Chapter 6

Connectivity index of m-polar fuzzy graphs

6.1 Introduction

Graph theory plays an egregious function, by providing a link in different fields such as computer technology, operations research, network routing, engineering and medical science, to relate certain values to certain parameters.The concepts of strength of connectedness in mPFG, mPF tree, mPF cut nodes are established by Mandal et al. [130]. Binu [23] introduced the connectivity index in fuzzy graphs. In this chapter, we described the connectivity index for mPFG. The upper and lower boundary of connectivity index for mPFG are discussed. If we delete an edge from a mPFG then the effects of the connectivity index in mPFG are given in this chapter. The average connectivity index in mPFG is provided here.

6.2 Connectivity index of mPFG

In a network system, connectivity is a major factor. Here, we have presented a mPFG connectivity index.

Definition 6.2.1. *The connectivity index of mPFG is characterised by* $CI_{mPF}(G)$*, defined as* $CI_{mPF}(G) = (p_1 \circ CI_{mPF}(G), p_2 \circ CI_{mPF}(G), \ldots p_m \circ CI_{mPF}(G)) = (\sum_{s,t\in V}(p_1 \circ A(s))(p_1 \circ A(t))(p_1 \circ CONN_G(s,t)), \sum_{s,t\in V}(p_2 \circ A(s))(p_2 \circ A(t))(p_2 \circ CONN_G(s,t)), \ldots, \sum_{s,t\in V}(p_m \circ A(s))(p_m \circ A(t))(p_m \circ CONN_G(s,t)))$ *where,* $p_i \circ CI_{mPF} = \sum_{s,t\in V}(p_i \circ A(s))(p_i \circ A(t))(p_i \circ CONN_G(s,t))$ *is the i th component of connectivity index of mPFG*

G and $p_i \circ CONN_G(s,t)$ is the i-th component of strength of connectedness between s and t.

Example 6.2.1. *Here we take the connected mPFG $G = (V, A, B)$ in Figure 6.1 where $V = \{q, r, s, t, u, v, w, x\}$. We now think that each vertex's membership value is $(1, 1, 1)$. Then the connectivity index of mPFG G is $CI_{mPF}(G) = (20.3, 14.7, 11.9)$.*

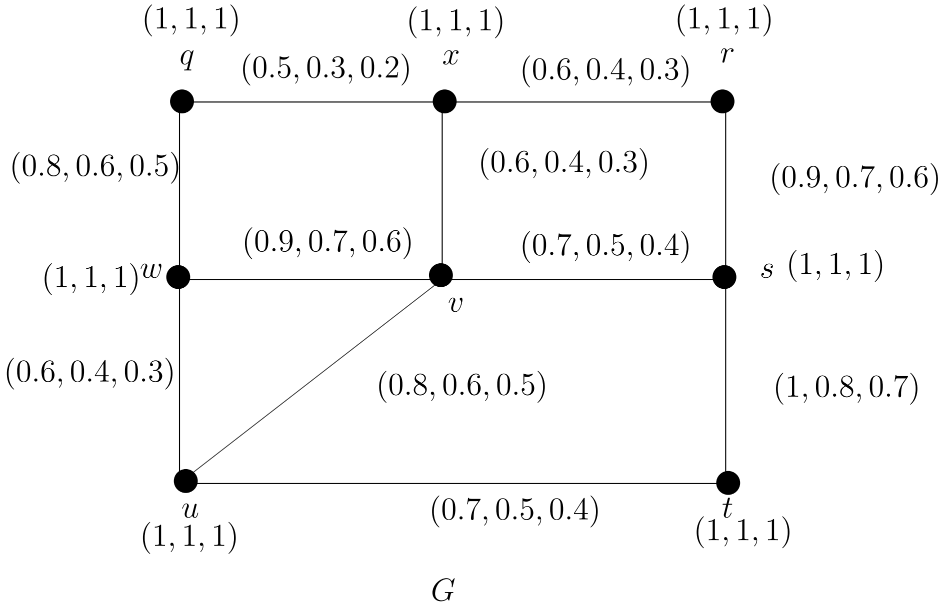

Figure 6.1: An mPFG G.

In the above example, we calculated the connectivity index of a 3PFG. Depending on this, we established the following Theorems. Different properties on the connectivity index of a mPFG are discussed below.

Theorem 6.2.1. *Let $G' = (V', A', B')$ be a partial mPFSG of a mPFG $G = (V, A, B)$ then $p_i \circ CI_{mPF}(G') \leq p_i \circ CI_{mPF}(G)$ $\forall$ $i = 1, 2, \ldots, m$.*

Proof. Here, G' is a partial mPFSG of G. So, $p_i \circ A'(s) \leq p_i \circ A$ $\forall$ $s \in G'$ and $p_i \circ B'(s,t) \leq p_i \circ B(s,t)$ $\forall$ $s, t \in G'$ and $\forall$ i.

