1+0.3)}{(1.1+0.3+0.8)} + \frac{(\frac{\pi}{2}+\frac{\pi}{2}+\pi)}{(2\pi+\pi+2\pi)}\right\} \\
&= \frac{0.9}{2.2} + \frac{2}{5} \approx 0.81.
\end{aligned}$$

Therefore, the ratio similarity between between $(\tilde{\tilde{F}}, E)$ and $(\tilde{\tilde{G}}, E)$ is as follows: $\hat{S}_R((\tilde{\tilde{F}}, E), (\tilde{\tilde{G}}, E)) = \frac{0.8\times0.77+0.9\times0.82+0.7\times0.81}{0.8+0.9+0.7} \approx 0.81.$

Theorem 7.2. Ratio similarity of two complex fuzzy soft sets satisfies all the properties of Definition 7.3.

Proof. Proof is same as the proof of Theorem 7.1.

7.4 Aggregation operators on complex fuzzy soft sets

Generally, by using aggregation operation, multiple number of items can be integrated into a single resultant item as a representative of all the items. Recently, in solving decision-making involving multiple experts, some new proposals of aggregation operations have been developed under different uncertain environments such as, fuzzy soft set [142], T-spherical fuzzy soft set [69], intuitionsitic fuzzy soft set [66], neutrosophic soft set [80], etc. But, there exist no work on aggregation operation of complex fuzzy soft sets. Therefore in this section, we have proposed the aggregation operation on complex fuzzy soft sets.

Axiomatic definition of aggregation operation of complex fuzzy soft sets.

Definition 7.5. Consider an universal set as, $X = \{x_1, x_2, .., x_m\}$ and n corresponding parameters as, $E = \{e_1, e_2, .., e_n\}$ which are in complex fuzzy sense. Now, let, $\wp_{CFSS}(X)$ be the set of all complex fuzzy soft sets over the universal set X. Then, a mapping $\tilde{\tilde{A}}$, defined as,
$\tilde{\tilde{A}} : \underbrace{\wp_{CFSS}(X) \times \wp_{CFSS}(X) \times .. \times \wp_{CFSS}(X)} \to \wp_{CFSS}(X)$, is called a complex fuzzy soft aggregation operator if, this function satisfies the following axiomatic properties.
Let, $(\tilde{\tilde{F}}^1, E), (\tilde{\tilde{F}}^2, E), .., (\tilde{\tilde{F}}^k, E)$ be a set of k complex fuzzy soft sets over $\wp_{CFSS}(X)$. Then,

(A1) $\tilde{\tilde{A}}\left((\tilde{\tilde{F}}, E)_{\bar{1}}, (\tilde{\tilde{F}}, E)_{\bar{1}}, .., (\tilde{\tilde{F}}, E)_{\bar{1}}\right) = (\tilde{\tilde{F}}, E)_{\bar{1}};$

where, $(\tilde{\tilde{F}}, E)_{\bar{1}}$ is the absolute complex fuzzy soft set on $\wp_{CFSS}(X)$.

(A2) $\tilde{\tilde{A}}\left((f_{\tilde{C}}, E)_{\bar{0}}, (\tilde{\tilde{F}}, E)_{\bar{0}}, .., (\tilde{\tilde{F}}, E)_{\bar{0}}\right) = (\tilde{\tilde{F}}, E)_{\bar{0}};$

where, $(\tilde{\tilde{F}}, E)_{\bar{0}}$ is the null complex fuzzy soft set over $\wp_{CFSS}(X)$.

(A3) If, $\forall l = 1, 2, .., k$, $(\tilde{\tilde{F}}^l, E) \leq (\tilde{\tilde{G}}^l, E)$, then,
$\tilde{\tilde{A}}\left((\tilde{\tilde{F}}^1, E), (\tilde{\tilde{F}}^2, E), .., (\tilde{\tilde{F}}^k, E)\right) \leq \tilde{\tilde{A}}\left((\tilde{\tilde{G}}^1, E), (\tilde{\tilde{G}}^2, E), .., (\tilde{\tilde{G}}^k, E)\right),$
where, $(\tilde{\tilde{G}}^1, E), (\tilde{\tilde{G}}^2, E), .., (\tilde{\tilde{G}}^k, E)$ be the another k complex fuzzy soft sets over $\wp_{CFSS}(X)$.

(A4) If, $(\tilde{\tilde{F}}^+, E)$ and $(\tilde{\tilde{F}}^-, E)$ be the best approx (max-valued) complex fuzzy soft set and worst approx (min-valued) complex fuzzy soft set among k complex fuzzy soft sets

then,
$(\tilde{\tilde{F}}^{-}, E) \leq \tilde{\tilde{A}}\left((\tilde{\tilde{F}}_1, E), (\tilde{\tilde{F}}_2, E), .., (\tilde{\tilde{F}}_k, E)\right) \leq (\tilde{\tilde{F}}^{+}, E)$.

In Figure 7.1, graphical representation of aggregation of k complex fuzzy soft sets has been given.

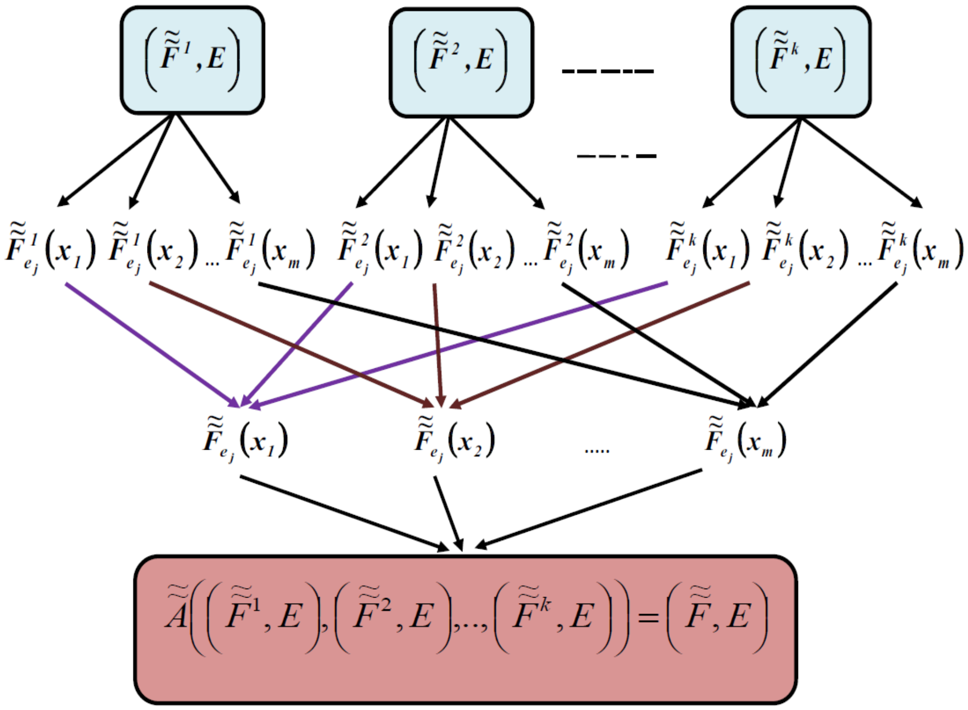

Figure 7.1: Aggregation of k complex fuzzy soft sets

Best approx (max-valued) and worst approx (min-valued) complex fuzzy soft sets.
Now, consider k complex fuzzy soft sets as, $(\tilde{\tilde{F}}^1, E), (\tilde{\tilde{F}}^2, E), .., (\tilde{\tilde{F}}^k, E)$ where,

$$(\tilde{\tilde{F}}^l, E) = \left\{\left(e_j, (x_s, p^l_{sj} e^{iu^l_{sj}})]\right) |\forall e_j \in E, x_s \in X\right\}, \; l = 1, 2, .., k.$$

$p^l_{sj} e^{iu^l_{sj}}$ indicates the complex fuzzy rating of an alternative x_s over a parameter e_j corresponding to the complex fuzzy soft set $(\tilde{\tilde{F}}^l, E)$.

Definition 7.6. The best approx (max-valued) complex fuzzy soft set over the k complex fuzzy soft sets $(\tilde{\tilde{F}}^1, E), (\tilde{\tilde{F}}^2, E), .., (\tilde{\tilde{F}}^k, E)$ is denoted by, $(\tilde{\tilde{F}}^{+}, E)$ and is defined as follows:
$(\tilde{\tilde{F}}^{+}, E) = \left\{\left(e_j, \left(x_s, \left(max(p^1_{sj}, p^2_{sj}, .., p^k_{sj}) e^{imax\left(u^1_{sj}, u^2_{sj}, .., u^k_{sj}\right)}\right)\right)\right) |\forall e_j \in E, x_s \in X\right\}.$

Tabular form of the best approx complex fuzzy soft set has been given in Table 7.3.

Table 7.3: Best approx complex fuzzy soft set $(\tilde{\tilde{F}}^{+}, E)$

	e_1	e_2	$\dots$	e_n
x_1	$max(p^1_{11}, p^2_{11}, .., p^k_{11})e^{imax(u^1_{11}, u^2_{11}, .., u^k_{11})}$	$max(p^1_{12}, p^2_{12}, .., p^k_{12})e^{imax(u^1_{12}, p^2_{12}, .., p^k_{12})}$	$\dots$	$max(p^1_{1n}, p^2_{1n}, .., p^k_{1n})e^{imax(u^1_{1n}, u^2_{1n}, .., u^k_{1n})}$
x_2	$max(p^1_{21}, p^2_{21}, .., p^k_{21})e^{imax(u^1_{21}, u^2_{21}, .., u^k_{21})}$	$max(p^1_{22}, p^2_{22}, .., p^k_{22})e^{imax(u^1_{22}, u^2_{22}, .., u^k_{22})}$	$\dots$	$max(p^1_{2n}, p^2_{2n}, .., p^k_{2n})e^{imax(u^1_{2n}, u^2_{2n}, .., u^k_{2n})}$
			$\dots$	
x_m	$max(p^1_{m1}, p^2_{m1}, .., p^k_{m1})e^{imax(u^1_{m1}, u^2_{m1}, .., u^k_{m1})}$	$max(p^1_{m2}, p^2_{m2}, .., p^k_{m2})e^{imax(u^1_{m2}, u^2_{m2}, .., u^k_{m2})}$	$\dots$	$max(p^1_{mn}, p^2_{mn}, .., p^k_{mn})e^{imax(u^1_{mn}, u^2_{mn}, .., u^k_{mn})}$

Definition 7.7. The worst approx (min-valued) complex fuzzy soft set over the k complex fuzzy soft sets $(\tilde{\tilde{F}}^1, E), (\tilde{\tilde{F}}^2, E), .., (\tilde{\tilde{F}}^k, E)$ is denoted by, $(\tilde{\tilde{F}}^{-}, E)$ and is defined as follows:

$$(\tilde{\tilde{F}}^{-}, E) = \left\{ \left(e_j, \left(x_s, \left(min(p^1_{sj}, p^2_{sj}, .., p^k_{sj})e^{imin\left(u^1_{sj}, u^2_{sj}, .., u^k_{sj}\right)}\right)\right)\right) |\forall e_j \in E, x_s \in X \right\}.$$

Tabular form of the worst approx complex fuzzy soft set has been given in Table 7.4.

