Chapter 3

Types of arcs in m-polar fuzzy graphs*

3.1 Introduction

Graph operations are a very important topic in graph theory. Also, they are conveniently used in many combinatorial applications, operations research, algebra, geometry, number theory. They are suitable for construction in different situations. For example, we handle complex objects in partition theory. A typical object is a fuzzy graph and fuzzy hypergraph with a large chromatic number that is not able to precisely measure the chromatic number of such graphs. In these cases the main role of these operations is to resolve problems. Hence, in this chapter, at first mPFP, mPFC in an mPFG are defined. The strength of a connectedness of mPFP is introduced. Next, the strongest and strong mPFP, mPFBs, mPFCNs, mPFT and mPFFs in an mPFG are considered. Also, it is proved that an arc of mPF tree is strong mPFE iff it is an mPFB. Actually, mPF end nodes are established in mPFG and certain characteristics are investigated. At the end, there is also the application of the strongest path problem. Also we presented the idea of δ^*-strong mPFE,δ-strong mPFE, β-strong mPFE and α-strong mPFE of mPFGs. Next we studied several properties on these arcs. At the end, there is also an application of a strong mPFP problem.

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3.2 m-polar fuzzy bridges and m-polar fuzzy cut nodes

In this section, m-polar fuzzy bridges(mPFBs) and m-polar fuzzy cut nodes (mPFCNs) are described on mPFGs and some features are provided.

Definition 3.2.1. *Let s', t' be two different nodes in mPFG G. Let the (s', t') edge be removed from G then it is a partial mPF subgraph G' of G. That means $G' = (V, A, B')$ in which $\forall\ i = 1, 2, 3, \ldots, m$, $p_i \circ B(s', t') = 0$ and $p_i \circ B'(q', r') = p_i \circ B(q', r')$ for all other pairs (q', r'). The edge (s', t') is a mPFB in G if $\forall\ i$, $(p_i \circ B'(q, r))^\infty < (p_i \circ B(q', r'))^\infty$ for some $q', r' \in V$.*

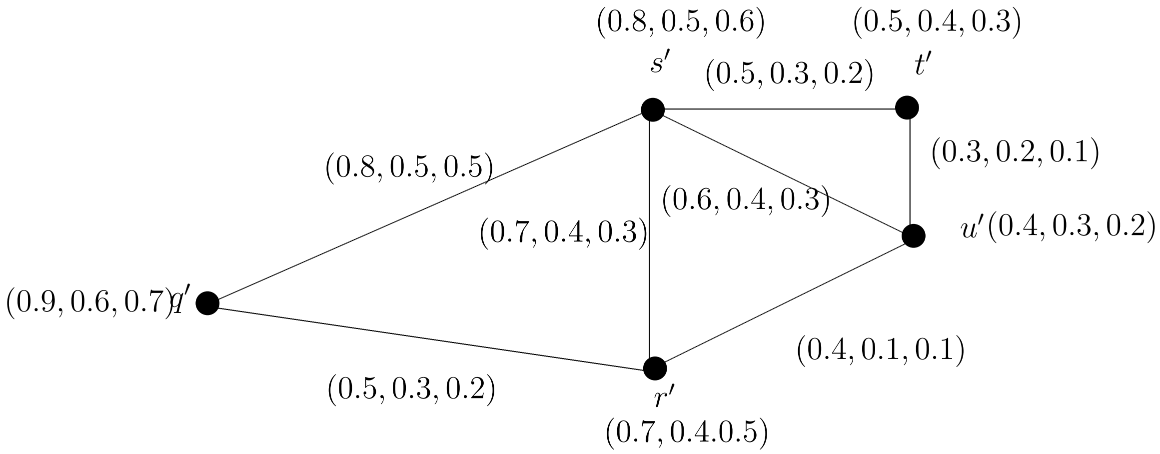


Figure 3.1: The arc (q', s') is bridge of G.

Example 3.2.1. *The Fig. 3.1 shows a 3PFG G of $G' = (V, E)$ where, $V = \{q', r', s', t', u'\}$ and $E = \{q's', s't', s'r', t'u', s'u', u'r', q'r'\}$.*

We consider all paths from q' to s'. They are $q'-r'-u'-t'-s'$, $q'-r'-u'-s'$, $q'-r'-s'$ and $q'-s'$ and strength of those paths are $(0.3, 0.1, 0.1)$, $(0.4, 0.1, 0.1)$, $(0.5, 0.3, 0.2)$ and $(0.8, 0.5, 0.5)$ respectively. So, $CONN_G(q', s') = (0.8, 0.5, 0.5)$ is the strength of connectedness between q' and s'. Now we are removing the (q', s') arc from G then the strength of connectedness between q' and s' in $G - (q', s')$ is $CONN_{G-(q',s')}(q', s') = (0.5, 0.3, 0.2)$. We see that $CONN_{G-(q',s')}(q', s') = (0.5, 0.3, 0.2) < (0.8, 0.5, 0.5) = CONN_G(q', s')$. So, (q', s') is a mPFB.

Definition 3.2.2. *A node $s' \in V$ is called the mPFCN of G if in the mPFG $G - s'$ getting from G by substituting $p_i \circ A(s') = 0 \ \forall \ i = 1, 2, 3, \ldots, m$, we have $p_i \circ CONN_{G-s'}(t', u') < p_i \circ CONN_G(t', u')$ for some $t', u' \in V$, $\forall \ i$ and $s' \neq t' \neq u'$.*

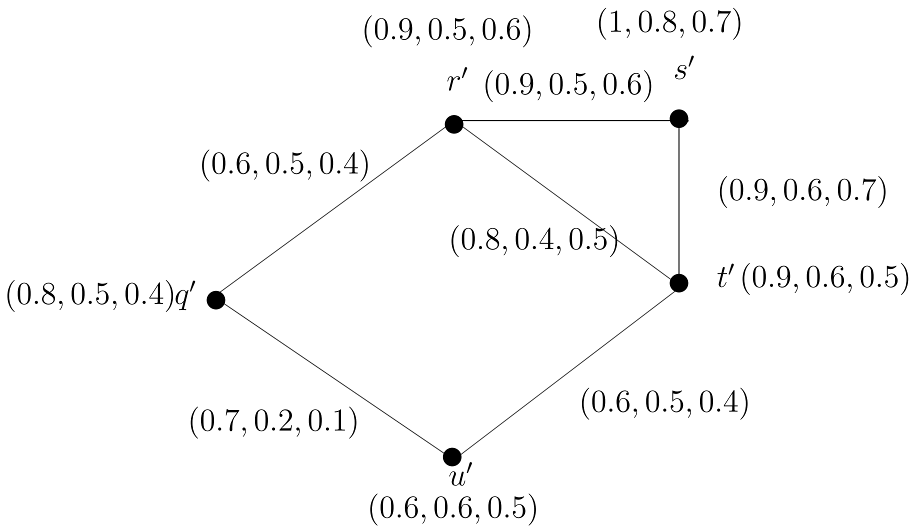


Figure 3.2: The vertex s' is a mPFCN of G.