Again $CONN_{G'}(x, y)$ is connectedness between s and t in G' where G' is the partial mPF subgraph of G and $p_i \circ B'(s,t) \leq p_i \circ B(s,t)$ $\forall$ $(s,t) \in G'$, so we have $p_i \circ CONN_{G'}(s,t) \leq p_i \circ CONN_G(s,t)$ $\forall$ $(s,t) \in G'$ and $i = 1, 2, \ldots, m$. Next we know $(p_i \circ A(s))(p_i \circ A(t)) \geq 0$ and $(p_i \circ A'(s))(p_i \circ A'(t)) \geq 0$ $\forall$ i. So, $(p_i \circ A'(s))(p_i \circ A'(t)) p_i \circ (CONN_{G'}(s,t)) \leq (p_i \circ A(s))(p_i \circ A(t))(p_i \circ CONN_G(s,t))$ because $p_i \circ A'(s) \leq p_i \circ A(s)$, which implies $\sum_{s,t\in V'}(p_i \circ A'(s))(p_i \circ A'(t))p_i \circ (CONN_{G'}(s,t)) \leq \sum_{s,t\in V'}(p_i \circ A(s))(p_i \circ A(t))(p_i \circ CONN_G(s,t)) \quad \forall\ i.$

Hence, $p_i \circ CI_{mPF}(G') \leq p_i \circ CI_{mPF}(G) \quad \forall\ i$ i.e. $CI_{mPF}(G') \leq CI_{mPF}(G)$.

An mPFSG is a partial mPFG. Then the Theorem 6.2.1 holds for a mPFSG that is given in the next note.

Note 6.2.1. *For a mPFSG $G' = (V, A', B')$ of a mPFG $G = (V, A, B)$ $CI_{mPF} \leq CI_{mPF}(G)$.*

Theorem 6.2.2. *Let $G = (V, A, B)$ be a connected mPFG. If $G' = (V', A', B')$ is a mPFSG of G, where $V' = V - \{t\}$ $t \in V$, then $p_i \circ CI_{mPF}(G') < p_i \circ CI_{mPF}(G)$ $\forall$ $i = 1, 2, \ldots, m$.*

Proof. Suppose n is the total number of nodes in Gi.e. $|V| = n$ and $s \in V$. Then, $V' = V - \{s\}$ which implies $|V'| = n-1$. Next we consider $V = \{s = s_1, s_2, \ldots, s_n\}$. So, $V' = \{s_2, \ldots, s_n\}$. Again G' is a mPFSG then G' is also a partial mPFSG of G. Then $p_i \circ CImPF(G) = p_i \circ CI_{mPF}(G') + \sum_{j=2}^{n}(p_i \circ A(s_1))(p_i \circ A(s_j))(p_i \circ CONN_G(s_1, s_j))$ $\forall\ i = 1, 2, \ldots, m$.

From the above relation we easily say that $p_i \circ CI_m PF(G) \rangle p_i \circ CI_{mPF}(G')$ for all i.

Hence, $CI_{mPF}(G) \rangle CI_{mPF}(G')$.

6.3 Boundedness of connectivity index of m Polar fuzzy graph

In this section, we described boundedness implies lower and upper boundary of connectivity index of a mPFG. Next, we introduced some Theorems on connectivity index of mPFG.

Theorem 6.3.1. *Let n be the number of vertices in a connected mPFG $G = (V, A, B)$. If $G' = (V', A', B')$ is the complete mPFG spanned by the nodes of G, then $0 \leq p_i \circ CI_{mPF}(G) \leq p_i \circ CI_{mPF}(G')$ $\forall\ i = 1, 2, \ldots, m$.*

Proof. If G has only one vertex. Also G' has one vertex. Then, $CONN_G(s,t) = 0 = CONN_{G'}(s,t)$ which implies $p_i \circ CONN_G(s,t) = 0 = p_i \circ CONN_{G'}(s,t)$ $\forall\ i$ and $s, t \in V$.

So, $p_i \circ CI_{mPF}(G) = 0 = p_i \circ CI_{mPF}(G') \ \forall \ i$ which means $CI_{mPF}(G) = 0 = CI_{mPF}(g')$.

Let G contain more than one vertex. Then the complete mPFG $G' = (V', A', B')$ with $|V'| = n$ and $p_i \circ A'(s) = p_i \circ A(s) \ \forall \ i$ and $s \in V$ because G' is complete mPFG spanned by the node set of G.

Then for every edge (s,t) of G we have $p_i \circ B(s,t) \leq p_i \circ B'(s,t) \ \forall \ i$.

From the above relation, we say that $p_i \circ CONN_G(s,t) \leq p_i \circ CONN_{G'}(s,t) \ \forall \ i = 1, 2, \ldots, m$.

So $\forall \ i$ $(p_i \circ A(s))(p_i \circ A(t))(p_i \circ CONN_G(s,t)) \leq (p_i \circ A\prime(s))(p_i \circ A\prime(t))(p_i \circ CONN_{G\prime}(s,t))$

$\Rightarrow \sum\limits_{s,t \in V} (p_i \circ A(s))(p_i \circ A(t))(p_i \circ CONN_G(s,t)) \leq \sum\limits_{s,t \in V} (p_i \circ A\prime(s))(p_i \circ A\prime(t))(p_i \circ CONN_{G\prime}(s,t))$

$\Rightarrow p_i \circ CI_{mPF}(G) \leq p_i \circ CI_{mPF}(G\prime)$

Hence $CI_{mPF}(G\prime) \geq CI_{mPF}(G)$.

Thus, we have $0 \leq CI_{mPF} \leq CI_{mPF}$.

Theorem 6.3.2. *If G is complete mPFG with vertex set $V = \{u_1, u_2, \ldots, u_n\}$. The membership value of each vertex u_j is u_j is $A(u_j) = (k_j^1, k_j^2, \ldots, k_j^m)$ i.e. $p_i \circ A(u_j) = k_j^i$ for all $i = 1, 2, \ldots, n$, where $k_1^i \leq k_2^i \leq \ldots \leq k_n^i$ for all i. Then $p_i \circ CI_{mPF}(G) = \sum\limits_{s=1}^{n-1} (k_s^i)^2 \sum\limits_{j-s-1}^{n-2} (k_{j+2}^i) \ \forall \ i = 1, 2, \ldots, m$.*

proof. Let u_1 be a node in G. Here $p_i \circ A(u_1) = k_1^i \ \forall \ i$.