Table 7.4: Worst approx complex fuzzy soft set $(\tilde{\tilde{F}}^{-}, E)$

	e_1	e_2	$\dots$	e_n
x_1	$min(p^1_{11}, p^2_{11}, .., p^k_{11})e^{imin(u^1_{11}, u^2_{11}, .., u^k_{11})}$	$min(p^1_{12}, p^2_{12}, .., p^k_{12})e^{imin(u^1_{12}, p^2_{12}, .., p^k_{12})}$	$\dots$	$min(p^1_{1n}, p^2_{1n}, .., p^k_{1n})e^{imin(u^1_{1n}, u^2_{1n}, .., u^k_{1n})}$
x_2	$min(p^1_{21}, p^2_{21}, .., p^k_{21})e^{imin(u^1_{21}, u^2_{21}, .., u^k_{21})}$	$min(p^1_{22}, p^2_{22}, .., p^k_{22})e^{imin(u^1_{22}, u^2_{22}, .., u^k_{22})}$	$\dots$	$min(p^1_{2n}, p^2_{2n}, .., p^k_{2n})e^{imin(u^1_{2n}, u^2_{2n}, .., u^k_{2n})}$
			$\dots$	
x_m	$min(p^1_{m1}, p^2_{m1}, .., p^k_{m1})e^{imin(u^1_{m1}, u^2_{m1}, .., u^k_{m1})}$	$min(p^1_{m2}, p^2_{m2}, .., p^k_{m2})e^{imin(u^1_{m2}, u^2_{m2}, .., u^k_{m2})}$	$\dots$	$min(p^1_{mn}, p^2_{mn}, .., p^k_{mn})e^{imin(u^1_{mn}, u^2_{mn}, .., u^k_{mn})}$

Complex fuzzy soft weighted geometric mean aggregation operator.

In 2013, Yager and Abbasov [176] introduced the aggregation operation for k pythagorean fuzzy numbers as follows:

Definition 7.8 [176]. Let, $A_1, A_2, .., A_k$ be a set of k criteria where, each of which has pythagorean fuzzy membership degree as, $A_l(x) = r_l(x)e^{i\theta_l(x)}$, $l = 1, 2, .., k$. It is also assumed that, $\{w_1, w_2, .., w_k\}$ are the associated weights of the criteria such that, each $w_l \in [0, 1]$ and $\sum\limits_{l=1}^{k} w_l = 1$. Then, the geometric mean aggregation of these k pythagorean fuzzy numbers is as follows:

$$A(x) = \prod_{l=1}^{k}(A_l(x))^{w_l} = \prod_{l=1}^{k}(r_l(x)e^{i\theta_l(x)})^{w_l} = \prod_{l=1}^{k}(r_l(x))^{w_l} e^{i\sum\limits_{l=1}^{k}\theta_l(x)w_l}$$

Now, since, each of the evaluations in a complex fuzzy soft set is in terms of complex fuzzy membership i.e., in the form as, $p^l_{sj}e^{iu^l_{sj}}$ where, p^l_{sj} is fuzzy-valued and u^l_{sj} is real-valued, therefore, we can follow this same manoeuvre to introduce geometric mean aggregation operator for complex fuzzy soft sets.

Definition 7.9. Let, $(\tilde{\tilde{F}}^1, E), (\tilde{\tilde{F}}^2, E), .., (\tilde{\tilde{F}}^k, E)$ be a set of k complex fuzzy soft sets over X and E is the parameter set. The l^{th} complex fuzzy soft set has been given below:

$$\begin{aligned}(\tilde{\tilde{F}}^l, E) &= \left\{\left(e_j, \tilde{\tilde{F}}^l(e_j)\right) |\forall e_j \in E\right\} = \left\{\left(e_j, \left(x_s, \tilde{\tilde{F}}^l_{e_j}(x_s)\right)\right) |\forall e_j \in E, x_s \in X\right\} \\ &= \left\{\left(e_j, \left(x_s, p^l_{sj}e^{iu^l_{sj}}\right)\right) |\forall e_j \in E, x_s \in X\right\}.\end{aligned}$$

Consider that, $W = \{W_l, W_2, .., W_k\}$ be the associated weights of the k complex fuzzy soft sets where, $W_l \in [0, 1], \forall j$.
Then, the complex fuzzy soft weighted geometric mean aggregation of k complex fuzzy soft sets $(\tilde{\tilde{F}}^1, E), (\tilde{\tilde{F}}^2, E), .., (\tilde{\tilde{F}}^k, E)$ is denoted by,
$\tilde{\tilde{A}}^{\{W_1,W_2,..,W_k\}}_{WGM}\left((\tilde{\tilde{F}}^1, E), (\tilde{\tilde{F}}^2, E), .., (\tilde{\tilde{F}}^k, E)\right)$ and is defined as follows:

$$\tilde{\tilde{A}}^{\{W_1,W_2,..,W_k\}}_{WGM}\left((\tilde{\tilde{F}}^1, E), (\tilde{\tilde{F}}^2, E), .., (\tilde{\tilde{F}}^k, E)\right) = (\tilde{\tilde{F}}, E) =$$

$$\left\{\left(e_j, \left(x_s, \left(\prod_{l=1}^{k}(p^l_{sj}e^{iu^l_{sj}})^{W_l}\right)^{1/\sum\limits_{l=1}^{k}W_l}\right)\right) |\forall e_j \in E, x_s \in X\right\} \tag{7.4}$$

where,

$$\begin{aligned}\left(\prod_{l=1}^{k}(p^l_{sj}e^{iu^l_{sj}})^{W_l}\right)^{1/\sum\limits_{l=1}^{k}W_l} &= \left((p^1_{sj}e^{iu^1_{sj}})^{W_1} \times (p^2_{sj}e^{iu^2_{sj}})^{W_2} \times ..(p^k_{sj}e^{iu^k_{sj}})^{W_k}\right)^{1/\sum\limits_{l=1}^{k}W_l} \\ &= \left(\left(\prod_{l=1}^{k}(p^l_{sj})^{W_l}\right)^{1/\sum\limits_{l=1}^{k}W_l}\right) e^{i\left(\frac{\sum\limits_{l=1}^{k}W_l u^l_{sj}}{\sum\limits_{l=1}^{k}W_l}\right)}\end{aligned}$$

Since, each $p^l_{sj} \in [0,1]$ and each $W_l \in [0,1]$ then, it is obvious that,
$\left(\prod_{l=1}^{k}(p^l_{sj})^{W_l}\right)^{1/\sum\limits_{l=1}^{k} W_l} \in [0,1]$. Moreover, since, each $u^l_{sj} \in [0,2\pi]$ then, clearly it is obtained that, $\frac{\sum\limits_{l=1}^{k} W_l u^l_{sj}}{\sum\limits_{l=1}^{k} W_l} \in [0,2\pi]$.

- If, $\sum\limits_{l=}^{k} w_l = 1$ then, Equation 7.4 takes the form as follows:

$$\tilde{\tilde{A}}_{GM}^{\{w_1,w_2,..,w_k\}}\left((\tilde{\tilde{F}}^1,E),(\tilde{\tilde{F}}^2,E),..,(\tilde{\tilde{F}}^k,E)\right) = (\tilde{\tilde{F}},E)$$
$$= \left\{\left(e_j, \left(x_s, \prod_{l=1}^{k}(p^l_{sj}e^{iu^l_{sj}})^{w_l}\right)\right)|\forall e_j \in E, x_s \in X\right\}$$

Theorem 7.3. Complex fuzzy soft weighted geometric mean aggregation operator satisfies all the properties of Definition 7.5.

Proof. **(A1)** We have, $(\tilde{\tilde{F}},E)_{\tilde{1}} = \left\{(e_j,(x_s,1e^{i2\pi}))|\forall e_j \in E, x_s \in X\right\}$.
Then, from Equation 7.4,
$\tilde{\tilde{A}}_{WGM}^{\{W_1,W_2,..,W_k\}}\left((\tilde{\tilde{F}},E)_1,(\tilde{\tilde{F}},E)_1,..,(\tilde{\tilde{F}},E)_1\right) =$
$$\left\{\left(e_j, \left(x_s, \left(\prod_{l=1}^{k}(1e^{i2\pi})^{W_l}\right)^{1/\sum\limits_{l=1}^{k} W_l}\right)\right)|\forall e_j \in E, x_s \in X\right\}$$
which implies that,
$$\left(\prod_{l=1}^{k}(1e^{i2\pi})^{W_l}\right)^{1/\sum\limits_{l=1}^{k} W_l} = \left(\left(\prod_{l=1}^{k}(1)^{W_l}\right)^{1/\sum\limits_{l=1}^{k} W_l}\right) e^{i\left(\frac{\sum\limits_{l=1}^{k} W_l 2\pi}{\sum\limits_{l=1}^{k} W_l}\right)} = 1e^{i2\pi}.$$
So, it is obtained that, $\tilde{\tilde{A}}_{WGM}^{\{W_1,W_2,..,W_k\}}\left((\tilde{\tilde{F}},E)_1,(\tilde{\tilde{F}},E)_1,..,(\tilde{\tilde{F}},E)_1\right) = (\tilde{\tilde{F}},E)_1$.

(A2) It is straightforward.

(A3) Now consider that, $(\tilde{\tilde{G}}^l,E) = \{(e_j,(x_s,P^l_{sj}e^{iU^l_{sj}}))|\forall e_j \in E, x_s \in X\}$.
Then, from the definition of complex fuzzy soft subset we have,
$(\tilde{\tilde{F}}^l,E) \leq (\tilde{\tilde{G}}^l,E) \Rightarrow p^l_{sj} \leq P^l_{sj}$ and $u^l_{sj} \leq U^l_{sj}$, for each, $s = 1,2,..,m$, $j = 1,2,..,n$ and $l = 1,2,..,k$.
Then, we get the result that,
$\tilde{\tilde{A}}_{WGM}^{\{W_1,W_2,..,W_k\}}((\tilde{\tilde{F}}^1,E),(\tilde{\tilde{F}}^2,E),..,(\tilde{\tilde{F}}^k,E)) \leq \tilde{\tilde{A}}_{WGM}^{\{W_1,W_2,..,W_k\}}((\tilde{\tilde{G}}^1,E),(\tilde{\tilde{F}}^2,E),..,(\tilde{\tilde{G}}^k,E))$.

