Chapter 1

Introduction

As a modeling method, the graph is very important in order to solve many practical issues. For instance, a network of towns depicted in nodes and their associates makes a graph.The issue of the travelers wants to make the shortest journey possible, which takes you through every city precisely once. Graphic theory was founded in 1736 by the paper of Euler, which solved the Konigsberg bridge problem too. This issue guides the idea of the Eulerian graph. The concept of the graph and the bipartite graph was fully provided by Mobious in 1840. In addition, the research of recreational mathematics and games inspired a big proportion of graph theory. Graphs represent the relations between objects represented by vertices in a very convenient way. In this situation, relations between vertices are depicted by links. In particular, a graph or a hypergraph can be called any mathematical object containing points and relations between them. Such pictorial depictions can lead to a solution to a huge variety of issues. Instances containing organic molecules, databases, physical networks, map colours, mapped maps, internet charts, signal-flow charts as well as computer programs and ecosystem flow. Graphs can be used as mathematical models in order to solve a suitable graph-theoretical issue. Currently, both theory and applications have a vibrant field in graph theory.

Many kinds of graphs depict actual issues in the world. These are discussed below.

1.1 Some preliminaries on graph

A graph has a collection of points or vertices (representing objects) and arcs (representing links) between them. The formal definition is given below:

Definition 1.1.1. (Graph) *A graph is an ordered couple $G = (V, E)$ of two sets E and V, where V is the set of nodes or vertices each representing the objects and E is the set of arcs or edges which is a subset of $V \times V$, i.e. a relation defined on V.*

A multigraph [13] is a graph with multiple arcs between each two nodes but without self-loops. The graphical presentation of a graph on any surface without any intersection between arcs is named embedding [13].

The graph has several variants like infinite graph, finite graph, simple graph, undirected and directed graph, etc. The relation described in V in a directed graph is not symmetric, even though symmetrical in an undirected graph. The vertex has a connection to itself in a graph loop. There may be loops in a graph that implies a vertex has a relation to itself. More than one edge may also exist between two nodes, called parallel arcs. There are no loops and multiple arcs in simple graphs. If a graph has a limited number of arcs and a limited number of nodes, then this graph is considered a finite graph. If not, it is infinite.

In a directed graph $\overrightarrow{G} = (V, \overrightarrow{E})$, a *walk* is an alternating sequence $W = s_1 \overrightarrow{e_1} s_2 \overrightarrow{e_2} \ldots s_{l-1} \overrightarrow{e_l} s_l$ of vertices s_k and arcs $\overrightarrow{e_k}$ of $\overrightarrow{G}$ where tail of $\overrightarrow{e_k}$ is s_k and head is s_{k+1} for every $k = 1, 2, \ldots, l-1$. If $s_1 = s_l$ then walk is called closed. A *path* is a walk in which all vertices are distinct. A path $s_1, s_2, \ldots, s_l$ with $l \geq 3$ is a cycle if $s_1 = s_l$. The number of loop's or path's edges is the length of this loop or path.

If a node x_i is an end node of an arc e_j, x_i and e_j are named incidents with each other. Two nonparallel edges are said to be adjacent if they are incident on a common vertex. Also, the end nodes of the same arc are named adjacent. The number of arcs that occur on node x_i is said to be degree $d(x_i)$ of the node x_i. Sometimes a node's degree is also known as its valence. Now we have to look at a graph G with e arc and n nodes $x_1, x_2, \ldots, x_n$. In G, the sum of all nodes is double the number of arcs $\sum_{i=1}^{n} d(x_i) = 2e$.

Definition 1.1.2. *A graph that carries the same degree as all nodes is known to be a regular (or just regular) graph.*

It is said that a node with no incident arc is an isolated node. It implies that nodes with zero degrees are referred to as isolated nodes. If G is regular of degree 1, then each component contains precisely one line. If it is regular with degree 2, then every component is a cycle. There are no lines in a regular graph with degree 0.