Again we know, G is complete mPFG, then $p_i \circ CONN_G(u_s, u_j) = p_i \circ B(u_s, u_j) \ \forall \ i$ and for every $u_s, u_j \in V$.

Thus $p_i \circ B(u_1, u_j) = k_1^j \ \forall \ i - 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

So, $(p_i \circ A(u_1))(p_i \circ A(u_j))(p_i \circ CONN_G(u_1, u_j)) = k_1^i k_j^i k_1^i = (k_1^i)^2 k_j^i \ \forall \ i$ and $j = 2, 3, \ldots, n$

This implies $\sum\limits_{j=2}^{n} (p_i \circ A(u_1))(p_i \circ A(u_j))(p_i \circ CONN_G(u_1, u_j)) = \sum\limits_{j=2}^{n} (k_1^i)^2 k_j^i$

Then, $\sum\limits_{s=1}^{n} \sum\limits_{j=2}^{n} (p_i \circ A(u_1))(p_i \circ A(u_j))(p_i \circ CONN_G(u_1, u_j)) = \sum\limits_{s=1}^{n} \sum\limits_{j=2}^{n} (k_1^i)^2 k_j^i \ \forall \ i$.

Then, $p_i \circ CI_{mPF}(G) = \sum\limits_{s=1}^{n-1} (k_s^i)^2 \sum\limits_{j=s-1}^{n-2} k_{j+1}^i \ \forall \ i = 1, 2, \ldots, m$.

Example 6.3.1. *Here G is a complete $3m$PFG in Figure 6.2, where $V = \{s_1, s_2, s_3, s_4\}$.*

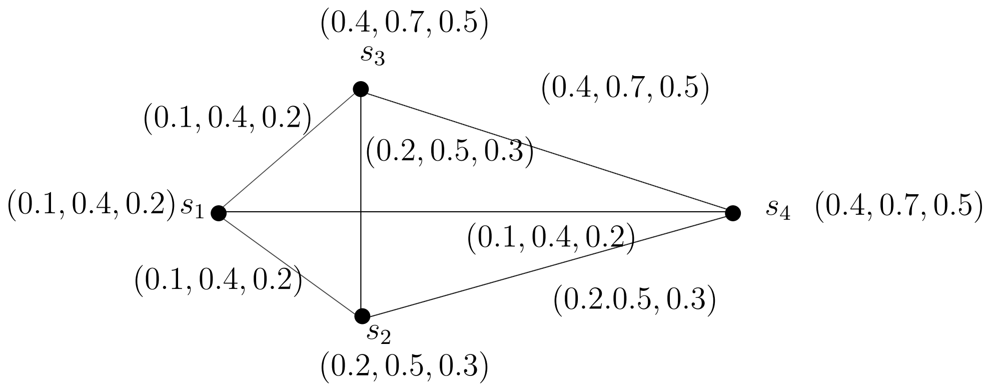

Figure 6.2: An mPFG G.

Here $A(u_j) = (k_j^1, k_j^2, k_j^3)$ *for* $j = 1, 2, 3, 4.$

$A(u_1) = (k_1^1, k_1^2, k_1^3) = (0.1, 0.4, 0.2)$

$A(u_2) = (k_2^1, k_2^2, k_2^3) = (0.2, 0.5, 0.3)$

$A(u_3) = (k_3^1, k_3^2, k_3^3) = (0.4, 0.7, 0.5)$

$A(u_4) = (k_4^1, k_4^2, k_4^3) = (0.4, 0.7, 0.5).$

Here we see that, $k_4^1 \geq k_3^1 \geq k_2^1 \geq k_1^1$, $k_4^2 \geq k_3^2 \geq k_2^2 \geq k_1^2$ *and* $k_4^3 \geq k_3^3 \geq k_2^3 \geq k_1^3$.

Now,

$$\begin{aligned}
& p_1 \circ CI_{mPF}(G) \\
= & \sum_{s=1}^{4} (k_s^1)^2 \sum_{j=s-1}^{n-2} k_{j+2}^1 \\
= & (k_1^1)^2 k_2^1 + (k_1^1)^2 k_3^1 + (k_1^1)^2 k_4^1 + (k_2^1)^2 k_3^1 + (k_2^1)^2 k_4^1 + (k_3^1)^2 k_4^1 \\
= & (0.1)^2(0.2) + (0.1)^2(0.4) + (0.1)^2(0.4) + (0.2)^2(0.4) + (0.2)^2(0.4) + (0.4)^2(0.4) \\
= & 0.106.
\end{aligned}$$

$$\begin{aligned}
& p_2 \circ CI_{mPF}(G) \\
= & \sum_{s=1}^{4} (k_s^2)^2 \sum_{j=s-1}^{n-2} k_{j+2}^2 \\
= & (k_1^2)^2 k_2^2 + (k_1^2)^2 k_3^2 + (k_1^2)^2 k_4^2 + (k_2^2)^2 k_3^2 + (k_2^2)^2 k_4^2 + (k_3^2)^2 k_4^2 \\
= & (0.4)^2(0.5) + (0.4)^2(0.7) + (0.4)^2(0.7) + (0.5)^2(0.7) + (0.5)^2(0.7) + (0.7)^2(0.7) \\
= & 0.997.
\end{aligned}$$

$$
\begin{aligned}
& p_3 \circ CI_{mPF}(G) \\
= & \sum_{s=1}^{4} (k_s^2)^3 \sum_{j=s-1}^{n-2} k_{j+2}^3 \\
= & (k_1^3)^2 k_2^3 + (k_1^3)^2 k_3^3 + (k_1^3)^2 k_4^3 + (k_2^3)^2 k_3^3 + (k_2^2)^3 k_4^3 + (k_3^3)^2 k_4^3 \\
= & (0.2)^2(0.3) + (0.2)^2(0.5) + (0.2)^2(0.5) + (0.3)^2(0.5) + (0.3)^2(0.5) + (0.5)^2(0.5) \\
= & 0.262.
\end{aligned}
$$

Therefore, $CI_{mPF}(G) = (0.106, 0.997, 0.262)$.