Example 3.2.2. *The Fig. 3.2 shows a 3PFG G of G', where $V = \{q', r', s', t', u'\}$ and $E = \{q'r', r's', s't', r't', t'u', q'u'\}$. The paths from $r'-t'$, $r'-s'-t'$ and $r'-q'-u'-t'$ and strength of those paths are $(0.8, 0.4, 0.5)$, $(0.9, 0.5, 0.6)$ and $(0.6, 0.2, 0.1)$ respectively. So the strength of connectedness between r' and t' in G and $G-s'$ are $CONN_G(r', t') = (0.9, 0.5, 0.6)$ and $CONN_{G-s'}(r', t') = (0.8, 0.4, 0.5)$ respectively. So, s' is a mPFCN of G.*

Proposition 3.2.1. *If $G_1 = (C, D)$ is an mPFSG of $G = (A, B)$, then $\forall \ s, t \in V$ we have $p_i \circ CONN_{G_1}(s, t) \leq p_i \circ CONN_G(s, t)$.*

Theorem 3.2.1. *Let G be mPFG. Then the following statements are equivalent.*

(i) (s', t') *is an mPFB.*

(ii) $\left(p_i \circ B'(s', t')\right)^{\infty} < (p_i \circ B(s', t')) \ \forall \ i = 1, 2, 3, \ldots, m$. *Here $G = (V, A, B')$ is a partial mPFSG of G obtained by removing the edge (s', t').*

(iii) (s', t') *is not the weakest mPFE of any mPFC.*

Proof. (2) → (1)

Suppose (x', y') is not an mPFB, then $\forall\ i = 1, 2, 3, \ldots, m$,

$$(p_i \circ B'(s', t'))^{\infty} = (p_i \circ B(s', t'))^{\infty} \geq p_i \circ B(s', t').$$

It is a contradiction.

So, (s', t') is an mPFB.

(1) → (3)

If (s', t') is a weakest edge of an mPFE, then path with edge (s', t') can be transformed into a path not containing (s', t') but as strong at least, using the entire rest of the cycle as a path between s' and t'. Thus (s', t') could not be an mPFB.

(3) → (2)

If $\forall\ i$, $(p_i \circ B'(s', t'))^{\infty} \geq (p_i \circ B(s', t'))$, there is a path between s' and t' which does not contain (s', t'), $B_i^n(s', t') \geq p_i \circ B(s', t')$ and this path along with (s', t') is an mPFC whose (s', t') ia an weakest mPFE. □

Theorem 3.2.2. *Every mPFB in an mPFG G is a strong mPFE.*

Proof. When (s', t') is not strong, then $p_i \circ B(s', t') < p_i \circ CONN_{G-(s',t')}(s', t')\ \forall\ i$. Let P be the strongest mPFP between s' and t' in $G - (s', t')$. The strength of this path is $CONN_{G-(s',t')}(s', t')$. If we add (s', t') to P then we get a mPFC where (s', t') is the weakest mPFE of this mPFC, hence (s', t') is not an mPFB of G (by Theorem 4.5). This indicates that an mPFB must be a strong mPFE. □

Theorem 3.2.3. *If (s', t') is a strong mPFE in mPFG G iff $p_i \circ B(s', t') = p_i \circ CONN_G(s', t')\ \forall\ i$.*

Proof. We know, $p_i \circ CONN_G(s', t') \geq p_i \circ B(s', t')\ \forall\ i$. when an mPFP from s' to t' includes (s', t'), i-th component of strength of connectedness $\leq p_i \circ B(s', t')$. That is, $p_i \circ CONN_G(s', t') \geq p_i \circ B(s', t')\ \forall\ i$. If it does not have (s', t'), that implies it is in $G - (s', t')$. So i-th component of strength of connectedness $\leq p_i \circ CONN_{G-(s',t')}(s', t')$ $\leq p_i \circ B(s', t')$, since (s', t') is strong. Hence in each case the strength of a path between s' and t' is at most $B(s', t')$, so that $p_i \circ CONN_G(s', t') \leq p_i \circ B(s', t')\ \forall\ i$. Conversely, if $\forall\ i$, $p_i \circ B(s', t') = p_i \circ CONN_G(s', t')$ we get $p_i \circ B(s', t') \geq p_i \circ CONN_{G-(s',t')}(s', t')$. So (s', t') is a strong mPFE. □

Theorem 3.2.4. *Any two vertices s' and t' are connected by a strong mPFP in a connected mPFG G.*

Proof. Since G is connected mPFG, $\exists$ a path $P : s' = s_0, s_1, \ldots, s_n = t'$ from s' to t' s.t $p_i \circ B(s_{k-1}, s_k) > 0 \; \forall \; i = 1, 2, 3, \ldots, m$ and $1 \leq k \leq n$. If (s_{k-1}, s_k) is not strong then we get $p_i \circ B(x_{k-1}, x_k) < p_i \circ CONN_{H-(s_{k-1},s_k)(s_{k-1},s_k)}, \; \forall \; i$. Hence, a path P_j from s_{k-1} to s_k exist whose i-th component of strength of connectedness is larger than $p_i \circ B(s_{k-1}, s_k) \; \forall \; i$. If the path P_j does not have a strong mPFE then this statement can be repeated. The argument obviously can not arbitrarily be replicated frequently; So we can figure out that the s and t vertices link by a strong mPFP. □

Theorem 3.2.5. *At least two strong mPF neighbors are included in a mPFCN.*

Proof. Let the vertex s^* be deleted from G then $CONN_G(q^*, r^*)$ is reduced; this indicates there exists a strongest mPFP P from q^* to r^* which must be passes through s^*, say $q^*, \ldots, t^*, s^*, v^*, \ldots, r^*$. If (t^*, s^*) is not strong mPFP then we have $\forall \; i$, $p_i \circ B(t^*, s^*) < p_i \circ CONN_G(t^*, s^*)$ after deletion (t^*, s^*); so there is a path P' from t^* to s^*, except the (t^*, s^*) edge, whose i-th component of strength of connectedness is stronger than $p_i \circ B(t^*, s^*) \; \forall \; i$. Let the preceding node of s^* be t on P'; as the i-th strength of connectedness of P' is at most $p_i \circ B(t, s^*)$, then $p_i \circ B'(t^*, s^*) > p_i \circ B(t^*, s^*)$ must be provided. The claim would return if (t, s^*) is not strong mPFE. We eventually find t^* s.t (t', s) is strong mPFE because it can not endlessly be repeated. Similarly, we also found that v^* s.t (s, v') is strong mPFE. When $t' = v'$, we obtain a path P'' from q^* to r^* containing $t' = v'$ and the i-th component of strength of connectedness of P'' is stronger than P, this is means that deletion of s^* would not reduce $CONN_G(q^*, r^*)$, which contradict our statement. Hence s^* has at least two strong mPF neighbors. □

3.3 m-polar fuzzy trees and m-polar fuzzy forests

m-polar fuzzy trees(mPFTs) and m-polar fuzzy forests(mPFFs) on mPFG are described in the following section. In addition, some properties of mPFT and mPFFs on mPFGs are added .

Definition 3.3.1. *An mPFSG H of $G' = (V, E)$ is defined by an mPFSS $A : V \longrightarrow [0, 1]^m$ of V and an mPFSS $B : V \times V \longrightarrow [0, 1]^m$ of E s.t $\forall \; i = 1, 2, 3, \ldots, m$; $p_i \circ B(s', t') \leq min\{p_i \circ A(s'), p_i \circ A(t')\} \; \forall \; s', t' \in V$. H is called full mPFSG of G'*

if its mPF support is all of G', i.e. if for at least one i; $p_i \circ A(s') > 0 \ \forall \ s' \in V$ and $p_i \circ B(s', t') > 0 \ \forall \ (s', t') \in E$.