(A4) Based on the properties of weighted geometric mean aggregation operator, it is straightforward.

7.5 Complex fuzzy soft multi-expert decision-making

Recently, decision-making by using soft set theory has become a common propensity to the researchers because of its flexible parameter selection process to define an alternative. In such type of decision-making problems, the main goal is to sort out the best decision alternative from a bunch of alternatives over some parameters accepted by a single expert or multiple experts. Consequently, many algorithms [22, 50] have been constructed from different backgrounds to assess these decision-making under different uncertain environments. Now, we have solved a complex fuzzy soft set based decision-making involving multiple experts.

7.5.1 Problem description

Let us consider m alternatives as, $X = \{x_1, x_2, .., x_m\}$ and n corresponding parameters as, $E = \{e_1, e_2, .., e_n\}$. Now assume that, k experts ($D = \{d_1, d_2, .., d_k\}$) have been employed to define these m alternatives over the n considered complex fuzzy parameters. Then, the provided opinions of the k experts about the alternatives over n parameters have been given in k complex fuzzy soft sets $(\tilde{\tilde{F}}^1, E), (\tilde{\tilde{F}}^2, E), .., (\tilde{\tilde{F}}^k, E)$ as follows:

$$
\begin{aligned}
\forall l = 1, 2, .., k,\ (\tilde{\tilde{F}}^l, E) \quad &= \left\{ \left(e_j, \tilde{\tilde{F}}^l(e_j) \right) \right\} \\
&= \left\{ \left(e_j, \left(x_s, \tilde{\tilde{F}}^l_{e_j}(x_s) \right) \right) | \forall e_j \in E, x_s \in X \right\} \\
&= \left\{ \left(e_j, \left(x_s / p^l_{sj} e^{iu^l_{sj}} \right) \right) | \forall e_j \in E, x_s \in X \right\}.
\end{aligned}
$$

Tabular form of these k complex fuzzy soft sets is given in 7.5.

Table 7.5: Tabular form of k complex fuzzy soft sets (in general case)

	$d_1,$	$(\tilde{\tilde{F}}^1, E)$			$d_2,$	$(\tilde{\tilde{F}}^2, E)$		
	e_1	e_2	$\dots$	e_n	e_1	e_2	$\dots$	e_n
x_1	$p^1_{11} e^{iu^1_{11}}$	$p^1_{12} e^{iu^1_{12}}$	$\dots$	$p^1_{1n} e^{iu^1_{1n}}$	$p^2_{11} e^{iu^2_{11}}$	$p^2_{12} e^{iu^2_{12}}$	$\dots$	$p^2_{1n} e^{iu^2_{1n}}$
x_2	$p^1_{21} e^{iu^1_{21}}$	$p^1_{22} e^{iu^1_{22}}$	$\dots$	$p^1_{2n} e^{iu^1_{2n}}$	$p^2_{21} e^{iu^2_{21}}$	$p^2_{22} e^{iu^2_{22}}$	$\dots$	$p^2_{2n} e^{iu^2_{2n}}$
			$\dots$				$\dots$	
x_m	$p^1_{m1} e^{iu^1_{m1}}$	$p^1_{m2} e^{iu^1_{m2}}$	$\dots$	$p^1_{mn} e^{iu^1_{mn}}$	$p^2_{m1} e^{iu^2_{m1}}$	$p^1_{m2} e^{iu^2_{m2}}$	$\dots$	$p^2_{mn} e^{iu^2_{mn}}$

	d_k,	$(\tilde{\tilde{F}}^k, E)$		
	e_1	e_2	$\ldots$	e_n
..	$p^k_{11}e^{iu^k_{11}}$	$p^k_{12}e^{iu^k_{12}}$	$\ldots$	$p^k_{1n}e^{iu^k_{1n}}$
..	$p^k_{21}e^{iu^k_{21}}$	$p^k_{22}e^{iu^k_{22}}$	$\ldots$	$p^k_{2n}e^{iu^k_{2n}}$
..			$\ldots$	
..	$p^k_{m1}e^{iu^k_{m1}}$	$p^k_{m2}e^{iu^k_{m2}}$	$\ldots$	$p^k_{mn}e^{iu^k_{mn}}$

From this mathematical illustration, our main objective is to select the best decision alternative based on all the associated k complex fuzzy soft sets $(\tilde{\tilde{F}}^1, E)$, $(\tilde{\tilde{F}}^2, E)$,..,$(\tilde{\tilde{F}}^k, E)$.

7.5.2 Optimality criteria

The considered problem of this chapter is a multi-expert decision-making based on complex fuzzy soft sets. Now, the optimality criteria of this decision-making have been given point wise in the following:

(a) The optimal alternative will have maximum degree of satisfaction with respect to most of the associated parameters for every corresponding complex fuzzy soft set.

(b) Optimal alternative should have highest degree of distance from the anti-ideal alternative and smallest degree of distance from the ideal alternative.

7.5.3 Solution framework of our considered multi-expert decision-making

Now, we have proposed a stepwise algorithmic approach to solve our considered complex fuzzy soft set based multi-expert decision-making. Our solution methodology is divided into four phases as given in Figure 7.2. In Phase I, we have equalized all the decision evaluations to make them comparable to each other. In Phase II, we have derived the weight of an expert. Then, in Phase III, a collective complex fuzzy soft set has been constructed as a resultant representative of all the k complex fuzzy soft sets by using complex fuzzy soft weighted geometric mean aggregation operator. Finally in Phase IV, the best alternative has been selected from the collective complex fuzzy soft set.

Algorithm:

Phase I: ***Equalization of all the decision evaluations***
In a real-life based decision-making, not every considered parameter carries the same nature. Negative character may be subsisted in some considered parameters. Then, to overcome this unequalness behaviour of the parameters, we have taken complex fuzzy complement (given

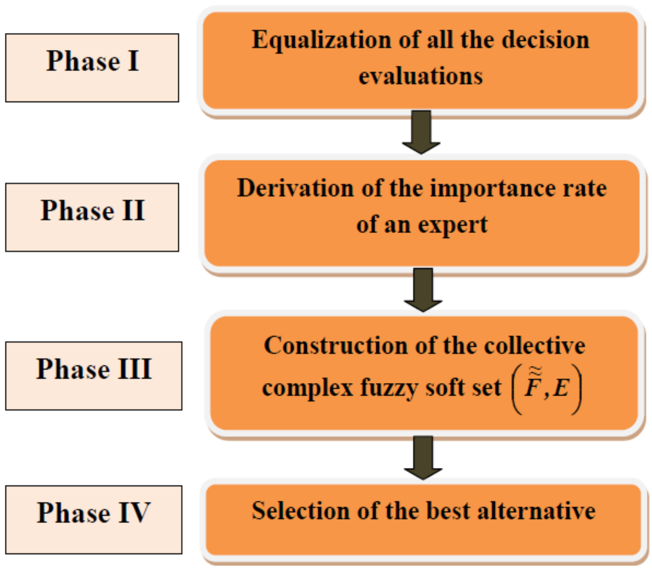

Figure 7.2: Schematic framework of our proposed approach

in Preliminary Section) of all evaluations with respect to every negative parameter so that the evaluation of every alternative can accomplish the same feature (positive nature).

Step 1. Input alternative set, $X = \{x_1, x_2, .., x_m\}$ and corresponding parameter set, $E = \{e_1, e_2, .., e_n\}$. Input k complex fuzzy soft sets $(\tilde{\tilde{F}}^1, E)$, $(\tilde{\tilde{F}}^2, E)$,..,$(\tilde{\tilde{F}}^k, E)$ provided by the k experts, $D = \{d_1, d_2, .., d_k\}$ as given in Table 7.5. Input the primary given weights of the associated parameters as, $W = \{w_1, w_2, .., w_n\}$ where, each $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

Step 2. Classify all the parameters according to their positive nature and negative nature. Suppose A and B are the set of positive parameters and set of negative parameters where, $A \cup B = E$ and $A \cap B = \phi$.

Step 3. Now, take complex fuzzy complement of all the evaluations over every negative parameter as follows:
$\forall e_j \in A$, there will be no change in decision evaluation $p^l_{sj} e^{iu^l_{sj}}$ of an alternative x_s.
$\forall e_j \in B$, take the complex fuzzy complement of the evaluation $p^l_{sj} e^{iu^l_{sj}}$ as, $(p^l_{sj})^c$ where, c is the complex fuzzy complement operator; $\forall s = 1, 2, .., m$, $j = 1, 2, .., n$.

Phase II: ***Determination of the weight ($\varpi(d_l)$) of an expert d_l***
In reality, in multi-expert decision-making, all the experts do not provide similar opinion about the alternatives because of their having different knowledge levels, different choice

strategies, and different satisfaction levels, which reflects the disparity between their importance in making a decision about the best alternative. Therefore, it is required to derive the weight of the experts individually in a multi-expert decision-making problem to obtain a better decision result. Now, we have derived the weight of an expert d_l with respect to his/her provided complex fuzzy soft set $(\tilde{\tilde{F}}^l, E)$.

Step 1. Derivation of the best-approx complex fuzzy soft set

Since, according to the optimality criteria of our considered decision-making problem, final decision solution will be taken based on the maximum satisfaction of the alternatives with respect to the parameters over all the k complex fuzzy soft sets therefore, we have constructed the best-approx complex fuzzy soft set $(\tilde{\tilde{F}}^+, E)$ by taking union among all the k complex fuzzy soft sets since, union of complex fuzzy soft sets takes maximum evaluation as given in Definition 7.6.

Step 2. Evaluation of the proximity index of an expert d_l

The proximity index of an expert d_l with respect to his/her associated complex fuzzy soft set $(\tilde{\tilde{F}}^l, E)$ is denoted by, $P_I(d_l)$ and is evaluated as follows:

$$P_I(d_l) = \hat{S}((\tilde{\tilde{F}}^+, E)(\tilde{\tilde{F}}^l, E)) \tag{7.5}$$

where, $\hat{S}$ indicates the similarity measure of two complex fuzzy soft sets.