6.4 Find connectivity index of edge deleted m-polar fuzzy subgraph

In a mPFG, if a vertex is deleted from G then the connectivity index reduces [from Theorem 6.2.2]. But if we remove an edge from G, it could decrease or stay the same connectivity index. In this section we will present the connectivity index of edge deleted mPFG.

Next, we find out the connectivity index of edge deleted mPFSG of mPFG using the following example.

Example 6.4.1. *Consider an mPFG G (see in fig 6.3).* $G' = G - (s_1, s_5)$, $G'' = G - (s_3, s_4)$ *be two edge deleted mPFSG of G. Here membership value of all vertices is* $(1, 1, 1)$.

Here,

$$
\begin{aligned}
CI_{mPF}(G) &= (p_1 \circ CI_{mPF}(G), p_2 \circ CI_{mPF}(G), p_3 \circ CI_{mPF}(G)) \\
&= (7.5, 6.5, 4.5).
\end{aligned}
$$

$$
\begin{aligned}
CI_{mPF}(G') &= (p_1 \circ CI_{mPF}(G'), p_2 \circ CI_{mPF}(G'), p_3 \circ CI_{mPF}(G')) \\
&= (7.5, 6.5, 4.5).
\end{aligned}
$$

$$
\begin{aligned}
CI_{mPF}(G'\prime) &= (p_1 \circ CI_{mPF}(G'\prime), p_2 \circ CI_{mPF}(G'\prime), p_3 \circ CI_{mPF}(G'\prime)) \\
&= (7.3, 6.3, 4.3).
\end{aligned}
$$

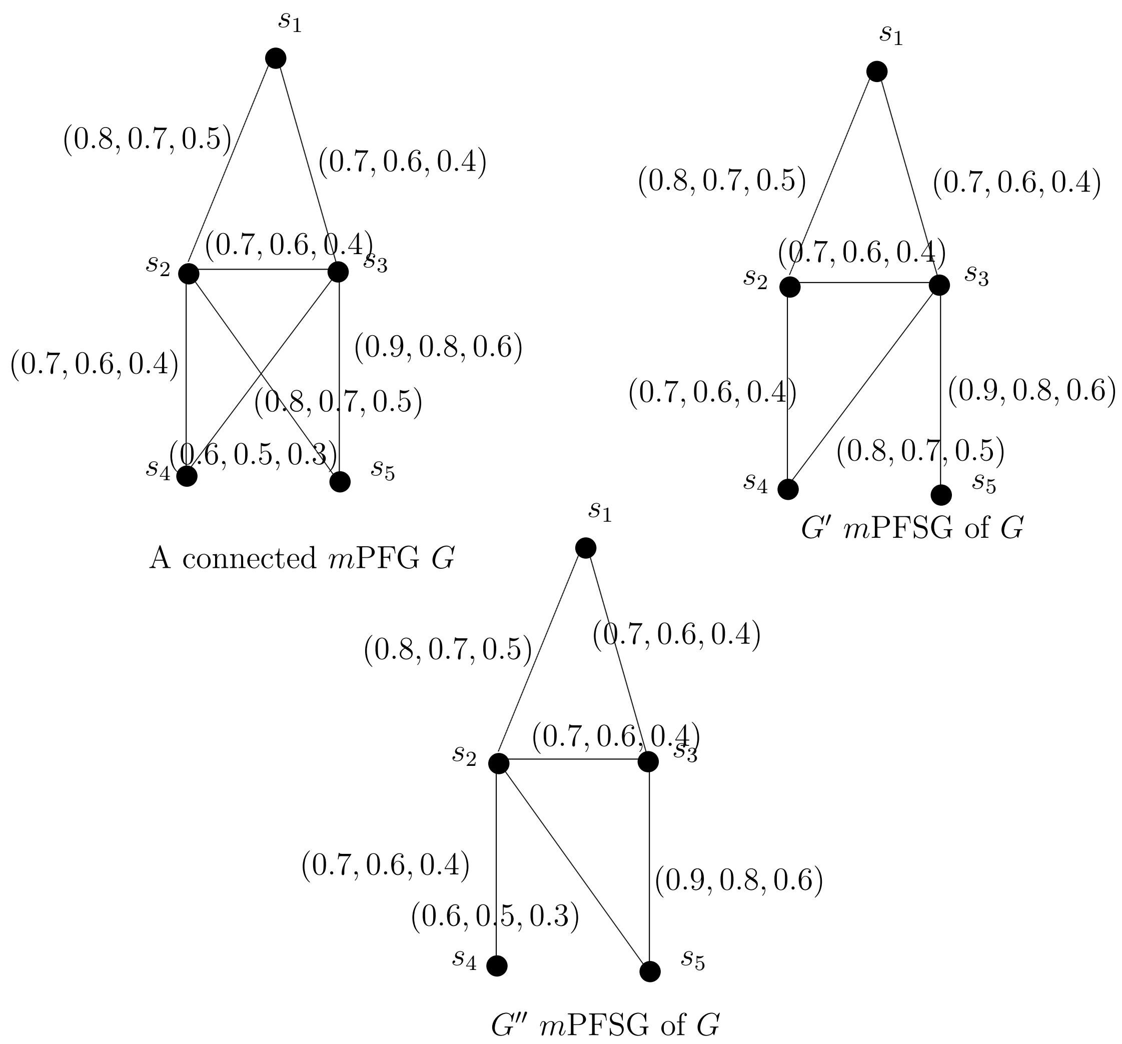

Figure 6.3: A Connected 3PF graph G with its mPFSG G' and G''.