In geometry, it is said that two figures are equivalent (congruent) if they have the same geometrical properties. In terms of graph-theoretical property, two graphs are considered to be identical (and to be called isomorphic) if they have the same characteristics. More specifically, if there is one-to-one mapping between their nodes and the arcs then two graphs G_1 and G_2 are considered as isomorphic. In another words, assuming that edge e_1 exists in G_1 on vertices s_1 and s_2, so e_2 in G_2 must have an incident in s_3 and s_4 that correspond to s_1 and s_2 respectively.

Definition 1.1.3. *If a graph G is plotted in the plane with intersection in its arcs only with nodes in G then it is a planar graph. If without arc crossing it can not be drawn in the plane then the graph is a non-planar.*

Kuratowski [69] developed several major features in planar graphs in 1930. A planar graph with cycles is also said to be faces that can divide the plane within a number of regions. The face length in planar graph G is the full length of the closed walk(s) in G that bounds the face. For a specific Embedding in G, it has an arc linking two regions which are neighboring for each arc in G. Whitney's planarity criterion [140] characterizes the existence of an algebraic dual. The planarity criterion of MacLane's [71] offers an algebraic description of finite planar graphs. The planarity criterion of Fraysseix Rosenstiehl's [50] provides a definition on the basis that there is a bipartition of the co-tree edges of a depth-first search tree. In the terms of partial order dimensions, the Schnyders theorem [135] has described its planarity characters.

A graph is referred to as an intersection graph if the structure of the intersection between families of sets is represented. The overlap of multiset intervals in real line is an interval graph (IG). IG are important for operational analysis of resource allocation. Interval graph is also widely used in theory of organisation, ecological modeling, archaeology, mathematical sociology and mathematical modelling.

Another essential graph is the tolerance graph [60]. In order to generalize several well proven implementations of IGs, tolerance graphs have been added. The main motivation was to map the allocation of resources and some scheduled issues, which could tolerate sharing among users of resources like rooms and vehicles. Obviously, tolerance graphs can be used for biological and bioinformatics applications. The tolerance graphs can be found in numerous applications for restricted time reasoning, network data transmission to effective planes and crews, etc. as they lead to genetic analysis and brain testing. The tolerance graph description is described below.

Definition 1.1.4. *[60] Tolerance graphs are generalization of interval graphs in which each vertex can be represented by an interval and a tolerance such that an edge occurs if and only if the overlap of corresponding intervals is at least as large as the tolerance associated with one of the vertices. Hence, a graph $G = (V, E)$ is a tolerance graph if there is a set $I = \{I_v : v \in V\}$ of closed real intervals and a set $\{T_v : v \in V\}$ of positive real numbers such that $(x, y) \in E$ if $|I_x \cap I_y| \geq min\{T_x, T_y\}$. The collection $\langle I, T\rangle$ of intervals and tolerances is called tolerance representation of the graph G.*

The proper and unit tolerance graphs were introduced by Bogart [25] et al. Brigham et al. [27] demonstrated a wide range of results of tolerance competition graphs. Tolerance and bounded tolerance graphs were introduced by Mertzios and Zaks [84]. Threshold graphs have a major preface in scheduling theory, computer science and psychology, etc. These graphs monitor information flow between processors similar to the lighting used for traffic control.

The open neighborhood graphs were presented by Acharya and Vartak [2]. Chvatal and Hammer [41] explained problems of set-packing and described threshold graphs and their properties. Some notes on threshold graphs were discussed by Andelic and Simic [12]. The threshold graph description is described below.

Definition 1.1.5. *[41] A graph G is a threshold graph where non-negative reals $u_s(s \in V)$ and t exists with $U(w) \leq t$ iff $W \subseteq V$ is stable set where $U(W) = \sum\limits_{s \in W} u_s$.*

Here, G is a threshold graph, which allows the weight of a single vertex to be allocated so that a set of vertices is stable iff if the sum of its weight is less than a certain threshold. A minimum number k of thresholds subgraphs $T_1, T_2, \ldots, T_k$ of G that cover the set of edges of G is the threshold dimension $(t(G))$ of the graph G.