Step 3. Measuring the closeness index of an expert d_l

The closeness index of an expert d_l with respect to his/her corresponding complex fuzzy soft set $(\tilde{\tilde{F}}^l, E)$ is denoted by, $C_I(d_l)$ and is defined by measuring his/her total similarity degree with the other experts. Mathematically, it is defined as follows:

$$C_I(d_l) = \frac{\sum\limits_{l \neq l'; l'=1}^{k} \hat{S}\left((\tilde{\tilde{F}}^l, E), (\tilde{\tilde{F}}^{l'}, E)\right)}{k-1} \tag{7.6}$$

Step 4. Determination of the weight of an expert d_l

Now, the weight of an expert d_l with respect to his/her provided complex fuzzy soft set $(\tilde{\tilde{F}}^l, E)$ is denoted by, $\varpi_I(d_l)$ and can be derived from the following equation:

$$\varpi(d_l) = \frac{(P_I(d_l) \oplus C_I(d_l))}{\sum\limits_{l=1}^{k} (P_I(d_l) \oplus C_I(d_l))} \tag{7.7}$$

Phase III: ***Construction of the collective complex fuzzy soft set*** $(\tilde{\tilde{F}}, E)$

Now, we have derived a collective complex fuzzy soft set $(\tilde{\tilde{F}}, E)$ as a representation of all the k complex fuzzy soft sets with the help of complex fuzzy soft weighted geometric mean aggregation operator (given in Definition 7.9) as follows:

$$\tilde{\tilde{A}}_{WGM}^{\{\varpi(d_1),\varpi(d_2),..,\varpi(d_k)\}}\left((\tilde{\tilde{F}}^1, E), (\tilde{\tilde{F}}^2, E), .., (\tilde{\tilde{F}}^k, E)\right) = (\tilde{\tilde{F}}, E)$$

where, $\varpi(d_l)$ is the weight of an expert d_l with respect to his/her provided complex fuzzy soft set $(\tilde{\tilde{F}}^l, E)$. Tabular form of the collective complex fuzzy soft set $(\tilde{\tilde{F}}, E)$ has been provided in Table 7.6.

Table 7.6: Collective complex fuzzy soft set $(\tilde{\tilde{F}}, E)$ (in general case)

	e_1	e_2	$\dots$	e_n
x_1	$P_{11}e^{iU_{11}}$	$P_{12}e^{iU_{12}}$	$\dots$	$P_{1n}e^{iU_{1n}}$
x_2	$P_{21}e^{iU_{21}}$	$P_{22}e^{iU_{22}}$	$\dots$	$P_{1n}e^{iU_{1n}}$
			$\dots$	
x_m	$P_{m1}e^{iU_{m1}}$	$P_{m2}e^{iU_{m2}}$	$\dots$	$P_{mn}e^{iU_{mn}}$

Phase IV: ***Selection of the best alternative***
This is the last phase of our proposed approach. Here, we have selected the best alternative based on the collective complex fuzzy soft set $(\tilde{\tilde{F}}, E)$ (given in Table 7.6). This phase contains the following steps.

Step 1. Determination of combined weight of a parameter e_j
Weight of an associated parameter in a decision-making has a significant disposition in practice. Usually, weight of a parameter is of two types:

- *Subjective weight:* Subjective weight of a parameter is basically the given weight of a parameter in a decision-making problem.
  $W = \{w_1, w_2, .., w_n\}$ are the subjective weights (given weights) of the parameters where, each $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.
- *Objective weight*: Objective weight of a parameter can be the derived from the corresponding decision evaluations.

Each of these two weights has an individual impact on the final decision of a decision-making problem. Therefore, now we have evaluated a combined weighted value of an associated parameter by using both the subjective weight and objective weight together.

Objective weight of a parameter e_j:
Objective weight of a parameter e_j is denoted by, w_j^{Ob} and is derived by using maximizing deviation method [159] where, the key idea is, the parameter for which the total deviation in the evaluations of the alternatives is high, will have highest objective importance or objective weight for this decision-making. Its mathematical formulation is as follows:

$$w_j^{Ob} = \frac{1}{m(m-1)} \times \frac{\sum\limits_{s'=1, s \neq s'}^{m} \sqrt{\frac{1}{2}\left((P_{sj} - P_{s'j})^2 + \frac{1}{4\pi^2}(U_{sj} - P_{s'j})^2\right)}}{\sum\limits_{s=1}^{m} \sum\limits_{s'=1, s \neq s'}^{m} \sqrt{\frac{1}{2}\left((P_{sj} - P_{s'j})^2 + \frac{1}{4\pi^2}(U_{sj} - U_{s'j})^2\right)}} \tag{7.8}$$

Combined weight of a parameter e_j:
Now, the combined weight of a parameter e_j is denoted by, w_j^{Cb} and is defined as follows:

$$w_j^{Cb} = \frac{\alpha(w_j) \oplus (1-\alpha)(w_j^{Ob})}{\sum_{j=1}^{n} \left(\alpha(w_j) \oplus (1-\alpha)(w_j^{Ob})\right)} \tag{7.9}$$

where, $\alpha \in [0,1]$ is the influence factor.

Step 2. Obtain the ideal alternative and anti-ideal alternative
The ideal alternative is denoted is $\bar{x}$ and is derived by taking complex fuzzy union (given in the Preliminary Section) over all the alternatives for every parameter corresponding to the collective complex fuzzy soft set $(\tilde{\tilde{F}}, E)$ as follows:

$$\begin{aligned} \bar{x} \quad &= \left\{\left(e_1, (P_{11}e^{iU_{11}} \cup P_{21}e^{iU_{21}} \cup .. \cup P_{m1}e^{iU_{m1}})\right),\right. \\ &\left(e_2, (P_{12}e^{iU_{12}} \cup P_{22}e^{iU_{22}} \cup .. \cup P_{m2}e^{iU_{m2}})\right), .., \\ &\left.\left(e_n, (P_{1n}e^{iU_{1n}} \cup P_{2n}e^{iU_{2n}} \cup .. \cup P_{mn}e^{iU_{mn}})\right)\right\} \\ &= \left\{\left(e_1, \bar{P}_1 e^{i\bar{U}_1}\right), \left(e_2, \bar{P}_2 e^{i\bar{U}_2}\right), .., \left(e_n, \bar{P}_n e^{i\bar{U}_n}\right)\right\} \end{aligned} \tag{7.10}$$

In the similarly way, the anti-ideal alternative is denoted is $\underline{x}$ and is defined by taking complex fuzzy intersection (given in the Preliminary Section) over all the alternatives for every parameter corresponding to the collective complex fuzzy soft set $(\tilde{\tilde{F}}, E)$ as follows:

$$\begin{aligned} \underline{x} \quad &= \left\{\left(e_1, (P_{11}e^{iU_{11}} \cap P_{21}e^{iU_{21}} \cap .. \cap P_{m1}e^{iU_{m1}})\right),\right. \\ &\left(e_2, (P_{12}e^{iU_{12}} \cap P_{22}e^{iU_{22}} \cap .. \cap P_{m2}e^{iU_{m2}})\right), .., \\ &\left.\left(e_n, (P_{1n}e^{iU_{1n}} \cap P_{2n}e^{iU_{2n}} \cap .. \cap P_{mn}e^{iU_{mn}})\right)\right\} \\ &= \left\{\left(e_1, \underline{P}_1 e^{i\underline{U}_1}\right), \left(e_2, \underline{P}_2 e^{i\underline{U}_2}\right), .., \left(e_n, \underline{P}_n e^{i\underline{U}_n}\right)\right\} \end{aligned} \tag{7.11}$$

Step 3. Derivation of the separation levels of an alternative x_s from $\bar{x}$ and $\underline{x}$
The separation levels of an alternative x_s from the corresponding ideal alternative ($\bar{x}$) and the

anti-ideal alternative ($\underline{x}$) are denoted by, $D(x_s/\bar{x})$ and $D(x_s/\underline{x})$ and are defined as follows: $\forall s = 1, 2, .., m$

$$D(x_s/\bar{x}) = \Big(\frac{1}{2}\{\big(w_1^{Cb} \times (P_{s1} - \bar{P}_1)^2 + w_2^{Cb} \times (P_{s2} - \bar{P}_2)^2 + .. + w_n^{Cb} \times (P_{sn} - \bar{P}_n)^2\big) + \frac{1}{4\pi^2}\big(w_1^{Cb} \times (U_{s1} - \bar{U}_1)^2 + w_2^{Cb} \times (U_{s2} - \bar{U}_2)^2 + .. + w_n^{Cb} \times (U_{sn} - \bar{U}_n)^2\big)\}\Big)^{1/2}$$

$$D(x_s/\underline{x}) = \Big(\frac{1}{2}\{\big(w_1^{Cb} \times (P_{s1} - \underline{P}_1)^2 + w_2^{Cb} \times (P_{s2} - \underline{P}_2)^2 + .. + w_n^{Cb} \times (P_{sn} - \underline{P}_n)^2\big) + \frac{1}{4\pi^2}\big(w_1^{Cb} \times (U_{s1} - \underline{U}_1)^2 + w_2^{Cb} \times (U_{s2} - \underline{U}_2)^2 + .. + w_n^{Cb} \times (U_{sn} - \underline{U}_n)^2\big)\}\Big)^{1/2}$$

Step 4. Derive the ranking index of an alternative x_s

Finally, the ranking index of an alternative x_s is denoted by, $\tilde{R}(x_s)$ and is derived by the following equation:

$$\tilde{R}(x_s) = \frac{D(x_s/\underline{x})}{D(x_s/\underline{x}) + D(x_s/\bar{x})};\ s = 1, 2, .., m \tag{7.12}$$

Step 5. Determine the best alternative based on the ranking indices

The alternative with maximum ranking index ($\tilde{R}$) will be selected as the optimal alternative or best alternative for this multi-expert decision-making.

If, this is not unique then, we have to select any one of them as an optimal solution.

Example 7.3. Let the universal set as, $X = \{x_1, x_2, x_3, x_4\}$ and the parameter set as, $E = \{e_1, e_2, e_3\}$ which are in complex fuzzy sense where, e_1 and e_2 are the positive parameters and e_3 is the negative parameter. Now, consider three complex fuzzy soft sets $(\tilde{\tilde{F}}^1, E)$, $(\tilde{\tilde{F}}^2, E)$ and $(\tilde{\tilde{F}}^3, E)$ over X provided by three experts, $D = \{d_1, d_2, d_3\}$ as given in Tables 7.7, 7.8 and 7.9.

Assume that, the given weights or subjective weights of the parameters for this multi-expert decision-making are, $W = \{w_1 = 0.40, w_2 = 0.37, w_3 = 0.23\}$. Now, we have solved this complex fuzzy soft multi-expert decision-making by applying our approach.