So, Here we see that $\forall\ i = 1, 2, 3,\ p_i \circ CI_{mPF}(G) = p_i \circ CI_{mPF}(G')$ *for the arc deleted mPFSG* G' *of* G.

Again, $\forall\ i,\ p_i \circ CI_{mPF}(G) > p_i \circ CI_{mPF}(G'')$ *for the arc deleted mPFSG* G'' *of* G.

Theorem 6.4.1. *Let* $G' = G - (a, b)$ *be a mPFSG of mPFG* G *where* (a, b) *be an edge in* G. *Then for all* i, $p_i \circ CI_{mPF}(G) > p_i \circ CI_{mPF}(G')$ *iff* (a, b) *is a mPFB of* G.

proof. Let (s, t) be a mPFB of G. Using the concept of mPFB, there exist one edge $(s_1, t_1) \in G$ s.t. $p_i \circ CONN_G(s_1, t_1) > p_i \circ CONN_{G'}(s_1, t_1)\ \forall\ i$.

Then, $\forall\ i$

$$(p_i \circ A(s_1))(p_i \circ A(t_1))(p_i \circ CONN_G(s_1, t_1)) > (p_i \circ A(s_1))(p_i \circ A(t_1))(p_i \circ CONN_{G'}(s_1, t_1))$$

which implies,

$$\sum_{s_1,t_1\in V}(p_i\circ A(s_1))(p_i\circ A(t_1))(p_i\circ CONN_G(s_1,t_1)) > \sum_{s_1,t_1\in V}(p_i\circ A(s_1))(p_i\circ A(t_1))(p_i\circ CONN_{G'}(s_1,t_1))$$

Hance, $p_i\circ CI_{mPF}(G) > p_i\circ CI_{mPF}(G')$, $\forall\ i$.

Conversely, suppose $p_i\circ CI_{mPF}(G) > p_i\circ CI_{mPF}(G')$ for all i. Next we suppose that, the edge (s,t) is not a mPFB. Then for each pair of vertices (s_1,t_1) in G, we have, $p_i\circ CONN_G(s_1,t_1)\leq p_i\circ CONN_{G'}(a_1,b_1)\ \forall\ i$.

This implies

$$\sum_{s_1,t_1\in V}(p_i\circ A(s_1))(p_i\circ A(t_1))(p_i\circ CONN_G(s_1,t_1)) \leq \sum_{s_1,t_1\in V}(p_i\circ A(s_1))(p_i\circ A(t_1))(p_i\circ CONN_{G'}(s_1,t_1))$$

Hence, $p_i\circ CI_{mPF}(G)\leq p_i\circ CI_{mPF}(G')$, $\forall\ i$.

This is a contradiction. So (s,t) is mPFB of G.

Example 6.4.2. *In the above example, $p_i\circ CI_{mPF}(G) > p_i\circ CI_{mPF}(G')$, $\forall\ i$. So the edge (s_3,s_4) is a mPFB of G.*

Theorem 6.4.2. *Let $G' = G-(s,t)$ be a spanning mPFSG of mPFG G, where (s,t) is an edge of G. Then G is a mPFT iff $p_i\circ CI_{mPF} > p_i\circ CI_{mPF}(G')$.*

Proof. Suppose, G be a mPFT. Then the arcs of spanning mPFSG G' of G are the mPFB of G. So by Theorem 6.4.1, we have $p_i\circ CI_{mPF}(G) > p_i\circ CI_{mPF}(G')\ \forall\ i$.

Conversely, let $p_i\circ CI_{mPF}(G) > p_i\circ CI_{mPF}(G')\ \forall\ i$.

Then from the Theorem 6.4.1, we have the edge (s,t) is a mPFB and also $p_i\circ B(s,t) = p_i\circ CONN_(G)(s,t)\ \forall\ i$. Therefore G is a mPFT.

Example 6.4.3. *From Example 6.3, we see that $\forall\ i$, $p_i\circ CI_{mPF}(G) > p_i\circ CI_{G''}$. Then $G'\prime$ is a spanning of G. So, G is a mPFT.*

Corollary 6.4.1. *Let G be mPFG and $G' = G-(s,t)$ be its mPFSG, where (s,t) is an arc of G. Then $p_i\circ CI_{mPF}(G) = p_i\circ CI_{mPF}(G')$ iff $\forall\ i$, $p_i\circ B(x,y)\leq p_i\circ CONN_{G'}(s,t)$.*

Example 6.4.4. *From Example 6.3, we see that for the mPFSG G' of G, $B(s_2,s_5) = (0.6,0.5,0.3)$ and $CONN_{G'}(s_2,s_5) = (0.7,0.6,0.4)$. This implies $B(s_2,s_5) < CONN_{G'}$. Also, $CI_{mPF}(G) = (7.5,6.5,4.5) = CI_{mPF}(G')$. So corollary 6.4.1 is satisfied.*