Definition 3.3.2. *A fuzzy subgraph $H = (V, A', B')$ is a partial mPFSG of an mPFG $G = (V, A, B)$ if $A' \subseteq A$ and $B' \subseteq B$. If $\forall \ i$, $p_i \circ A'(x') = p_i \circ A(s') \ \forall \ s'$ then H is said to be spanning mPFSG of G .*

Definition 3.3.3. *An mPFG G is an mPFF if it has an partial spanning mPFSG $H = (V, A, D)$ which is a forest, where for each edges (s', t') not in H (i.e. $D(s', t') = 0$), we get $p_i \circ B(s', t') < (p_i \circ D(s', t'))^\infty \ \forall \ i$. To put it another way, if (s', t') is in G but (s', t') is not in H and then a path in H exits between s' and t' whose i-th component of strength of connectedness is larger than $p_i \circ B(s', t') \ \forall \ i$.*

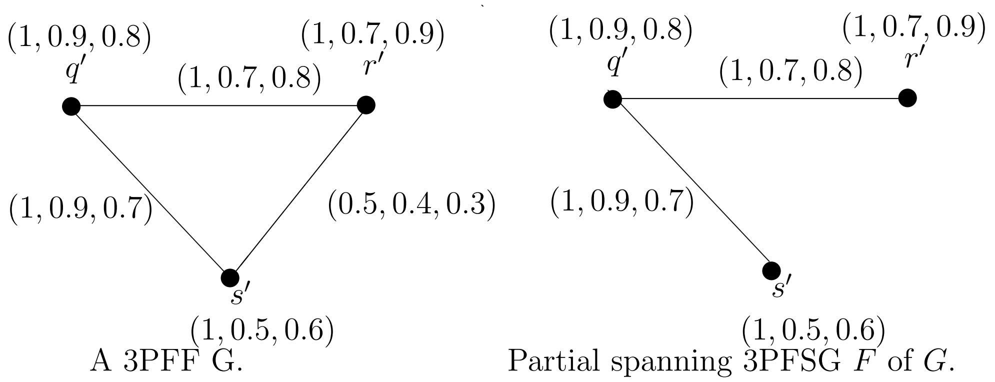


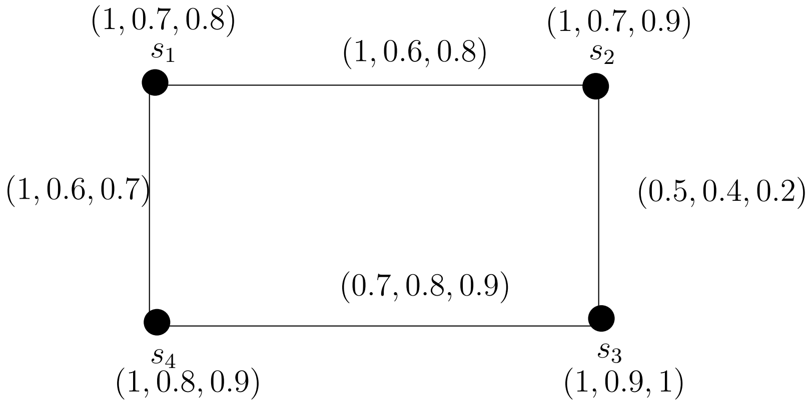
Figure 3.3: Illustration of example 3.3.1.

Example 3.3.1. *The Fig. 3.3 shows an 3PFG G of $G' = (V, E)$, where $V = \{q', r', s'\}$ and $E = \{q'r', r's', s'q'\}$. $F = (V, A, D)$ be the partial 3PFSG of G where (r', s') is not in F and $D(q', r') = (1, 0.7, 0.8)$ and $D(q', s') = (1, 0.9, 0.7)$ respectively. And now we see that clearly, $B(r', s') = (0.5, 0.4, 0.3) < CONN_F(r', s') = (1, 0.7, 0.7)$. So, G is an mPFF.*

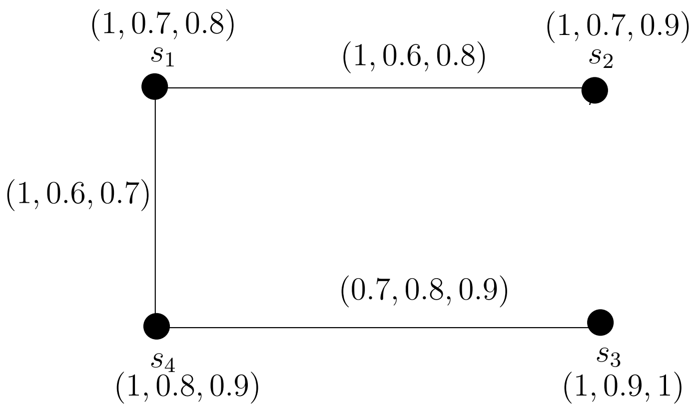
Definition 3.3.4. *A full mPFSG of G' is referred to as an m-polar F-tree or m-polar F-cycle if G' is a tree or cycle respectively.*

Let us say there are at least two vertices in a nontrivial tree and three vertices in a cycle.

Definition 3.3.5. *An mPFG G is an mPFT if it has a spanning mPFSG H' that is an m-polar F-tree, and is s.t* $p_i \circ B'(s,t) = 0$ *implies* $p_i \circ B(s,t) < p_i \circ CONN_{H'}(s,t)$ $\forall$ i,.



A 3PF tree G



Spanning 3PFSG H' of G

Figure 3.4: Illustration of example 3.3.2.

Example 3.3.2. *The Fig. 3.4 shows an 3PFG G of* $G' = (V, E)$ *where* $V = \{s_1, s_2, s_3, s_4\}$ *and* $E = \{s_1s_2, s_2s_3, s_3s_4, s_1s_4\}$.. $H' = (V, A, B')$ *be the spanning 3PFSG of G where* (s_2, s_3) *is not in* H' *and* $B'(s_1, s_4) = (1, 0.6, 0.7)$, $B'(s_1, s_2) = (1, 0.6, 0.8)$ *and* $B'(s_4, s_3) = (0.7, 0.8, 0.9)$. *Here,* H' *be a 3PFT and* $B(s_2, s_3) = (0.5, 0.4, 0.2) < (0.7, 0.6, 0.7) = CONN_{H'}(s_2, s_3)$. *Then G is an mPFT using the concept of mPFT.*

Theorem 3.3.1. *G be an mPFF iff in any mPFC of G, there is an arc (s', t') s.t $\forall\ i$, $p_i \circ B(s', t') < (p_i \circ B'(s', t'))^\infty$, where $G' = (V, A, B')$ is the partial mPFSG obtained by removal of the edge (s', t') from G.*

Proof. Let (s', t') be the edge. Assume the edge (s', t') belongs to an mPFC having the property that $p_i \circ B(s', t')$ is least $\forall\ i$. The resulting partial mPFSG fulfils the property of an mPFF if (s', t') is removed from G. If cycles are present in this graph, we may repeat the above procedure. Now, the edge that has been removed previously is not stronger than the current edge at every step. Thus, only edges which have not still been removed include the path guaranteed by the Theorem 's property. When there are no cycles in G, the getting partial mPFSG becomes an mPFF F. Let (s', t') edge is not in F, then (s', t') edge is removed to create F and between s' and t', there is an mPFP which is more stronger than $B(s', t')$ and which is not involving (s', t') or any of the edges removed before it. If the mPFP described above has induced edges that are later removed, they can be transformed around it using an mPFP of still stronger mPFE; The path can be diverted further if one of them was removed later and so on. At last, this method ultimately stabilizes with a path consisting entirely of edges of F. Thus G be an mPFF.