Table 7.7: CFSS $(\tilde{\tilde{F}}^1, E)$ (Example 7.3)

	e_1	e_2	e_3
x_1	$0.8e^{i\pi/2}$	$0.3e^{i\pi/2}$	$0.5e^{i\pi}$
x_2	$0.5e^{i7\pi/4}$	$0.6e^{i3\pi/2}$	$0.4e^{i0}$
x_3	$1e^{i4\pi/3}$	$0.7e^{i3\pi/2}$	$0.6e^{i\pi}$
x_4	$0.7e^{i3\pi/4}$	$0.8e^{i2\pi/3}$	$0.6e^{i7\pi/4}$

Table 7.8: CFSS $(\tilde{\tilde{F}}^2, E)$ (Example 7.3)

	e_1	e_2	e_3
x_1	$1e^{i\pi/4}$	$0.5e^{i2\pi/3}$	$0.4e^{i3\pi/2}$
x_2	$0.4e^{i3\pi/2}$	$0.6e^{i4\pi/3}$	$0.8e^{i\pi}$
x_3	$0.7e^{i2\pi}$	$0.1e^{i\pi/2}$	$0.2e^{i2\pi/3}$
x_4	$1e^{i\pi}$	$0.6e^{i\pi/2}$	$0.5e^{i5\pi/3}$

Table 7.9: CFSS $(\tilde{\tilde{F}}^3, E)$ (Example 7.3)

	e_1	e_2	e_3
x_1	$1e^{i\pi/4}$	$0.8e^{i\pi/5}$	$0.3e^{i3\pi/2}$
x_2	$0.6e^{i\pi}$	$0.6e^{i2\pi/3}$	$0.3e^{i\pi/2}$
x_3	$0.4e^{i\pi}$	$0.7e^{i2\pi/3}$	$0.7e^{i\pi}$
x_4	$0.7e^{i\pi/6}$	$0.8e^{i\pi/2}$	$0.7e^{i7\pi/4}$

Solution:

Phase I:

Step 1. Three complex fuzzy soft sets have been given in Tables 7.7, 7.8 and 7.9.

Step 2. Since, e_3 is a negative parameter therefore, before solving this problem, we have equalized all the evaluations by taking the complex fuzzy complement of every evaluation of every alternative over e_3 parameter of each of the complex fuzzy soft sets as given in Tables 7.10, 7.11 and 7.12.

Table 7.10: CFSS $(\tilde{\tilde{F}}^1, E)$ after equalization (Example 7.3)

	e_1	e_2	e_3
x_1	$0.8e^{i\pi/2}$	$0.3e^{i\pi/2}$	$0.5e^{i\pi}$
x_2	$0.5e^{i7\pi/4}$	$0.6e^{i3\pi/2}$	$0.6e^{i2\pi}$
x_3	$1e^{i4\pi/3}$	$0.7e^{i3\pi/2}$	$0.4e^{i\pi}$
x_4	$0.7e^{i3\pi/4}$	$0.8e^{i2\pi/3}$	$0.4e^{i\pi/4}$

Table 7.11: CFSS $(\tilde{\tilde{F}}^2, E)$ after equalization (Example 7.3)

	e_1	e_2	e_3
x_1	$1e^{i\pi/4}$	$0.5e^{i2\pi/3}$	$0.6e^{i\pi/2}$
x_2	$0.4e^{i3\pi/2}$	$0.6e^{i4\pi/3}$	$0.2e^{i\pi}$
x_3	$0.7e^{i2\pi}$	$0.1e^{i\pi/2}$	$0.8e^{i4\pi/3}$
x_4	$1e^{i\pi}$	$0.6e^{i\pi/2}$	$0.5e^{i\pi/3}$

Phase II:

Step 1. By using Definition 7.6, best-approx complex fuzzy soft set $(\tilde{\tilde{F}}^+, E)$ is given in Table 7.13.

Step 2. Now, by using Equation 7.5, proximity indices of the experts are,
$P_I(d_1) = 0.91$; $P_I(d_2) = 0.86$; $P_I(d_3) = 0.79$.

Step 3. By applying Equation 7.6, closeness indices of the experts are,

Table 7.12: CFSS $(\tilde{\tilde{F}}^3, E)$ after equalization (Example 7.3)

	e_1	e_2	e_3
x_1	$1e^{i\pi/4}$	$0.8e^{i\pi/5}$	$0.7e^{i\pi/2}$
x_2	$0.6e^{i\pi}$	$0.6e^{i2\pi/3}$	$0.7e^{i3\pi/2}$
x_3	$0.4e^{i\pi}$	$0.7e^{i2\pi/3}$	$0.3e^{i\pi}$
x_4	$0.7e^{i\pi/6}$	$0.8e^{i\pi/2}$	$0.3e^{i\pi/4}$

Table 7.13: Best-approx complex fuzzy soft set $(\tilde{\tilde{F}}^{+}, E)$ (Example 7.3)

	e_1	e_2	e_3
x_1	$1e^{i\pi/2}$	$0.8e^{i2\pi/3}$	$0.7e^{i\pi}$
x_2	$0.6e^{i7\pi/4}$	$0.6e^{i3\pi/2}$	$0.7e^{i2\pi}$
x_3	$1e^{i2\pi}$	$0.7e^{i3\pi/2}$	$0.8e^{i4\pi/3}$
x_4	$1e^{i\pi}$	$0.8e^{i2\pi/3}$	$0.6e^{i\pi/3}$

$C_I(d_1) = 0.82$; $C_I(d_2) = 0.80$; $C_I(d_3) = 0.78$.
Step 4. Then, with the help of Equation 7.7, weights of the experts are as follows:
$\varpi(d_1) = 0.35$; $\varpi(d_2) = 0.33$; $\varpi(d_3) = 0.32$.

Phase III:

Now, by using complex fuzzy soft weighted geometric mean aggregation operator, the collective complex fuzzy soft set $(\tilde{\tilde{F}}, E)$ from the three complex fuzzy soft sets $(\tilde{\tilde{F}}^1, E)$, $(\tilde{\tilde{F}}^2, E)$, $(\tilde{\tilde{F}}^3, E)$ (Table 7.10, Table 7.11, 7.12) has been provided in Table 7.14.

Table 7.14: Collective complex fuzzy soft set $(\tilde{\tilde{F}}, E)$ (Example 7.3)

	e_1	e_2	e_3
x_1	$0.92e^{i0.34\pi}$	$0.49e^{i0.46\pi}$	$0.59e^{i0.68\pi}$
x_2	$0.49e^{i0.43\pi}$	$0.60e^{i1.18\pi}$	$0.44e^{i1.51\pi}$
x_3	$0.66e^{i1.45\pi}$	$0.37e^{i0.90\pi}$	$0.46e^{i1.11\pi}$
x_4	$0.79e^{i0.64\pi}$	$0.73e^{i0.56\pi}$	$0.39e^{i0.28\pi}$

Phase IV:

Step 1. By using Equation 7.8, objective weights of the parameters are,
$w_1^{Ob} = 0.36$, $w_2^{Ob} = 0.29$, $w_3^{Ob} = 0.35$.
Then, from Equation 7.9, combined weights of the parameters ar as follows,
$w_1^{Cb} = 0.38$; $w_2^{Cb} = 0.33$; $w_3^{Cb} = 0.29$ ($\alpha = 0.5$ has been considered here).
Step 2. Then, base on Table 7.14, ideal alternative and anti-ideal alternative for this multi-expert decision-making are, $\bar{x} = \{0.92e^{i1.45\pi}, 0.73e^{i1.18\pi}, 0.59e^{i1.51\pi}\}$;
$\underline{x} = \{0.49e^{i0.34\pi}, 0.37e^{i0.46\pi}, 0.39e^{i0.28\pi}\}$.
Step 3. After that, separation levels of an alternative from $\bar{x}$ and $\underline{x}$ with respect to the collective complex fuzzy soft set $(\tilde{\tilde{F}}, E)$ are given below:
$D(x_1/\bar{x}) = 0.3382$; $D(x_2/\bar{x}) = 0.2030$; $D(x_3/\bar{x}) = 0.2138$; $D(x_4/\bar{x}) = 0.3330$.
$D(x_1/\underline{x}) = 0.2216$; $D(x_2/\underline{x}) = 0.3765$; $D(x_3/\underline{x}) = 0.3125$; $D(x_4/\underline{x}) = 0.2078$.
Step 4. Then, the final ranking index of each of the alternatives is,
$\tilde{R}(x_1) = 0.3959$; $\tilde{R}(x_2) = 0.6497$; $\tilde{R}(x_3) = 0.5938$; $\tilde{R}(x_4) = 0.3842$.
Step 5. So, the ranking order of the associated alternatives is, $x_2 > x_3 > x_1 > x_4$.
Hence, it can be concluded that, x_2 is the best decision alternative for this multi-expert decision-making.

7.6 Experimental analysis

A case Study on medical science.
Usually, one symptom may be the cause of various diseases. For instance, the symptom fever is a common symptom of several different diseases such as, acute viral hepatitis, influenza, peptic ulcer, food poisoning, etc. Moreover, for a particular disease, multiple symptoms may be responsible. Therefore, accurate diagnosis and treatment are very emergent in medical diagnosis system.

In reference [22], Basu et al. developed a balanced solution approach to detect exact affected disease of a patient by using fuzzy soft set theory. Moreover, Tang [155], Li et al. [99] and Wang et al. [161] proposed some disease diagnosis decision-making algorithms through fuzzy soft set theory. In these existing approaches, we have seen that, authors only focused on the belongingness degree of symptom to a patient. But, in a disease diagnosis system, how long a symptom is seen in the patient is also an inescapable information. Therefore, to find out the exact solution of a disease diagnosis problem, both of the information of a symptom such as, time duration of a symptom and belongingness degree are necessary.

Then in order to fulfill this research gap, we have used complex fuzzy soft set framework instead of fuzzy soft set framework so that, belongingness degree of a symptom and time duration of the symptom can be taken together via complex fuzzy membership in terms of its amplitude term and phase term. In the following, we have illustrated a disease diagnosis decision-making problem based on complex fuzzy soft set theory.

Example 7.4. Consider five diseases as, malaria, pneumonia, gastric ulcer, viral fever, typhoid. Now, some common related symptoms of these five diseases are, headache, stomach pain, cough, chest pain, temperature, weight loss. Consider that, a set of three decision makers or experts have been assigned to conduct this disease diagnosis decision-making. Now, to express the opinions of the three experts, we have used complex fuzzy soft set theory where, 'belongingness degree of a symptom' represents by amplitude part and 'time period of the symptom' represents by phase part.

Now, to express this problem mathematically, assume that, $X = \{malaria(x_1), pneumonia(x_2), gastric\ ulcer(x_3), viral\ fever(x_4), typhoid(x_5)\}$ be a universal set and $E = \{headache(e_1), stomach\ pain(e_2),\ cough(e_3),\ chest\ pain(e_4),$ $temperature(e_5), weight\ loss(e_6)\}$ be a parameter set each of which is in complex fuzzy sense. Let, $D = \{d_1, d_2, d_3\}$ be the three associated experts. Now, based the opinions of the three experts, the three complex fuzzy soft sets $(\tilde{\tilde{F}}^1, E)$, $(\tilde{\tilde{F}}^2, E)$ and $(\tilde{\tilde{F}}^3, E)$ have been given in Tables 7.15, 7.16 and 7.17 respectively.