Example 6.4.5. *From Example 6.3, we see that for the mPFSG G'' of G. Here $B(s_3, s_4) = (0.8, 0.7, 0.5)$ and $CONN_{G''}(s_3, s_4) = (0.7, 0.6, 0.4)$. This implies that, $B(s_3, s_4) > CONN_{G''}(s_3, s_4)$. So $CI_{mPF}(G) = (7.5, 6.5, 4.5) \neq (7.3, 6.3, 4.3) = CI_{mPF}(G'')$.*

Theorem 6.4.3. *Let $G = (V, A, B)$ and $G' = (V', A', B')$ be two isomorphic mPFGs. Then $p_i \circ CI_{mPF}(G) = p_i \circ CI_{mPF}(G')\ \forall\ i$.*

proof. Here, two isomorphic mPFGs are G and $G' = (V', A', B')$. Then, there a bijection mapping $h : G \to G'$ exists and so $\forall\ i$, $p_i \circ A(s) = p_i \circ A'(h(s))\ \forall\ s \in V$ and $p_i \circ B(s,t) = p_i \circ B'(h(s), h(t))\ \forall\ (s,t) \in \tilde{V}^2$. Since G is isomorphic to G', so we have $CONN_G(s,t) = CONN'_G(h(s), h(t))$, $\forall s, t \in V$. Therefore,
$\sum_{s,t\in V} (p_i \circ A(s))(p_i \circ A(t))(p_i \circ CONN_G(s,t)) = \sum_{h(s),h(t)\in V'} (p_i \circ A'(h(s)))(p_i \circ A'(h(t)))(p_i \circ CONN'_G(h(s), h(t)))$.
Hence, $CI_{mPF}(G) = CI_{mPF}(G')$.

Theorem 6.4.4. *Let G be a mPFG. Then $CI_{mPF}(G^{r_1}) \geq CI_{mPF}(G^{r_2})$ where $0 \leq r_1 \leq r_2 \leq 1$.*

proof. Let G be mPFG. Since $0 \leq r_1 \leq r_2 \leq 1$, G^{r_1} and G^{r_2} are the r_1 and r_2 cuts of G respectively. So G^{r^1} is a partial mPFSG of G^{r_2}. Then $CONN_{G^{r_1}}(s,t) \geq CONN_{G^{r_2}}(s,t)$. Then by Theorem , we have $CI_{mPF}(G^{r_1}) \geq CI_{mPF}(G^{r_2})$.

6.5 Average connectivity index of a mPFG

In this section, we explain the average connectivity index of mPFG. In Theorem , we see that, if a node is removed from a connected mPFG, then its connectivity index reduces. But if a vertex is excluded from a connected mPFG then its average connectivity index may reduce or increase or same. Here certain types of nodes mPFCRN, mPFCEN,mPFCNN are introduced and some properties on those nodes have been established.

Definition 6.5.1. *The average connectivity index of mPFG G denoted by $ACI_{mPF}(G)$ is defined as $ACI_{mPF}(G) = (p_1 \circ ACI_{mPF}(G), p_2 \circ ACI_{mPF}(G), \ldots p_m \circ ACI_{mPF}(G)) = \frac{1}{\binom{n}{2}}\Big(\sum_{s,t\in V} (p_1 \circ A(s))(p_1 \circ A(t))(p_1 \circ CONN_G(s,t)), \sum_{s,t\in V} (p_2 \circ A(s))(p_2 \circ A(t))(p_2 \circ CONN_G(s,t)), \ldots, \sum_{s,t\in V} (p_m \circ A(s))(p_m \circ A(t))(p_m \circ CONN_G(s,t))\Big)$*

Here, $p_i \circ ACI_{mPF}(G)$ is the i th component of average mPF Connectivity index and $|V| = n$. Also, $0 \leq p_i \circ ACI_{mPF}(G) \leq 1\ \forall\ i$.

Example 6.5.1. *From Example* 6.2.1, *for the mPFG G in (see Fig. 6.1), then* $ACI_{mPF}(G) = (\frac{20.3}{\binom{11}{2}}, \frac{14.7}{\binom{11}{2}}, \frac{11.9}{\binom{11}{2}}) = (0.369, 0.267, 0.216)$.

Definition 6.5.2. *Let $G = (V, A, B)$ be a connected mPFG. A node v is said to be a*

i) *m polar fuzzy Connectivity reducing node (mPFCRN) if* $p_i \circ ACI_{mPF}(G - v) < p_i \circ ACI_{mPF}(G)\ \forall\ i = 1, 2, 3, \ldots, m$.

ii) *m polar fuzzy Connectivity enhancing node (mPFCEN) if* $p_i \circ ACI_{mPF}(G - v) > p_i \circ ACI_{mPF}(G)\ \forall\ i = 1, 2, 3, \ldots, m$.

iii) *m polar fuzzy Connectivity neutral node (mPFCNN) if* $p_i \circ ACI_{mPF}(G - v) = p_i \circ ACI_{mPF}(G)\ \forall\ i = 1, 2, 3, \ldots, m$.

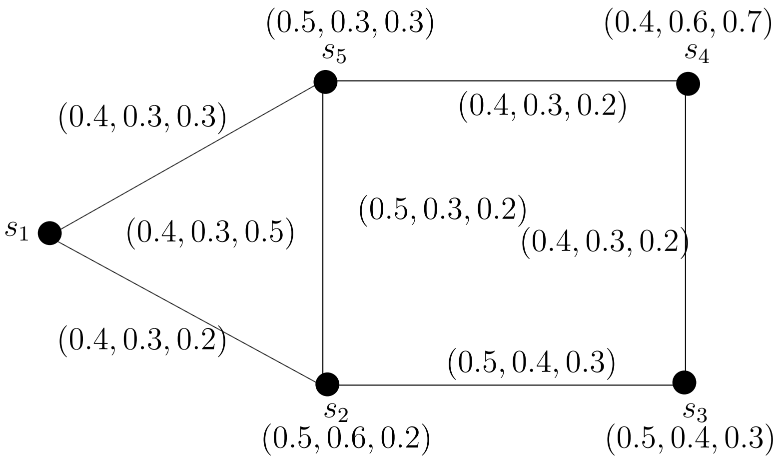

Figure 6.4: A Connected 3PFG G.