Conversely, if G is an mPFF and P is any mPFC, then some edge (s', t') of P is not belonging to F. Thus using the concept of an mPFF we have $\forall\ i$, $p_i \circ B(s', t') < (p_i \circ D(s', t'))^\infty \leq (p_i \circ B'(s', t'))^\infty$. □

Theorem 3.3.2. *When there are at most one strongest mPFP to any two vertices of G then the G must be an mPFF.*

Proof. Suppose G is not an mPFF. Then by the Theorem 5.8, there is an mPFC P in G s.t. $\forall\ i$, $p_i \circ B(s^*, t^*) \geq p_i \circ B'(s^*, t^*)$ $\forall$ arcs (s^*, t^*) of P. Thus (s^*, t^*) is a strongest mPFP from s^* to t^*. If we declare the edge (s^*, t^*) be a weakest mPFE of P, it means that the remaining P is also a strongest mPFP between s^* and t^*, a contradiction. So, if there is at most one strongest mPFP to any two vertices of G then the G must be an mPFF. □

Theorem 3.3.3. *The F edges would be just mPFBs of G, while G is a mPFF.*

Proof. An edge (s^*, t^*) that is not present in F cannot be an mPFB since $\forall\ i$, $p_i \circ B(s^*, t^*) < (p_i \circ D(s^*, t^*))^\infty \leq (p_i \circ B'(s^*, t^*))^\infty$. Assume that (s^*, t^*) is an arc in

F. If it was not an mPFB, we had an mPFP P between s^* and t^*, not belonging (s^*, t^*), then its i-th component of strength of connectedness $\geq p_i \circ B(s^*, t^*)\ \forall\ i$. The path must have no edges in F because F has no cycles and is an mPFF. However, by definition, any such (u_j, v_j) edge may be substituted by an mPFP F_j in F of i-th component of strength of connectedness $p_i \circ B(s^*, t^*)\ \forall\ i$. Now F_j is unable to include (s^*, t^*) since i-th strength of connectedness of all its edges are wholly stronger than $p_i \circ B(u, v) \geq p_i \circ B(s^*, t^*)$. Thus by changing every (u_j, v_j) by F_j, we can construct an mPFP in F from s^* to t^* that does not involve (a^*, b^*) which gives us an mPFC in F, a contradiction. □

Theorem 3.3.4. *If G is an mPFT, an arc of G is strong mPFE iff it is an arc of H' (spanning mPFSG of G).*

Proof. If (s^*, t^*) edge is not in H', we get $p_i \circ B(s^*, t^*) < p_i \circ CONN_{H'}(s^*, t^*)$; but because (s^*, t^*) is strong we must have $p_i \circ B(s^*, t^*) \geq p_i \circ CONN_{G-(s^*,t^*)}$ $(s^*, t^*) \geq p_i \circ CONN_{H'}(s^*, t^*)$ as the edge (s^*, t^*) does not involve to H', contraction. Conversely, suppose (s^*, t^*) is in H' but not a strong mPFE of G; thus $p_i \circ B(s^*, t^*) < p_i \circ CONN_{G-(s^*,t^*)}(s^*, t^*)$. The maximum strength of the from s^* to t^* in $G - (s^*, t^*)$ be P, be mPFP. The i-th strength of connectedness of P is $p_i \circ CONN_{G-(s^*,t^*)}(s^*, t^*)$, The weakest arc of the cycle, which is created by the adjacent (s^*, t^*) to P, is (s^*, t^*). According to the above theory, (s^*, t^*) is an mPFB, so by Theorem 4.4, (s^*, t^*) cannot make it the weakest mPFE of an mPFC. This is a contradiction. Hence (s^*, t^*) be the strong mPFE of G. □

Corollary 3.3.1. *An arc of mPFT is strong mPFE iff it is an mPFB.*

Proof. A strong mPFE of G must be an arc of H' (by Theorem 5.9), hence must be mPFB of G (by Theorem 4.3). By proposition 4.4, the converse is true by proposal 4.4,even if G is not a mPFT. □

Theorem 3.3.5. *G is an mPFT iff a unique mPFP is found in G between any two vertices of G.*

Proof. By Theorem 4.6, if the nodes s^* and t^* are in G then there a strong mPFP P exists between s^* and t^*. By the Theory 5.9, P is completely belonging in H', where

H' is the spanning m-polar F-tree. Since H' is an m-polar F-tree, a unique path in H' between s^* and t^* is available; therefore P is unique. Conversely, we noticed that a connected mPFG G is an mPFT iff in any mPFC of G $\exists$ an arc (s^*, t^*) for which $p_i \circ B(s^*, t^*) < CONN_{G-(s^*,t^*)}(s^*, t^*)$. Hance, if G is not an mPFT then an mPFC P exists in G s.t. $p_i \circ B(s^*, t^*) \geq CONN_{G-(s^*,t^*)}(s^*, t^*)$ for every edge (s^*, t^*) of P. That is every arc of P is strong mPFE. Thus two strong mPFPs exist between any two arbitrary vertices u^* and v^* on P, a contradiction. This leads to the result. □

3.4 Different types of arcs and their results

In this section, α-strong mPFE, β-strong mPFE and δ mPFE on mPFG are defined and some characterisation are given. some properties of α-strong mPFE, β-strong mPFE and δ mPFE on mPFG are introduced.

Definition 3.4.1. *Let G be an mPFG and (s, t) be an arc in G. If $\forall\ i = 1, 2, 3, \ldots, m$, $p_i \circ B(s, t) > p_i \circ CONN_{G-(s,t)}(s, t)$, $p_i \circ B(s, t) = p_i \circ CONN_{G-(s,t)}(s, t)$ and $p_i \circ B(s, t) < p_i \circ CONN_{G-(s,t)}(s, t)$ then the (s, t) arc is called α-strong mPFE, β-strong mPFE and δ mPFE respectively.*

Definition 3.4.2. *Let G be an mPFG and (r, s) be an arc in G. The arc (r, s) is a δ^*-mPFE if $\forall\ i = 1, 2, 3, \ldots, m$, $p_i \circ B(r, s) > p_i \circ B(p, q)$ where (p, q) is a weakest mPFE of G.*

Definition 3.4.3. *A path in an mPFG G is named an α-strong mPFP when all of the arcs in it are α-strong mPFE and is named a β-strong mPFP when all of the arcs in it are β-strong mPFE.*

Example 3.4.1. *The Fig. 3.5 shows a 3PFG G of the crisp graph $G' = (V, E)$ where $V = \{p, q, r, t\}$ and $E = \{pq, qr, rt, tp, pr, qt\}$. Here, (q, t) and (q, r) are α-strong mPFEs, (p, q) and (p, t) are β-strong mPFEs and (r, p) and (r, t) are δ-strong mPFEs. Again arc (r, t) is a δ^* arc as $B(r, t) = (0.3, 0.4, 0.5) > (0.1, 0.2, 0.3) = B(p, r)$, where (p, r) is a weakest mPFE of G.*

Definition 3.4.4. *A maximum spanning mPFT of a connected mPFFG G is an spanning mPFSG T of G, that is a m polar F-tree, s.t $CONN_G(s, t)$ is the strength of the unique strongest st mPFP in T $\forall\ s, t \in G$.*