All the above data have been considered based on 15 consecutive days. Then, to define every time duration data in the interval $[0, 2\pi]$, 2π has been assumed here instead of 15 days i.e., with respect to a parameter e_1 for the expert d_1, the evaluation $0.8e^{i\pi/4}$ of an alternative x_1 indicates that, belongingness degree of the symptom headache e_1 over the disease malaria

Table 7.15: CFSS $(\tilde{\tilde{F}}^1, E)$ given by d_1 (Example 7.4)

	e_1	e_2	e_3	e_4	e_5	e_6
x_1	$0.8e^{i\pi/4}$	$0.9e^{i\pi}$	$1e^{i\pi/2}$	$0.8e^{i\pi}$	$0.3e^{i\pi/6}$	$0.8e^{i2\pi/3}$
x_2	$0.4e^{i3\pi/2}$	$0.5e^{i\pi}$	$0.5e^{i4\pi/3}$	$0.4e^{i\pi/2}$	$0.6e^{i\pi}$	$0.6e^{i\pi}$
x_3	$0.5e^{i7\pi/4}$	$0.6e^{i2\pi}$	$0.7e^{i2\pi}$	$0.4e^{i2\pi}$	$0.5e^{i5\pi/3}$	$0.5e^{i2\pi}$
x_4	$0.6e^{i\pi/2}$	$0.8e^{i\pi/2}$	$0.8e^{i\pi}$	$0.8e^{i\pi/6}$	$0.5e^{i\pi}$	$0.7e^{i\pi/4}$
x_5	$0.1e^{i3\pi/2}$	$0.4e^{i2\pi}$	$0.3e^{i3\pi/2}$	$1e^{i4\pi/3}$	$0.7e^{i\pi}$	$0.3e^{i3\pi/2}$

Table 7.16: CFSS $(\tilde{\tilde{F}}^2, E)$ given by d_2 (Example 7.4)

	e_1	e_2	e_3	e_4	e_5	e_6
x_1	$0.7e^{i\pi/2}$	$0.8e^{i\pi/2}$	$0.6e^{i\pi}$	$0.7e^{i3\pi/2}$	$0.7e^{i\pi/4}$	$0.9e^{i\pi}$
x_2	$0.6e^{i\pi/2}$	$0.7e^{i2\pi/3}$	$0.7e^{i\pi}$	$0.6e^{i4\pi/3}$	$0.6e^{i\pi}$	$0.8e^{i\pi/2}$
x_3	$0.5e^{i3\pi/2}$	$0.5e^{i3\pi/2}$	$0.7e^{i7\pi/4}$	$0.25e^{i7\pi/5}$	$0.45e^{i2\pi}$	$0.4e^{i2\pi}$
x_4	$0.6e^{i2\pi/3}$	$0.4e^{i4\pi/5}$	$0.6e^{i\pi/2}$	$0.5e^{i\pi/4}$	$0.6e^{i\pi/5}$	$0.7e^{i\pi}$
x_5	$0.5e^{i7\pi/4}$	$0.6e^{i6\pi/5}$	$0.4e^{i4\pi/3}$	$0.3e^{i\pi}$	$0.5e^{i\pi}$	$0.5e^{i2\pi/3}$

(x_1) is 0.8 and the time period of the symptom headache e_1 for this disease is approx 2 days $(\pi/4)$.

Nature of the considered parameters:
Since, in this disease diagnosis decision-making, with respect to each of the considered parameters, highest rating of an alternative is best i.e., with respect to each of the symptoms, higher evaluation of a disease indicates that, the patient has abundant chance to be infected by this disease. Therefore, it can be concluded that, in this problem, all the considered parameters are in positive nature.
Weights of the parameters:
Assume that, give weight to each of the parameters (symptom) is equal i.e., $W = \{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\}$.

Now, our target is to detect the disease by which the patient is suffered.

Solution: This is a complex fuzzy soft set based multi-expert decision-making. Now, we will handle this disease diagnosis decision-making problem by applying our algorithm.
Phase I: Since, all the parameters are in positive sense, therefore, we have skipped Phase I.
Phase II:
Step 1. The best-approx complex fuzzy soft set has been given in Table 7.18.
Step 2. By using Equation 7.5, proximity indices of the experts are as follows:
$P_I(d_1) = 0.91$, $P_I(d_2) = 0.89$, $P_I(d_3) = 0.86$.
Step 3. Then, the closeness indices of the experts are given below:
$C_I(d_1) = 0.82$, $C_I(d_2) = 0.83$, $C_I(d_3) = 0.82$.
Step 4. Finally, by using Equation 7.7, weights of the experts are as follows:

Table 7.17: CFSS $(\tilde{\tilde{F}}^3, E)$ given by d_3 (Example 7.4)

	e_1	e_2	e_3	e_4	e_5	e_6
x_1	$0.9e^{i\pi}$	$0.4e^{i\pi/5}$	$0.8e^{i\pi/4}$	$0.5e^{i\pi/3}$	$0.7e^{i\pi/2}$	$0.8e^{i\pi}$
x_2	$0.4e^{i2\pi}$	$0.7e^{i\pi}$	$0.3e^{i\pi/2}$	$0.4e^{i\pi/4}$	$0.7e^{i3\pi/2}$	$0.5e^{i\pi}$
x_3	$0.6e^{i3\pi/2}$	$0.5e^{i4\pi/3}$	$0.5e^{i2\pi}$	$0.7e^{i2\pi}$	$0.5e^{i5\pi/3}$	$0.4e^{i7\pi/4}$
x_4	$0.6e^{i\pi}$	$0.5e^{i\pi/2}$	$0.4e^{i\pi/5}$	$0.6e^{i\pi/4}$	$0.8e^{i\pi}$	$0.2e^{i\pi/2}$
x_5	$0.5e^{i2\pi}$	$0.7e^{i\pi}$	$0.5e^{i3\pi/2}$	$0.4e^{i\pi}$	$0.3e^{i\pi}$	$0.7e^{i\pi}$

Table 7.18: Best-approx complex fuzzy soft set $(\tilde{\tilde{F}}^+, E)$ (Example 7.4)

	e_1	e_2	e_3	e_4	e_5	e_6
x_1	$0.9e^{i\pi}$	$0.9e^{i\pi}$	$1e^{i\pi}$	$0.8e^{i3\pi/2}$	$0.7e^{i\pi/2}$	$0.9e^{i\pi}$
x_2	$0.6e^{i2\pi}$	$0.7e^{i\pi}$	$0.7e^{i4\pi/3}$	$0.6e^{i4\pi/3}$	$0.7e^{i3\pi/2}$	$0.8e^{i\pi}$
x_3	$0.6e^{i7\pi/4}$	$0.6e^{i2\pi}$	$0.7e^{i2\pi}$	$0.7e^{i2\pi}$	$0.5e^{i5\pi/3}$	$0.5e^{i2\pi}$
x_4	$0.6e^{i\pi}$	$0.8e^{i4\pi/5}$	$0.8e^{i\pi}$	$0.8e^{i\pi/4}$	$0.8e^{i\pi}$	$0.7e^{i\pi}$
x_5	$0.5e^{i2\pi}$	$0.7e^{i2\pi}$	$0.5e^{i3\pi/2}$	$1e^{i4\pi/3}$	$0.7e^{i\pi}$	$0.7e^{i3\pi/2}$

$\varpi_I(d_1) = 0.34$, $\varpi_I(d_2) = 0.33$, $\varpi_I(d_3) = 0.33$.

Phase III:

Based on our proposed complex fuzzy soft weighted geometric mean aggregation operator (given in Equation 7.4), the collective complex fuzzy soft set $(\tilde{\tilde{F}}, E)$ as a representative of all the three complex fuzzy soft sets $(\tilde{\tilde{F}}^1, E)$, $(\tilde{\tilde{F}}^2, E)$ and $(\tilde{\tilde{F}}^3, E)$ has been given in Table 7.19.

Table 7.19: Collective complex fuzzy soft set $(\tilde{\tilde{F}}, E)$ (Example 7.4)

	e_1	e_2	e_3	e_4	e_5	e_6
x_1	$0.80e^{i0.58\pi}$	$0.70e^{i0.57\pi}$	$0.80e^{i0.58\pi}$	$0.67e^{i0.94\pi}$	$0.56e^{i0.30\pi}$	$0.83e^{i0.89\pi}$
x_2	$0.47e^{i1.34\pi}$	$0.63e^{i0.89\pi}$	$0.50e^{i0.95\pi}$	$0.47e^{i0.69\pi}$	$0.63e^{i1.16\pi}$	$0.63e^{i0.84\pi}$
x_3	$0.53e^{i1.58\pi}$	$0.53e^{i1.62\pi}$	$0.63e^{i1.92\pi}$	$0.45e^{i1.80\pi}$	$0.48e^{i1.78\pi}$	$0.43e^{i1.92\pi}$
x_4	$0.60e^{i0.72\pi}$	$0.57e^{i0.60\pi}$	$0.60e^{i0.57\pi}$	$0.64e^{i0.22\pi}$	$0.63e^{i0.74\pi}$	$0.54e^{i0.58\pi}$
x_5	$0.36e^{i1.75\pi}$	$0.56e^{i1.41\pi}$	$0.40e^{i1.44\pi}$	$0.57e^{i1.11\pi}$	$0.50e^{i\pi}$	$0.50e^{i1.06\pi}$

Phase IV:

Step 1. By using Equation 7.8, objective weights of the parameters are,
$w_1^{Ob} = 0.28$; $w_2^{Ob} = 0.28$; $w_3^{Ob} = 0.30$; $w_4^{Ob} = 0.29$; $w_5^{Ob} = 0.25$; $w_6^{Ob} = 0.27$.
Then, the combined weights of the parameters are, $w_1^{Cb} = 0.17$; $w_2^{Cb} = 0.14$; $w_3^{Cb} = 0.18$; $w_4^{Cb} = 0.18$; $w_5^{Cb} = 0.16$; $w_6^{Cb} = 0.17$ ($\alpha = 0.5$ has been considered).