Example 6.5.2. *For the mPFG in fig6.4. we have,*

$ACI_{mPFG}(G) = (\frac{0.844}{10}, \frac{.603}{10}, \frac{0.328}{10}) = (0.0844, 0.0603, 0.0328)$

$ACI_{mPFG}(G - s_1) = (\frac{0.59}{6}, \frac{.396}{6}, \frac{0.154}{6}) = (0.098, 0.066, 0.0256)$

$ACI_{mPFG}(G - s_2) = (\frac{0.484}{6}, \frac{.279}{6}, \frac{0.247}{6}) = (0.0806, 0.0465, 0.0412)$

$ACI_{mPFG}(G - s_3) = (\frac{0.444}{6}, \frac{.351}{6}, \frac{0.202}{6}) = (0.074, 0.0585, 0.0336)$

$ACI_{mPFG}(G - s_4) = (\frac{0.615}{6}, \frac{.303}{6}, \frac{0.122}{6}) = (0.102, 0.0505, 0.0203)$

$ACI_{mPFG}(G - s_5) = (\frac{0.509}{6}, \frac{.42}{6}, \frac{0.202}{6}) = (0.0848, 0.07, 0.0336)$

Thus the node s_5 is mPFCEN of G.

Theorem 6.5.1. *Let $G = (V, A, B)$ be a connected mPFG and $|V| = n \geq 3$ and $r_i = \frac{p_i \circ CI_{mPF}(G)}{p_i \circ CI_{mPF}(G-s)}$, $s \in V$. The node s is a*

(i) *mPFCEN iff $r_i < \frac{n}{n-2}$ $\forall\, i$.*

(ii) *mPFCRN iff $r_i > \frac{n}{n-2}$ $\forall\, i$.*

(iii) *mPFCNN iff $r_i = \frac{n}{n-2}$ $\forall\, i$.*

Proof. **(i)** Suppose, s is a mPFCEN. Then, $\forall\, i$,

$$\begin{aligned}
& p_i \circ ACI_{mPF}(G) < p_i \circ ACI_{mPF}(G-s) \\
\Rightarrow\; & \frac{p_i \circ ACI_{mPF}(G)}{\binom{n}{2}} < \frac{p_i \circ ACI_{mPF}(G-s)}{}\binom{n-1}{2} \\
\Rightarrow\; & \frac{p_i \circ ACI_{mPF}(G)}{p_i \circ ACI_{mPF}(G-s)} < \frac{\binom{n}{2}}{\binom{n-1}{2}} \\
\Rightarrow\; & r_i < \frac{n}{n-2}.
\end{aligned}$$

Conversely, let $\forall\, i$,

$$\begin{aligned}
& r_i < \frac{n}{n-2} \\
\Rightarrow\; & \frac{p_i \circ ACI_{mPF}(G)}{p_i \circ ACI_{mPF}(G-s)} < \frac{\binom{n}{2}}{\binom{n-1}{2}} \\
\Rightarrow\; & \frac{p_i \circ ACI_{mPF}(G)}{\binom{n}{2}} < \frac{p_i \circ ACI_{mPF}(G-s)}{\binom{n-1}{2}} \\
\Rightarrow\; & p_i \circ ACI_{mPF}(G) < p_i \circ ACI_{mPF}(G-s) \\
\Rightarrow\; & s \text{ is a } mPFCEN \text{ of } G.
\end{aligned}$$

Similarly, *(ii)* and *(iii)* can be proved. □

Corollary 6.5.1. *From the above theory, we get,*

(i) If s is a mPFCNN then $r_1 = r_2 = \ldots = r_m$.

Theorem 6.5.2. *Let G be a connected mPFG, where $|V| = n \geq 3$. Let $p_i \circ = \sum_{s \in V-y} (p_i \circ A(s))(p_i \circ A(t))(p_i \circ CONN_G)(s,t)$ $\forall\, i$,*

(i) *$p_i \circ k < \frac{2}{n-2}$ iff t is an mPFCEN.*

(ii) *$p_i \circ k > \frac{2}{n-2}$ iff t is an mPFCRN.*

(iii) *$p_i \circ k = \frac{2}{n-2}$ iff t is a mPFCNN.*

Proof. Let $p_i \circ k < \frac{2}{n-2}$. Now

$$\begin{aligned} & p_i \circ CI_{mPF}(G) = p_i \circ CI_{mPF}(G-t) \\ & \qquad + \sum_{s \in V-t} (p_i \circ A(s))(p_i \circ A(t))(p_i \circ CONN_G(s,t)) \\ \Rightarrow\ & p_i \circ CI_{mPF}(G) = p_i \circ CI_{mPF}(G-t) + p_i \circ k \\ \Rightarrow\ & \frac{p_i \circ CI_{mPF}(G)}{\binom{n}{2}} = \frac{p_i \circ CI_{mPF}(G-t)}{\binom{n}{2}} + \frac{p_i \circ k}{\binom{n}{2}} \\ \Rightarrow\ & p_i \circ ACI_{mPF}(G) < \frac{p_i \circ CI_{mPF}(G-t)}{\binom{n-1}{2}} \frac{n-2}{n} + \frac{\frac{2}{n-2}}{\binom{n}{2}} \\ \Rightarrow\ & p_i \circ ACI_{mPF}(G) < p_i \circ CI_{mPF}(G-t) - \frac{2}{n}[p_i ACI_{mPF}(G-y) - \frac{2}{(n-1)(n-2)}] \\ \Rightarrow\ & p_i \circ ACI_{mPF}(G) < p_i \circ CI_{mPF}(G-t). \end{aligned}$$