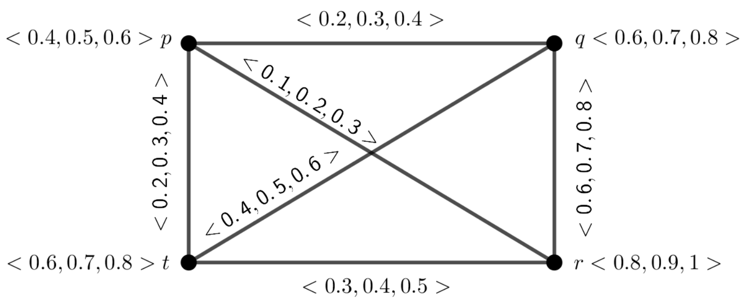


Figure 3.5: Different types of arc on mPFG G

Theorem 3.4.1. *An arc (s,t) in an mPFG G is an strongest $s-t$ mPFP iff (s,t) is either α-strong mPFE or β-strong mPFE.*

Proof. Let G be an mPFG and (s,t) be an arc in G. Consider P be a path between s and t. Then using the concept of strength of an mPFP, $\forall\ i = 1, 2, 3, \ldots, m$

$$ith\ component\ of\ strength\ of\ P\ =\ p_i \circ B(s,t). \tag{3.1}$$

Let P^* is a strongest mPF path, then the i th component of strength of connectedness of $P^* = p_i \circ CONN_G(s,t)$. From 3.1, $\forall\ i$

$$p_i \circ B(s,t) = p_i \circ CONN_G(s,t). \tag{3.2}$$

The i th component of strength of connectedness of $P^* \geq i$ th component of strength of connectedness of all other uv paths. In particular, $\forall\ i$, i th component of strength of connectedness of $P^* \geq CONN_{G-(s,t)}(s,t)$. Thus $\forall\ i$

$$p_i \circ CONN_G(s,t) \geq p_i \circ CONN_{G-(s,t)}(a,b). \tag{3.3}$$

Now from 3.2 and 3.3 we have,

$p_i \circ B(s,t) \geq\geq p_i \circ CONN_{G-(s,t)}(s,t)$

$\Rightarrow$ Arc (s,t) is either α-strong mPFE or β-strong mPFE.

Conversely, assume that arc (a,b) is either β-strong mPFE or α-strong mPFE. Then $\forall\ i$, $p_i \circ B(s,t) \geq p_i \circ CONN_{G-(s,t)}(s,t)$. $\Rightarrow p_i \circ CONN_G(s,t) = p_i \circ B(s,t)$.

i.e, $p_i \circ CONN_G(s,t)$ is the i- th component of strength of connectedness of P^*.

So, P^* is a mPFP in G, which is the strongest mPFP. □

Theorem 3.4.2. *Let an mPFG be G and P^* be a s_0s_n mPFP. Let (s,t) be any arc in P^* such that i th component of strength of $P^* = p_i \circ B(s,t)$. Then P^* is a strongest s_0s_n mPFP if (s,t) is a strong mPFE as well as it is the only one weakest arc of P^*.*

Proof. Here, G is an mPFG. Let $P^* : s_0 - s_1 - s_2 - s_3 - ... - s_n$ be a a_0a_n mPFP in G with ith component strength of $P^* = p_i \circ B(s_{j-1}, s_j)$ for some $j = 1, 2, 3, \dots, n$ and $i = 1, 2, 3, \dots, m$. Let a strong mPFE be (s_{j-1}, s_j) and which is an the unique weakest arc in P^*.

To prove P^* is the strongest s_0s_n mPFP. Let P^* is not the strongest s_0s_n mPFP. Let $P1 : s_0 - t_1 - t_2 - t_3 - ... - t_{n-1} - s_n$ be a strongest s_0s_n mPFP in G, in which every of $s_k,\ k = 1, 2, 3, \dots, n-1$ and t_j, $j = 1, 2, 3, \dots, n-1$ may be same. As ith component of strength of P_1 is greater than ith component of strength of P^*, we have ith component of strength of each arc of $P_1 > p_i \circ B(s_{j-1}, s_j)$. Also remark that arc (s_{j-1}, s_j) is an uncommon arc of P^* and P_1. Therefore $P^* \bigcup P1$ will contain at least one mPFC and let C be one similar mPFC, where (s_{i-1}, s_i) is the only weakest mPFE. Consider a $s_{j-1}s_j$ path P' in C not having the arc (s_{j-1}, s_j). Obviously $p_i \circ B(s_{j-1}, s_j) <$ ith component of strength of P' and ith component of strength of $P' \leq p_i \circ CONN_{G-(s_{j-1},s_j)}(s_{j-1}, s_j)$. $p_i \circ B(s_{j-1}, s_j) < CONN_{G-(s_{j-1},s_j)}(s_{j-1}, s_j)$, which implies (s_{j-1}, s_j) is a δ- mPFE, that contradicts that (s_{j-1}, s_j) is a strong mPFE. Therefore, P^* is the strongest s_0s_n mPFP in G. □

Theorem 3.4.3. *An arc (s, t) in an mPFG G is a δ-strong mPFP iff (s, t) is the unique weakest arc of at least one cycle in G.*

Proof. Suppose G is an mPFG. Also, let (s, t) arc be a δ-strong mPFE in G. Therefore, using the definition, $p_i \circ B(s, t) < p_i \circ CONN_{G-(s,t)}(s, t)$. i.e, there at least a path P exists between s and t and which does not contain the arc (s, t) with ith component of the strength of $P > p_i \circ B(s, t)$. This path P together with the arc (s, t) makes a mPFC where (s, t) is the unique weakest arc. Conversely, let (s, t) be the only one weakest arc of a cycle C in G. Let P be the st path in C not having the arc (s, t). Then,

$$p_i \circ B(s, t) < i - th\ component\ of\ strength\ of\ P \tag{3.4}$$

Let (s, t) be not a δ-arc in G. Then from definition we have,

$$p_i \circ B(s, t) \geq p_i \circ CONN_{G-(s,t)}(s, t) \tag{3.5}$$

Also remark that

$$i - th\ component\ of\ strength\ of\ P \leq p_i \circ CONN_{G-(s,t)}(s, t) \tag{3.6}$$

From 3.5 and 3.6, we get $p_i \circ B(s,t) \geq i$th component of strength of P, which contradicts 3.4.

Hence, (s,t) is a δ-strong mPFE in G. □

Theorem 3.4.4. *A strong mPFP P_1 from s_1 to t_1 is a strongest s_1t_1 mPFP if P_1 contains only α- strong mPFEs.*

Proof. Let G be an mPFG. Here P_1 be a strong mPFP between s_1 and t_1 and P_1 contains only α- strong mPFEs. At first we thought that P_1 is not the strongest mPFP. Let Q_1 be a strongest s_1t_1 mPFP in G. Then $P_1 \bigcup Q_1$ will have at least one cycle C and each arc of $C - P_1$ will have strength which is larger than the strength of P_1. Thus a weakest arc of C is also an arc of P_1. Suppose C contains an arc (q,r). Let C_1 be the $q-r$ mPFP in C where C_1 does not contain the arc (q,r). Then,

$p_i \circ B(q,r) \leq i \; th \; component \; of \; strength \; C_1 \leq p_i \circ CONN_{G-(q,r)}(q,r)$.

This means that (q,r) is not a α-strong mPFE, which is a contradiction. Thus P_1 is the strongest x_1y_1 mPFP. □

Theorem 3.4.5. *A strong mPFP P_1 from x_1 to y_1 is a strongest x_1y_1 mPFP if P_1 is the unique strong x_1y_1 mPFP.*

Proof. Let P_1 be a unique strong x_1y_1 mPFP in an mPFG G. If P_1 is not the strongest x_1y_1 mPFP in G. Let Q_1 be the strongest x_1y_1 mPFP in G. Then, ith component of strength of $Q_1 > i$th component of strength of P_1. i.e. for any arc (u_1,v_1) in Q_1, $p_i \circ B(u_1,v_1) > p_i \circ B(x_1^*,y_1^*)$, where (x_1^*,y_1^*) is a weakest mPFE of P_1.