Step 2. Now, the ideal and anti-ideal alternatives based on the collective complex fuzzy soft set $(\tilde{\tilde{F}}, E)$ are as follows:

$\bar{x} = (0.80e^{i1.75\pi}, 0.70e^{i1.62\pi}, 0.80e^{i1.92\pi}, 0.67e^{i1.80\pi}, 0.63e^{i1.78\pi}, 0.83e^{i1.92\pi})$;
$\underline{x} = (0.36e^{i0.58\pi}, 0.53e^{i0.57\pi}, 0.40e^{i0.57\pi}, 0.45e^{i0.22\pi}, 0.48e^{i0.30\pi}, 0.43e^{i0.58\pi})$.
Step 3. Then, the separation levels of an alternative from ideal and anti-ideal alternatives over the collective complex fuzzy soft set $(\tilde{\tilde{F}}, E)$ (Table 7.19) are as follows:
$D(x_1/\bar{x}) = 0.4149$; $D(x_2/\bar{x}) = 0.3449$; $D(x_3/\bar{x}) = 0.1766$; $D(x_4/\bar{x}) = 0.4617$;
$D(x_5/\bar{x}) = 0.2705$.
$D(x_1/\underline{x}) = 0.2268$; $D(x_2/\underline{x}) = 0.2154$; $D(x_3/\underline{x}) = 0.4904$; $D(x_4/\underline{x}) = 0.1585$;
$D(x_5/\underline{x}) = 0.2705$.
Step 4. So, with the help of Equation 7.12, ranking indices of the alternatives are,
$\tilde{R}(x_1) = 0.3535$; $\tilde{R}(x_2) = 0.3844$; $\tilde{R}(x_3) = 0.7352$; $\tilde{R}(x_4) = 0.2555$; $\tilde{R}(x_5) = 0.4497$.
Then from the above ranking values, final ranking order of the corresponding alternatives is,
$x_3 > x_5 > x_2 > x_1 > x_4$.
Hence, it can be concluded that, the patient is affected by the disease gastric ulcer (x_3).

7.7 Comparative discussion and sensitivity analysis

7.7.1 Comparative analysis

In this section, a comparative analysis has been interpreted to examine the validity and effectiveness of our complex fuzzy soft multi-expert decision-making approach. Since, in the existing literature, no algorithm exists for solving complex fuzzy soft multi-expert decision-making problems therefore, in order to verify the validity and effectiveness of our proposed approach, we have derived the ranking order of the alternatives by using our proposed approach when each phase term is equals to 0 i.e., when $u^l_{sj} = 0$ then, the corresponding complex fuzzy soft sets are transformed into fuzzy soft sets. So, then we can compare our proposed approach with the fuzzy soft set based algorithms (Roy and Maji's approach [139], Feng's approach [61], Alcantud's approach [4]).

Comparison based on Example 7.3

Now, we have derived the best decision solution of Example 7.3 by using different methods including our proposed approach when, each of the phase terms is equals to 0. i.e., each $u^l_{sj} = 0$.

- **Solution by using our proposed approach**

Firstly, put each $u^l_{sj} = 0$ in the given Tables 7.7, 7.8 and 7.9. After that, values have been given in Tables 7.20, 7.21 and 7.22.
Step 1. Since, e_3 is a negative parameter therefore, by using fuzzy complement operator (given in Chapter 2), we have equalized all the evaluations as given in Tables 7.23, 7.24 and

Table 7.20: CFSS $(\tilde{\tilde{F}}^1, E)$ (Example 7.3) (each $u^l_{sj} = 0$)

	e_1	e_2	e_3
x_1	0.8	0.3	0.5
x_2	0.5	0.6	0.4
x_3	1	0.7	0.6
x_4	0.7	0.8	0.6

Table 7.21: CFSS $(\tilde{\tilde{F}}^2, E)$ (Example 7.3) (each $u^l_{sj} = 0$)

	e_1	e_2	e_3
x_1	1	0.5	0.4
x_2	0.4	0.6	0.8
x_3	0.7	0.1	0.2
x_4	1	0.6	0.5

Table 7.22: CFSS $(\tilde{\tilde{F}}^3, E)$ (Example 7.3) (each $u^l_{sj} = 0$)

	e_1	e_2	e_3
x_1	1	0.8	0.3
x_2	0.6	0.6	0.3
x_3	0.4	0.7	0.7
x_4	0.7	0.8	0.7

7.25.

Table 7.23: $(\tilde{\tilde{F}}^1, E)$ after equalization (each $u^l_{sj} = 0$)

	e_1	e_2	e_3
x_1	0.8	0.3	0.5
x_2	0.5	0.6	0.6
x_3	1	0.7	0.4
x_4	0.7	0.8	0.4

Table 7.24: $(\tilde{\tilde{F}}^2, E)$ after equalization (each $u^l_{sj} = 0$)

	e_1	e_2	e_3
x_1	1	0.5	0.6
x_2	0.4	0.6	0.2
x_3	0.7	0.1	0.8
x_4	1	0.6	0.5

Table 7.25: $(\tilde{\tilde{F}}^3, E)$ after equalization (each $u^l_{sj} = 0$)

	e_1	e_2	e_3
x_1	1	0.8	0.7
x_2	0.6	0.6	0.7
x_3	0.4	0.7	0.3
x_4	0.7	0.8	0.3

Step 2. Weight of the experts are, $\varpi_I(d_1) = 0.34$; $\varpi_I(d_2) = 0.32$; $\varpi_I(d_3) = 0.34$.
Step 3. Then, the collective complex fuzzy soft set $(\tilde{\tilde{F}}, E)$ has been given in Table 7.26.

Table 7.26: Collective complex fuzzy soft set

	e_1	e_2	e_3
x_1	0.93	0.53	0.60
x_2	0.50	0.60	0.51
x_3	0.70	0.51	0.49
x_4	0.80	0.74	0.40

Step 4. Now, the combined weights of the parameters are,
$w_1^{Cb} = 0.46$; $w_2^{Cb} = 0.27$; $w_3^{Cb} = 0.22$.
Step 5. Then, the ideal and anti-ideal alternatives are,
$\bar{x} = \{0.93, 0.74, 0.60\}$; $\underline{x} = \{0.50, 0.51, 0.40\}$.
Step 6. After that, ranking indices of the alternatives are,
$\tilde{R}(x_1) = 0.72$; $\tilde{R}(x_2) = 0.19$; $\tilde{R}(x_3) = 0.40$; $\tilde{R}(x_4) = 0.65$.
So, from the above values, final ranking order of the alternatives is, $x_1 > x_4 > x_3 > x_2$.

- **Solution by using Roy and Maji's approach [139]**

Step 1. Now, by using minimum as AND operator, the resultant fuzzy soft set (F, E) from the three fuzzy soft sets (Tables 7.23, 7.24, 7.25) has been given in Table 7.27. Since, $w_1 = 0.40; w_2 = 0.37; w_3 = 0.23$ are the given weights of the parameters, we have just multiplied this weights to the associated evaluations as given in Table 7.28.
Step 2. Then, the comparison 4×4 matrix has been given in Table 7.29.

Table 7.27: Resultant fuzzy soft set (F, E)

	e_1	e_2	e_3
x_1	0.8	0.3	0.5
x_2	0.4	0.6	0.2
x_3	0.4	0.1	0.3
x_4	0.7	0.6	0.3

Table 7.28: Weighted resultant fuzzy soft set

	e_1	e_2	e_3
x_1	0.32	0.11	0.12
x_2	0.16	0.22	0.05
x_3	0.16	0.04	0.07
x_4	0.28	0.22	0.07

Step 3. Now, row sum, column sum and score value of every alternative have been given in Table 7.29.

Table 7.29: Comparison table

	x_1	x_2	x_3	x_4	Row sum	Column sum	Score value
x_1	3	2	3	2	10	5	5
x_2	1	3	1	1	6	9	-3
x_3	0	2	3	0	5	10	-5
x_4	1	2	3	3	9	6	3

Step 4. Then, the final ranking order of the associated alternatives is, $x_1 > x_4 > x_2 > x_3$.

- **Solution by using Feng's approach [61]**

In Table 7.28, we have given the weighted resultant fuzzy soft set (F, E). Then, the corresponding top-level soft set and mid-level soft set are given in Tables 7.30 and 7.31.

Table 7.30: Top-level soft set

	e_1	e_2	e_3	Choice value
x_1	1	0	1	2
x_2	0	1	0	1
x_3	0	0	0	0
x_4	0	1	0	1

Table 7.31: Mid-level soft set

	e_1	e_2	e_3	Choice value
x_1	0.32	0.11	0.12	2
x_2	0.16	0.22	0.05	1
x_3	0.16	0.04	0.07	0
x_4	0.28	0.22	0.07	2

So, by using top-level soft set, final ranking order of the corresponding alternatives is, $x_1 > x_2 = x_4 > x_3$ and by using mid-level soft set, final ranking order of the corresponding alternatives is, $x_1 = x_4 > x_2 > x_3$.

- **Solution by using Alcantud's approach [4]**

Step 1. By using product as AND operator, resultant fuzzy soft set has been given in Table 7.32. Since, $w_1 = 0.40; w_2 = 0.37; w_3 = 0.23$ are the given weights of the parameters, the corresponding weighted resultant fuzzy soft set has been given in Table 7.33.

Table 7.32: Resultant fuzzy soft set (F, E)

	e_1	e_2	e_3
x_1	0.80	0.12	0.21
x_2	0.12	0.22	0.08
x_3	0.28	0.05	0.10
x_4	0.49	0.38	0.06

Table 7.33: Weighted resultant fuzzy soft set

	e_1	e_2	e_3
x_1	0.32	0.04	0.05
x_2	0.05	0.08	0.02
x_3	0.11	0.02	0.02
x_4	0.20	0.14	0.01

Step 2. Then, the comparison table and the corresponding row sum, column sum and score values of the alternatives have given in Table 7.34.

Table 7.34: Comparison table

	x_1	x_2	x_3	x_4	Row sum	Column sum	Score value
x_1	0	1.44	1.40	1.18	4.02	0.99	3.03
x_2	0.28	0	0.43	0.20	0.91	2.03	-1.12
x_3	0	0.16	0	0.02	0.18	3	-2.82
x_4	0.71	0.43	1.17	0	2.31	1.40	0.99

Step 3. So, based on the above score values, ranking order of the associated alternatives is, $x_1 > x_4 > x_2 > x_3$.

Comparison based on Example 7.4

Now, we have derived the best decision solution of Example 7.4 by using different methods including our proposed approach when, each of the phase terms is equals to 0. i.e., each $u^l_{sj} = 0$.

- **Solution by using our proposed approach**

The final ranking order of the associated alternatives is, $x_1 > x_4 > x_2 > x_3 > x_5$.

- **Solution by using Roy and Maji' approach [139]**

The final ranking order of the associated alternatives is, $x_1 > x_2 > x_4 = x_3 > x_5$.

- **Solution by using Feng's approach [61]**

By using top-level soft set, final ranking order of the associated alternatives is,
$x_1 > x_2 > x_3 = x_4 > x_5$.
By using mid-level soft set, final ranking order of the associated alternatives is,
$x_1 = x_3 = x_4 > x_2 > x_5$.

- **Solution by using Alcantud's approach [4]**

The final ranking order of the associated alternatives is, $x_1 > x_4 > x_2 > x_3 > x_5$.

Discussion

The above resulting values have been summarized in Tables 7.35 and 7.36.