Definition 1.1.6. *[104] Ferrers digraph is a threshold graph connected to the digraph. A digraph $\overrightarrow{G}$ is a Ferrers digraph if it does not have vertices s, t, u, v which are not necessarily distinct, but satisfies $\overrightarrow{(s,t)}, \overrightarrow{(u,v)} \in \overrightarrow{E}$ and $\overrightarrow{(s,v)}, \overrightarrow{(u,t)} \notin \overrightarrow{E}$. For a digraph $\overrightarrow{G} = (V, \overrightarrow{E})$, the underlying loop less graph is $U(\overrightarrow{G}) = (V, E)$, where $E = \{(s,t) : s, t \in V, s \neq t, \overrightarrow{(s,t)} \in \overrightarrow{E}\}$.*

A split-graph separates the vertices into an independent clique and set.

For the graph G with distinct positive vertex degrees $\delta_1 < \delta_2 < \ldots < \delta_m$ and $\delta_0 = 0$ (even no vertex of degree 0 exists), $\delta_{m+1} = |V| - 1$ degree partition is the sequence $D_i = \{v \in V : \deg(v) = \delta_i\}$ for $i = 0, 1, \ldots, m$.

If the two vertices s and t do not belong to the same tree or if no path from s to t and no path from t to s is present then these vertices are incomparable.

The directed graphs are described in a similar way except they have directed edges. The formal definition is provided below.

Definition 1.1.7. *A directed graph (digraph) $\overrightarrow{G}$ composes of non-empty finite set $V(\overrightarrow{G})$ of elements known as nodes and a finite set $\overrightarrow{E}(\overrightarrow{G})$ of ordered pairs of different vertices known as arcs.*

The set $N^+(s) = \{t \in V - s : \overrightarrow{(s,t)} \in \overrightarrow{E}\}$ is the *out-neighborhood* [62] of a node s. Similarly, the set $N^-(s) = \{r \in V - s : \overrightarrow{(r,s)} \in \overrightarrow{E}\}$ is *in-neighborhood* [62] of a vertex v. The union of out-neighborhood and in-neighborhood of the vertex is called the *open neighborhood* of a vertex.

For an undirected graph, the set of all vertices adjacent to x is called *open-neighborhood* [2] and is denoted by $N(x)$. In 1968, Cohen [43] introduced the concept of competition graphs with respect to an ecological problem. The competition graph applies also in modelling of complex economic, coding, channel assignment and energy systems, etc. [107]. To represent ecological problems, Cable et al. [40] introduced niche graphs. With the concept of interval competition graph, Lundgren and Maybee [78] introduced food webs. The detailed work on competition graphs is available in [64–66, 127]. Cho et al. [39] introduced the m-step competition graph of a diagraph. Raychaudhuri and Roberts [107] described generalized competition graphs and applications for generalized competition graphs. Sano [124,125] studied various properties on the competition-common enemy graphs of digraphs. A pair $H^* = (X, E)$ is a (crisp) *hypergraph* on a set X where X is a finite set and E is a nonempty finite family of subset of X that comply with the following requirement: Every member of X is contained in some member of E. X and E are called the vertex set and the edge set of H^* respectively. Repeated or multiple edges are permitted. A hypergraph $H^* = (X, E)$ is *simple* if there has no repeated edges in E and then $E_1, E_2 \in E$ and $E_1 \subset E_2$, then $E_1 = E_2$. A hypergraph $H^* = (X; E_1, E_2, \ldots, E_k)$ where $X = \{x_1, x_2, \ldots, x_n\}$ can be mapped to a hypergraph $H^{**} = (E; x_1, x_2, \ldots, x_n)$ whose vertices are the points $e_1, e_2, \ldots, e_k$

(corresponding to $E_1, E_2, \ldots, E_k$), and whose edges are the sets $X_1, X_2, \ldots, X_n$ (corresponding to $x_1, x_2, \ldots, x_n$ respectively) where $X_j = \{x_j \in E_i, i \leq k\}, j = 1, 2, \ldots, n$. The hypergraph H^{**} is called the *dual hypergraph* of H.