Now we claim that Q_1 is a strong x_1y_1 mPFP. For otherwise, if there an arc (u_1,v_1) exists in Q_1 which is a δ mPFE, then

$p_i \circ B(x,y) < p_i \circ CONN_{G-(u,v)}(u,v) \leq p_i \circ CONN_G(u,v)$ and hence $p_i \circ B(u,v) < p_i \circ CONN_G(u,v)$.

Then there is a path that exists from u_1 to v_1 in G whose ith component of strength is longer than $p_i \circ B(u,v)$. Let it be P_1'. Let w_1 be the next node after u_1, common to Q_1 and P_1' in the u_1w_1 sub mPFP of P_1' and w_1' be the node before v_1, common to Q_1 and P_1' in the $w_1'v$ sub mPFP of P_1'. (If P_1' and Q_1 are disjoint u_1v_1 mPFP then $w_1 = u_1$ and $w_1' = v$). Suppose the path P_1'' is consisting of the x_1w_1 mPFP of Q_1, w_1w_1' path of P_1' and $w_1'y_1$ mPFP of Q_1. Then P_1'' is an x_1y_1 mPFP in G such that ith component of strength of $P_1'' > i$th component of strength of Q_1, contradiction to the assumption that Q_1 is a strongest x_1y_1 mPFP in G. Thus (u_1, v_1) cannot be a δ mPFE and so Q_1 is a strong x_1y_1 mPFP in G.

Next, We have therefore another path from x_1 to y_1, other than P, which is a contradiction to the assumption that P is the unique strong x_1y_1 mPFP in G. Hence, P should be the strongest x_1y_1 mPFP in G.

□

Theorem 3.4.6. *A strong mPFP P_1 from s_1 to t_1 is a strongest s_1t_1 mPFP if all s_1t_1 mPFPs in G are of equal strength.*

Proof. If every mPFP from x_1 to y_1 have the same strength, then each such path is strongest x_1y_1 mPFP. In particular, a strong x_1y_1 mPFP is a strongest x_1y_1 mPFP. □

Theorem 3.4.7. *For an mPF bridge (x_1, y_1), then $p_i{\circ}B(x_1, y_1) = p_i{\circ}CONN_G(x_1, y_1)$ $\forall\ i = 1, 2, 3, \ldots, m$.*

Proof. Here (x_1, y_1) is an mPFB. So, ith component of $CONN_G(x_1, y_1)$ exceeds ith component of $B(x_1, y_1)$ $\forall\ i$. So there is a strongest x_1y_1 mPFP in which ith component of strength is longer than ith component of $B(x_1, y_1)$ and each arcs of the strongest x_1y_1 mPFP have ith component of strength is more than ith component of $B(x_1, y_1)$ $\forall\ i$. Now this path forms an mPFC together with the arc (x_1, y_1) where, (x_1, y_1) is the weakest mPFE. This contradicts that (x_1, y_1) is an mPFB. □

Theorem 3.4.8. *If w is a common vertex of at least two mPFBs, then w is an mPFCN.*

Proof. Suppose (t_1^*, w) and (w, t_2^*) are two arcs in G and these two arcs are mPFBs. So there is some s, t for which (t_1^*, w) is present on each strongest st mPFP. If the node w is different from s and t, then w is an mPFCN. Let one of s, t is w such that (t_1^*, w) is lie on each strongest sw mPFP or (w, t_2^*) is lie on each strongest wt mPFP. Next, suppose w is not an mPFCN, so there is at least one strongest mPFP between any two vertices which does not containing w. Especially, there at least one strongest mPFP

P exists between t_1^* and t_2^*, not containing w. That path forms an mPFC together with (t_1^*, w) and (w, t_2^*).

Here we consider two cases.

Case1: Let $t_1^* - w - t_2^*$ is a strongest mPFP between t_1^* and t_2^*. Then $p_i \circ CONN_G(t_1^*, t_2^*) = p_i \circ B(t_1^*, w) \wedge p_i \circ B(w, t_2^*)$, which is the strength of P. Hence, edges of P are strong from $B(t_1, w)$ and $B(w, t_2)$, which implies that (t_1^*, w) and (w, t_2^*) are both weakest mPFEs of an mPFC, which is a contradiction.

Case 2: Let $t_1^* - w - t_2^*$ is not the strongest mPFP. Now One of (t_1^*, w), (w, t_2^*) or both become weakest mPFEs of an mPFC as $t_1^* - w - t_2^*$ is not a strongest mPFP, which contradicts that (t_1^*, w) and (w, v_2^*) are mPFBs. Hence, the result follows. □

Theorem 3.4.9. *Let (s_1, t_1) be an arc in an mPFG G. Then (s_1, t_1) is an mPFB iff it is α-strong mPFE.*

Proof. Let G be an mPFG and (s_1, t_1) is an mPFB in G. Then by Theory 3.2.1, we have $\forall\ i = 1, 2, 3 \ldots, m$

$$p_i \circ CONN_{G-(s_1,t_1)}(s_1, t_1) < p_i \circ CONN_G(s_1, t_1) \tag{3.7}$$

By Theorem 3.4.7,

$$p_i \circ CONN_G(s_1, t_1) = p_i \circ B(s_1, t_1) \tag{3.8}$$

From 3.7 and 3.8

$$p_i \circ B(s_1, t_1) > p_i \circ CONN_{G-(s_1,t_1)}(s_1, t_1)$$

which shows that (s_1, t_1) is an α-strong mPFE.

Conversely, we consider that (s_1, t_1) is α-strong mPFE. Then using the definition, (s_1, t_1) is the unique strongest mPFP between s_1 and t_1 and the removal of (s_1, t_1) will reduces the strength of connectedness between s_1 and t_1. Thus (s_1, t_1) is an mPFB. □

Theorem 3.4.10. *If G is an mPFT. Now if we remove any mPFB from G then the strength of connectedness between its end vertices is reduced and the strength of connectedness between some other pair of vertices is also reduced.*

Proof. Let (s^*, t^*) be an mPFB in G. Then using the above Theorem, we say that s^*t^* is the edge of the maximum spanning mPFT T^* of G. These maximum spanning mPFT T^* contains unique strongest mPFPs and which strongest mPFPs joining each pair of nodes. Next if we remove (s^*, t^*) from G then the strength of connectedness between some other pair of vertices q^*, r^* is reduces where, q^* and r^* are adjacent with s^* and t^* respectively if an internal edge of T^* is (s^*, t^*) and $s^* = q^*$ or $t^* = r^*$ otherwise. □

Theorem 3.4.11. *The internal nodes of F are mPFCN of an mPFT G.*

Proof. Suppose w^* is not an mPFEN of F where w^* is in G . Then the node w^* is common node of at least two arcs in F, which are mPFBs in G and by Theorem 3.4.8, w^* is an mPFCN. Next, if w^* is an mPFEN of F, then w^* is not an mPFCN, else there would exist u_1 and v_1 distinct from w^* s.t. w^* lies on every u_1v_1 mPFP and one such path lies in F. But w^* is an mPFEN of F, which is not possible. □

Corollary 3.4.1. *An mPFCN of an mPFT is the common vertex of at least two mPFBs.*

Theorem 3.4.12. *If G is an mPFT. If any α-strong mPFE is removed from G then the strength of connectedness between its end vertices is reduced and the strength of connectedness between some other pair of vertices is also reduced.*