Table 7.35: Comparison of different methods based on Example 7.3

Methods	Values of the phase terms	Final solution
Our proposed approach	not all $u^l_{sj} \neq 0$	x_2
Our proposed approach	all $u^l_{sj} = 0$	x_1
Roy and Maji's approach [139]	all $u^l_{sj} = 0$	x_1
Feng's method [61] (by using top-level soft set)	all $u^l_{sj} = 0$	x_1
Feng's method [61] (by using mid-level soft set)	all $u^l_{sj} = 0$	x_1, x_4
Alcantud's approach [4]	all $u^l_{sj} = 0$	x_1

Table 7.36: Comparison of different methods based on Example 7.4

Methods	Values of the phase terms	Final solution
Our proposed approach	not all $u^l_{sj} \neq 0$	x_3
Our proposed approach	all $u^l_{sj} = 0$	x_1
Roy and Maji's approach [139]	all $u^l_{sj} = 0$	x_1
Feng's method [61] (by using top-level soft set)	all $u^l_{sj} = 0$	x_1
Feng's method [61] (by using mid-level soft set)	all $u^l_{sj} = 0$	x_1, x_3, x_4
Alcantud's approach [4]	all $u^l_{sj} = 0$	x_1

From Table 7.35 we have seen that, when all $u^l_{sj} = 0$, the final decision solution of Example 7.3 by using our proposed approach is x_1. Moreover, since, the value 0 of each of the phase terms ($u^l_{sj} = 0$) indicates that, all the corresponding complex fuzzy evaluations are transformed into fuzzy evaluations (for $u^l_{sj} = 0$, $p^l_{sj}e^{iu^l_{sj}} = p^l_{sj}$) therefore, we can apply different types of fuzzy soft set based approaches on this decision-making problem by putting 0 in each u^l_{sj}. Then it is observed that, by using three fuzzy soft set based approaches

(Roy and Maji's approach [139], Feng's method [61] and Alcantud's approach [4]), the solution is x_1 which is not differed from the solution by our proposed approach when each $u^l_{sj} = 0$. Besides, based on Table 7.36 we have seen that, in Example 7.4, when each $u^l_{sj} = 0$, then the solution (alternative x_1) by our proposed approach is same with the solution (alternative x_1) by each of the other three fuzzy soft set based approaches (Roy and Maji's approach [139], Feng's method [61], Alcantud's approach [4]). So, it can be concluded that, when each $u^l_{sj} = 0$, the result of our proposed approach is same with the existing approaches including, Roy and Maji's approach [139], Feng's method [61], Alcantud's approach [4], etc. which proves the validity of our proposed approach.

Further, we have also seen that, in each of the two examples (Example 7.3 and Example 7.4), we have received two different solutions for two different situations such as, when not all $u^l_{sj} \neq 0$ and when all $u^l_{sj} = 0$. For instance, in the case of Example 7.4, when, we have considered only the information 'belongingness degree of a symptom' (which has been expressed through amplitude part p^l_{sj}) i.e., when we have taken each $u^l_{sj} = 0$ then, by using each of the approaches including our proposed approach, it is obtained that, the patient is affected by the malaria (x_1) disease. But, when 'time duration of a symptom' (which has been expressed through phase part u^l_{sj}) has been taken together with the information 'belongingness of a symptom', then we can not apply the other three approaches such as, Roy and Maji's approach [139], Feng's method [61], Alcantud's approach [4]. In that case, our proposed approach can only able to solve the problem. Then, it has been seen that, the patient is affected by the disease gastric ulcer (x_3). So, when, we have considered two information 'belongingness degree' and 'time duration' of a symptom together instead of only the single information 'belongingness degree' of a symptom then, the final result may be varied which proves the effectiveness of our proposed approach.

Hence, from the aforementioned discussions, it can be concluded that, our approach is more flexible and has a commendable efficiency in solving multi-expert decision-making problems especially in disease diagnosis decision-making.

7.7.2 Sensitivity analysis

In our algorithm, we have used an influence factor $\alpha \in [0, 1]$ in deriving the combined weight w^{Cb}_j of a parameter e_j as given in Equation 7.9 by which objective weight (w^{Ob}_j) and given weight (w_j) (subjective weight) of parameter have been considered together. Actually, combined weight of the parameter indicates its importance in selecting the final solution. Now, by changing different values of α in the interval $[0, 1]$, we have investigated its effect on final ranking order of the associated alternatives as well as on the ranking order of the associated parameters. For different values of α, combined weights of the parameters have been given in Figures 7.3 and 7.4.

From Figure 7.4 it has been observed that, in Example 7.4, final ranking order of the parameters based on its combined weights and final ranking order of the associated alternatives based on its ranking indices remains same whether we take different values of α in the interval $[0, 1]$. But, in Example 7.3, as given in Figure 7.3, it has been seen that, the final ranking order of the parameters based on its combined weights is, $e_1 > e_3 > e_2$ for any value of α in the interval $[0, 0.3)$ but after the value 0.3, this ranking order of the parameters has been changed to $e_1 > e_2 > e_3$. Moreover, it is also observed that, up to the value 0.7 of α, ranking order of the associated alternatives is, $x_2 > x_3 > x_1 > x_4$ and after the value 0.7, the ranking order of the associated alternatives has been slightly changed to $x_2 > x_3 > x_4 > x_1$.

So, from this discussion it can be concluded that, changing the value of the influence factor $\alpha \in [0, 1]$ can effect on the final ranking order of the parameters and the final ranking order of the associated alternatives. Moreover, since, by using the parameter α, objective weight (w_j^{Ob}) and subjective weight (given weight) (w_j) of a parameter have been considered together in deriving its combined weight w_j^{Cb}, so the worth of the objective weight (w_j^{Ob}) and the worth of the subjective weight (w_j) can be altered according to the requirement of a decision-making problem. Therefore, our approach is stable and more realistic and reasonable.

7.8 Conclusion

This chapter proposes a methodological approach for solving multi-expert decision-making by using complex fuzzy soft set theory. The proposed approach has been used here in a disease diagnosis decision-making problem. Major contributions in this chapter is as follows:

- Firstly, we have constructed a new ratio similarity measure approach to complex fuzzy sets and complex fuzzy soft sets.
- Secondly, we have defined the axiomatic definition of aggregation operation for complex fuzzy soft sets and then, we have introduced the notion of complex fuzzy soft weighted geometric mean aggregation operator to aggregate multiple complex fuzzy soft sets.
- Finally, a novel a decision-making algorithm has been proposed to solve complex fuzzy soft set based multi-expert decision-making where, the main highlighted steps are, *derivation of the weight of an expert*, *derivation of the combined weight of a parameter* and *selection of the best alternative*. In particular, the combined weight of an associated parameter has been derived by integrating its subjective weight and objective weight through an influenced factor(α).
- Further, our proposed decision-making approach has been applied in a medical disease diagnosis decision-making problem.

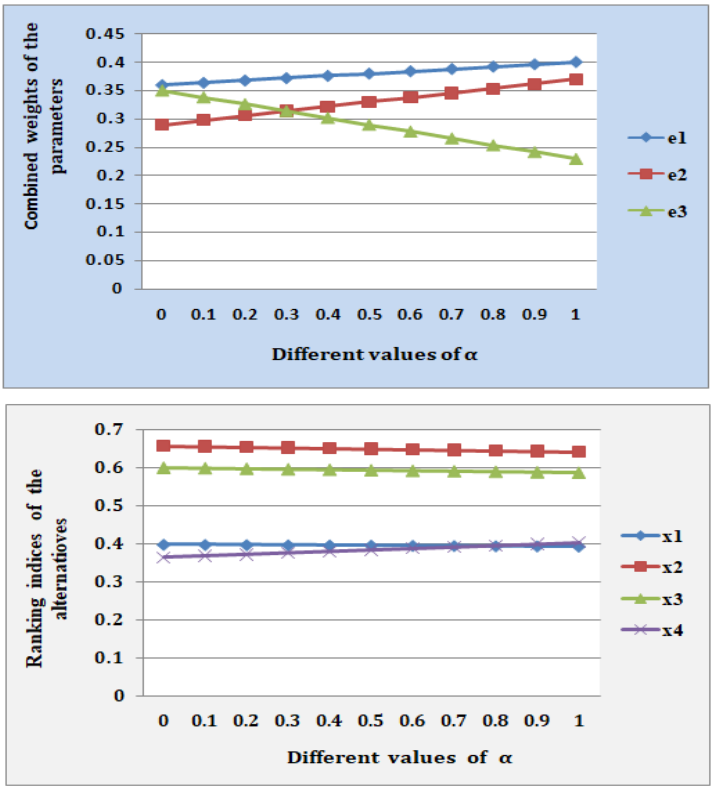

Figure 7.3: Sensitivity analysis based on α for Example 7.3

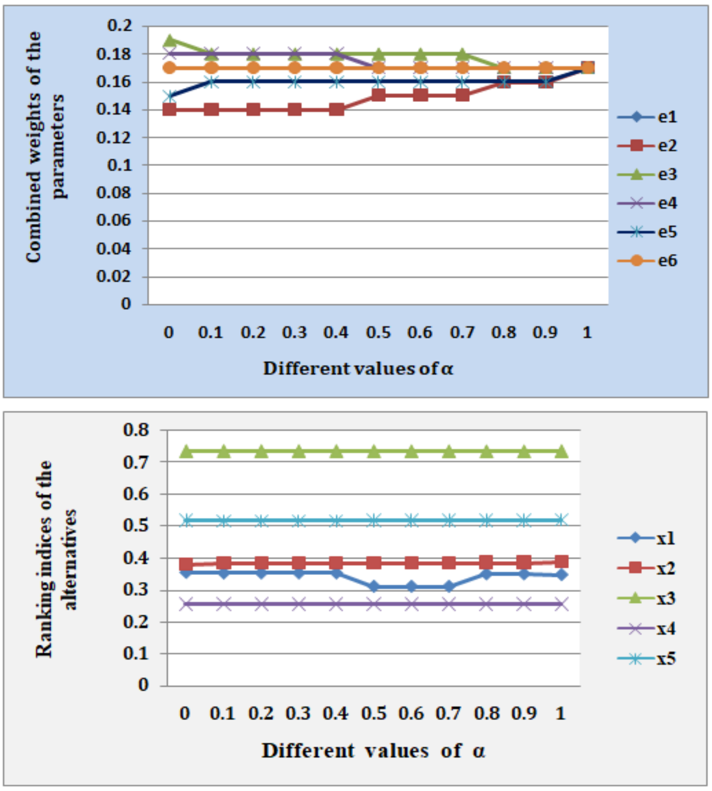

Figure 7.4: Sensitivity analysis based on α for Example 7.4

Since, complex fuzzy soft set is a newly proposed mathematical approach, so for further research, one can explore some well known algorithms such as, AHP (analytical hierarchy process) method, FUCOM (full consistency method) method, DEMATEL (decision-making trial and evaluation laboratory) method, etc. through this newly proposed approach. Moreover, one can work on its other generalizations like, complex intuitionsitic fuzzy soft set, complex neutrosophic soft set, etc. to solve more real-life related problems.