Similarly, (ii) and (iii) can be proved. □

Definition 6.5.3. *Let G be a mPFG.*

(i) *If G has at least one mPFCEN, then G is called Connectivity enhancing m-polar fuzzy graph (CEmPFG).*

(ii) *If G has no mPFCRN, then G is called Connectivity reducing m-polar fuzzy graph (CRmPFG).*

(iii) *If every node is of G mPFCNN, then G is called neutral m-polar fuzzy graph (NmPFG).*

Example 6.5.3. *For the mPFG, G in Example 6.4,we show that s_5 is an mPFCEN of G. Hence G is a CEmPFG and G does not contain any mPFCRN vertex then G is also called CRmPFG but G is not a NmPFG.*

Theorem 6.5.3. *For the number $(k_1, k_2, \ldots, k_m)$ where k_i is a positive real number $\forall\ i$, there always a mPFG G exists s.t. $p_i \circ CI_{mPF}(G) = k_i$, $|V| = n$*

Proof. Let $p_i \circ A(s) = 1$, $\forall x \in V$ $andi = 1, 2, \ldots, m$. Now we think of a path in G s.t. $p_i \circ B(s,t) = \frac{k_i}{\binom{n}{2}}$ $\forall$ edges $(s,t) \in G$. Then, $p_i \circ CI_{mPF}(G) = \sum_{s,t \in V} (p_i \circ A(s))(p_i \circ A(t))(p_i \circ CONN_G(s,t)) = \binom{n}{2} \frac{k_i}{\binom{n}{2}} = k_i$. □

Theorem 6.5.4. *Let G be a mPFT and $G - x = G_1 \bigcup G_2$, where $G_1 = (V, A, B)$ and $G_2 = (V_2, A_2, B_2)$ are the connected components of $G - s$, where s is not an mPFEN. Let $p_i \circ K = \sum_{t \in V-s} (p_i \circ A(s))(p_i \circ A(t))(p_i \circ CONN_G(s,t))$. Then*

(i) $\sum\limits_{k=1}^{2}\sum\limits_{t,r\in V_i}(p_i\circ A_k(s))(p_i\circ A_k(t))(p_i\circ CONN_G(s,t)) > (\frac{n-2}{2}(p_i\circ K)$ *iff the node s is a mPFCEN.*

ii) $\sum\limits_{k=1}^{2}\sum\limits_{t,r\in V_i}(p_i\circ A_k(s))(p_i\circ A_k(t))(p_i\circ CONN_G(s,t)) < (\frac{n-2}{2}(p_i\circ K)$ *iff the node s is a mPFCRN.*

iii) $\sum\limits_{k=1}^{2}\sum\limits_{t,r\in V_i}(p_i\circ A_k(s))(p_i\circ A_k(t))(p_i\circ CONN_G(s,t)) = (\frac{n-2}{2}(p_i\circ K)$ *iff the node s is a mPFCNN.*

Proof. Since G is a mPFT and x is not an mPFEN of G, $G-s$ is disconnected. Again, as G_1 and G_2 are connected components of $G-s$, So $G_1\bigcup G_2$ and $G_1\bigcap G_2=\phi$. First assume that the node x is a mPFCEN. Then $p_i\circ ACI_{mPF}(G) < p_i\circ ACI_{mPF}(G-s)$ $\forall\ i$. This implies that

$$\frac{1}{\binom{n}{2}}[\sum_{k=1}^{2}\sum_{t,r\in V_k}(p_i\circ A_k(t))(p_i\circ A_k(r))(p_i\circ CONN_{G_i}(t,r))]+(p_i\circ K)$$
$$<\frac{1}{\binom{n-1}{2}}[\sum_{k=1}^{2}\sum_{t,r\in V_k}(p_i\circ A_k(t))(p_i\circ A_k(r))(p_i\circ CONN_{G_i}(t,r))]$$
$$\Rightarrow\ \sum_{k=1}^{2}\sum_{t,r\in V_k}(p_i\circ A_k(t))(p_i\circ A_k(r))(p_i\circ CONN_{G_i}(t,r))+(p_i\circ K)$$
$$<\frac{n}{n-2}[\sum_{k=1}^{2}\sum_{t,r\in V_k}(p_i\circ A_k(t))(p_i\circ A_k(r))(p_i\circ CONN_{G_i}(t,r))]$$
$$\Rightarrow\ \sum_{k=1}^{2}\sum_{t,r\in V_k}(p_i\circ A_k(t))(p_i\circ A_k(r))(p_i\circ CONN_{G_i}(t,r)) > \frac{n-2}{2}(p_i\circ K).$$

The converse part is shown when all these steps are reversed. Similarly, (ii) and (iii) can be proved. □

6.6 Summary

In several areas including decision making, computer networking and management, the model of fuzzy graphs plays a major role. The connectivity index in mPFG is illustrated. The boundary of negative and positive connectivity index of a mPFG has been clarified. Connectivity index in vertex and edge deleted mPFG with some properties has been investigated. The average connectivity index in mPFG and the special nodes mPFCEN, mPFCRN, mPFCNN are recounted with their properties.