Proof. If G is an mPFT. An arc (s_1, t_1) of G is an α-strong mPFE then it is an mPFB in G(by Theorem 3.4.9). Again by Theorem 3.3.3, removing any mPFB decreases the strength of connectedness between its end vertices and also between some other pair of vertices in G. Then we easily say that if we remove an α-strong mPFE from G that means we delete an mPFB from G. So, if we remove any α-strong mPFE from G then the strength of connectedness between its end vertices is reduced and the strength of connectedness between some other pair of vertices is also reduced. □

Theorem 3.4.13. *A mPFCN of an mPFT is incident to at least two α-strong mPFEs.*

Proof. If G be an mPFT. An arc (s_1, t_1) of G is an mPFB then it is a α-strong mPFE in G(by Theorem 3.4.9). By Corollary 3.4.1, an mPFCN of an mPF tree is incident to at least two mPFBs. So an mPFCN of an mPFT is incident to at least two α-strong mPFE because an arc of G is an mPFB then it is a α-strong mPFE in G. □

Theorem 3.4.14. *Let G be an mPFT. An arc (s_1, t_1) in G is α-strong mPFE iff (s_1, t_1) represents an edge of the spanning tree F of G.*

Proof. Let (s_1, t_1) be an α-strong mPFE in G. Then $\forall\ i = 1, 2, 3, \ldots, m$

$$p_i \circ B(s_1, t_1) > p_i \circ CONN_{G-(s_1,t_1)}(s_1, t_1) \tag{3.9}$$

Suppose (s_1, t_1) does not belong to F. Then from the definition of an mPFT,

$$p_i \circ CONN_F(s_1, t_1) > p_i \circ B(s_1, t_1) \tag{3.10}$$

Now from Proposition 3.2.1, $\forall\ i$

$$p_i \circ CONN_F(s_1, t_1) \le p_i \circ CONN_{G-(s_1,t_1)}(s_1, t_1). \tag{3.11}$$

From 3.10 and 3.11 we get $p_i \circ B(s_1, t_1) < p_i \circ CONN_{G-(s_1,t_1)}(s_1, t_1)$ which contradicts to 3.9. Hence (s, t) is in F.

Conversely, let (s_1, t_1) be in F. Then (s_1, t_1) is an mPFB and arc (s_1, t_1) is the unique strongest s_1t_1 mPFP. Then, $\forall\ i = 1, 2, 3, \ldots, m$

$$p_i \circ CONN_{G-(s_1,t_1)}(s_1, t_1) < p_i \circ B(s_1, t_1)$$

which implies that (s_1, t_1) is α-strong mPFE. □

Theorem 3.4.15. *An mPFG G is an mPFT iff it has no β-strong mPFEs.*

Proof. Let G be an mPFT and let F be its maximal spanning mPFT. Here all edges in F are α-strong mPFE (by Theorem 3.4.9). Suppose (s_1, t_1) is a β-strong mPFE in G. Then (s_1, t_1) is not in F and by concept of an mPFT, we have

$$p_i \circ B(s_1, t_1) < CONN_F(s_1, t_1). \tag{3.12}$$

Now from Proposition 3.2.1, $\forall\ i = 1, 2, 3, \ldots, m$

$$p_i \circ CONN_F(x_1, y_1) \leq p_i \circ CONN_{G-(x_1,y_1)}(x_1, y_1). \quad (3.13)$$

From 3.12 and 3.13,

$$p_i \circ B(x_1, y_1) < p_i \circ CONN_{G-(x_1,y_1)}(x_1, y_1)$$

which says that (x_1, y_1) is a δ-strong mPFE, this is a contradiction. Thus, G contains no β-strong mPFEs.

Conversely, we consider that G does not contain any β-strong mPF arcs. If G has no mPFCs then G is an mPFT. Now assume that G has mPFCs. Let C_1 be an mPFC in G. Then C_1 will only contain α-strong mPFEs and δ-strong mPFEs. Also, all arcs of C_1 cannot be α-strong mPFEs since otherwise it contradicts the concept of α-strong mPFEs. So there exist at least one δ-strong mPFEs in C_1. Then applying the Theorem 3.3.1 then we get that G is an mPFT. □

3.5 An application

A fuzzy graph theory is now a few days essential to solve a lot of network-based problems, including networking of gas pipelines, social and road networks. At present social networks are growing in human life very quickly. People can exchange information very rapidly by the help of social networks and can be utilized for many purposes like spreading of news, sharing of data, thoughts, profession interests etc. These networks can be described as a graph in which each user is seen as vertices, and the relationship between two users consists of an edge.

1). We present here a 3PFG model which is used to detect the strong relationship between two users. Fig. 3.6 shows a model of the social network which is represented by a 3PFG $G = (V, A, B)$. Here each node represents one of the users of Whatsapp, Facebook, Instagram from a set of 9 and interrelationship between these users expressed by joining edges between them. We consider 9 users in this network denoted as $V = \{a, b, c, d, e, f, g, h, i\}$. The membership value of each edge is characterized by three criteria :{how much time they stay connected in Facebook per day, how much time they stay connected in Whatsapp per day, how much time they stay connected in Instagram per day}. Since all the above characteristics of an edge between two users are uncertain in real life. We can measure edge membership values, using the relation $p_i \circ B(s,t) \leq min\{p_i \circ A(s), p_i \circ A(t)\}$ for each $(s,t) \in E$, $i = 1,2,3$ where these values together represent the interconnections of the two users.

Here, the network contains 9 nodes and 15 arcs. From the graph below, it can be seen that every user is connected by certain paths. So, we want to check whether the relationship between them is α-strong, β-strong or δ-strong. At first we consider the edge (a,b). Now we want to check whether this arc is α-strong, β-strong or δ-strong. The strength of connectedness of (a,b) in $G-(a,b)$ is $CONN_{G-(a,b)}(a,b) = (0.3, 0.2, 0.1)$ and $B(a,b) = (0.5, 0.4, 0.3)$. So $B(a,b) = (0.5, 0.4, 0.3) > (0.3, 0.2, 0.1) = CONN_{G-(a,b)}(a,b)$ that means (a,b) is α-strong mPFE. So, the relation between a and b strong but which is α-strong type. In this way, we calculate whether these other arcs are α-strong, β-strong or δ-strong.

From the Table 3.1, we see that the relation between h and i is α-strong when they are connected to each other in social networks. This implies that h spends more time with i. If the relation between two users is β-strong which means that they spend an adequate time per day with each other. Similarly, if the relation between two users is δ-strong which means they spend very less time with each other. In this way we easily find out the relation between two users in a social network.

2). Next We present here a 3PFG model which is used to detect the strongest path between two cities. The model of the road network, represented with the 3PF graph $G = (V, A, B)$, is shown in Fig. 3.7. In this respect, the nodes represent towns in a country and the corresponding edges show the roads between two towns. We consider 6 cities of a country denoted as $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$. The membership value of each road is characterized by three criteria :{road hazards, traffic jam on road, quality of the road}. We can measure edge membership values, using the relation $p_i \circ B(s,t) \leq min\{p_i \circ A(s), p_i \circ A(t)\}$ for each $(s,t) \in E$, $i = 1,2,3$ where these values together represent the interconnections of the two cities.

Edge	Membership value	Strength of connectedness after deleting the edge	Types of Strong arc
(a, b)	$(0.5, 0.4, 0.3)$	$(0.3, 0.2, 0.1)$	α-strong
(a, h)	$(0.3, 0.2, 0.1)$	$(0.5, 0.4, 0.3)$	δ-strong
(b, c)	$(1, 0.9, 0.8)$	$(1, 0.9, 0.8)$	β-strong
(h, i)	$(1, 0.9, 0.8)$	$(0.6, 0.5, 0.4)$	α-strong
(i, c)	$(1, 0.9, 0.8)$	$(1, 0.9, 0.8)$	β-strong
(h, b)	$(0.6, 0.5, 0.4)$	$(1, 0.9, 0.8)$	δ-strong
(b, d)	$(0.5, 0.4, 0.3)$	$(0.3, 0.2, 0.1)$	α-strong
(c, d)	$(0.3, 0.2, 0.1)$	$(0.5, 0.4, 0.3)$	δ-strong
(d, e)	$(0.3, 0.2, 0.1)$	$(0.5, 0.4, 0.3)$	δ-strong
(f, e)	$(0.5, 0.4, 0.3)$	$(0.7, 0.6, 0.5)$	δ-strong
(i, f)	$(0.8, 0.7, 0.6)$	$(0.5, 0.4, 0.3)$	α-strong
(i, e)	$(0.7, 0.6, 0.5)$	$(0.5, 0.4, 0.3)$	α-strong
(g, h)	$(0.7, 0.6, 0.5)$	$(0.5, 0.4, 0.3)$	α-strong
(g, f)	$(0.5, 0.4, 0.3)$	$(0.7, 0.6, 0.5)$	δ-strong
(b, i)	$(1, 0.9, 0.8)$	$(1, 0.9, 0.8)$	β-strong

Table 3.1: Strong arcs in G.

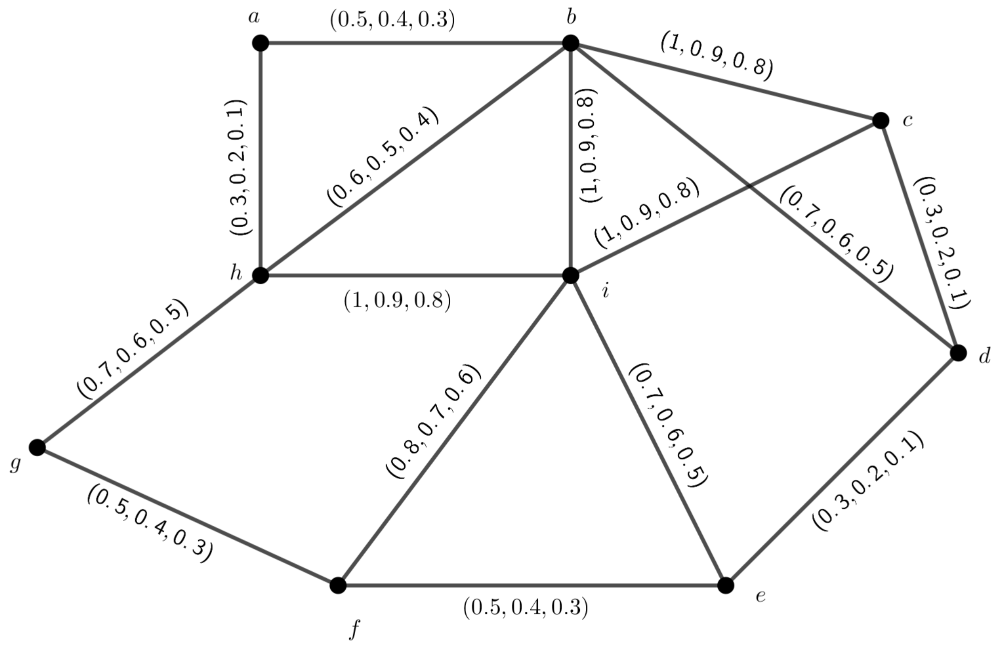


Figure 3.6: A 3PF graph G of a social network

Assume, a flood affects the town v_1. Then the disaster site needs different kinds of necessary things such as food, medical care kits, dry towels, tents, etc. That is why the strongest path between other cities and this disaster site will support this disaster site. Here, v_1 is a disaster site. The strongest path from $v_1$to other vertices is to find next. From Fig. 3.7, we see that:

- There are four paths between $v_1$and v_2. This four paths are $P_1 : v_1 \to v_2$, $P_2 : v_1 \to v_6 \to v_3 \to v_2$, $P_3 : v_1 \to v_6 \to v_5 \to v_2$ and $P_4 : v_1 \to v_6 \to v_5 \to v_4 \to v_3 \to v_2$. The strength of the paths P_1, P_2, P_3 and P_4 are $(0.7, 0.6, 0.6)$, $(0.6, 0.6, 0.6)$,(0.3,0.3,0.1) and $(0.5, 0.6, 0.4)$ respectively. Here $CONN_G(v_1, v_2) = (0.7, 0.6, 0.6)$ and P_4 be the strongest path between v_1 and v_2. So the city v_2 send necessary things to v_1 along path P_4.
- The strongest path from v_1 to v_3 is $P : v_1 \to v_2 \to v_3$.
- The strongest path from v_1 to v_4 is $P : v_1 \to v_6 \to v_5 \to v_4$.
- The strongest path from v_1 to v_5 is $P : v_1 \to v_6 \to v_5$.
- The strongest path from v_1 to v_6 is $P : v_1 \to v_6$.

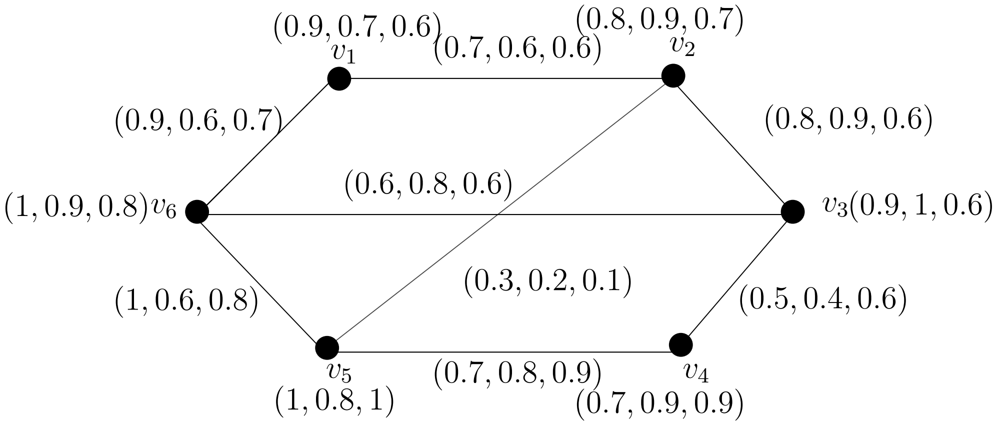


Figure 3.7: A 3PF graph G of a road network.

Likewise, other strongest paths can be found. The strongest route between two cities can easily be identified and using these strongest paths the other cities will help the disaster site in a 3PFG of a road network.

3.6 Summary

Fuzzy graph theory is widely used in computer science research, along with control theory , data collection, expert systems, database theory etc. In this chapter, at first we defined mPFP, mPFC in an mPFG. The strength of a connectedness of mPFP is introduced. Next, we defined the strongest and strong mPFP, mPFBs, mPFCNs, mPFT and mPFFs in an mPFG. Next, we discussed mPFP, mPFC in an mPFG. Here we defined strongest and strong mPFP, α-strong, β-strong, δ-strong and δ^*-strong mPFE of mPFGs and their related result.