Explicitness agrees that variables of a system are either part of the system or not. In fact, if the complexity of the systems increases, the explicit nature of the systems decreases. Uncertainty plays a crucial role in any attempt to optimize the efficiency of device or model parameters. One of the definitions applied to the word 'uncertainty' is "vagueness", i.e. the difficulty of making a sharp or precise distinction. Zadeh [144–147] proposed in the seminal paper entitled "Fuzzy Sets" a mathematical work to explain this phenomenon. Kosko [70], in his book, calls this a mismatch problem: The world is gray but science is black and white. Research on fuzzy sets is increasing to date due to its wide range of applications. The main applications of fuzzy set include image processing, neural networks, modeling and classification of video traffic, shortest path problem, network optimization, etc.

1.2 Fuzzy sets

In 1965, Zadeh, an Iranian-American mathematician and Professor of Computer Science, presented a theory of fuzzy set as a generalization of Cantor's set theory. The word fuzzy often stands for the word vague (formless, unclear) in the literature of fuzzy sets.

Definition 1.2.1. (Crisp set) *A classical set is a collection of well-defined set of items with a crisp boundary. A crisp set A is categorized by a characteristic function χ_A and is described by*

$$\chi_A(s) = \begin{cases} 1, & \text{if } s \in A \\ 0, & \text{otherwise.} \end{cases}$$

The concept of membership of a fuzzy set stems from the characteristic function of crisp set.

Definition 1.2.2. (Fuzzy set) *A fuzzy set A for the set X is a mapping $m_A : X \to [0, 1]$, called the membership function. $A = (X, m_A)$ is the notation for a fuzzy set.*

The membership degree of elements ranges over $[0, 1]$ in the fuzzy set. The degree of membership indicates the degree to which the elements belong to a fuzzy set. An element belong to its corresponding fuzzy set if its membership degree is 1 and an element that is not belong to the fuzzy set if its membership degree 0. In $(0,1)$ interval, the membership degree implies the partial belongingness to the fuzzy set. A (crisp) *multiset* is a mapping $d: V \to \mathbb{N}$ on a non-empty set V, where $\mathbb{N}$ is the natural numbers set. The term "fuzzy bag" was first described by Yager [141]. An element of a set V entity with probably the same or distinct membership value may occur more than once. The notion of *fuzzy multiset*, is a natural generalized multiset representation of V as a mapping $\tilde{C}: V \times [0,1] \to \mathbb{N}$, where V is a non-empty set. The membership values of $v \in V$ are denoted by $v_{\mu^j}, j = 1, 2, \ldots, p$ where $p = \max\{j : v_{\mu^j} \neq 0\}$. This allows us to describe a fuzzy multiset as $M = \{(v, v_{\mu^j}), j = 1, 2, \ldots, p | v \in V\}$.

Operations on fuzzy sets

Definitions for set theoretic operations extend the fuzzy set theory. The basic operations were initially specified by Zadeh. Over time , additional and alternative operations have been suggested by other authors. In order to provide general understanding of fuzzy set theories, the following concepts provide an overview of a variety of important operations on the fuzzy set and characteristics. In addition, different types of set operations are introduced that combine fuzzy sets.

Definition 1.2.3. *[144] Let us consider two fuzzy sets $A = (X, m_A)$ and $B = (X, m_B)$ in X. Then,*

(i) $A \subseteq B$ iff $m_A(s) \leq m_B(s)$ $\forall$ $s \in X$ (sometimes $A \subseteq B$ is denoted as $A \leq B$).

(ii) $A = B$ iff $m_A(s) = m_B(s)$ $\forall$ $s \in X$.

(iii) $A \cup B$ is the union of the fuzzy sets A and B and it is described by the membership function $m_{A\cup B}(s) = \max\{m_A(s), m_B(s)\}$ $\forall$ $s \in X$.

(iv) $A \cap B$ is the intersection of the fuzzy sets A and B and it is described by the membership function $m_{A\cap B}(s) = \min\{m_A(s), m_B(s)\}$ $\forall$ $s \in X$.

Definition 1.2.4. (Cut level set) *[90] Let $A = (X, m_A)$ be a fuzzy set. The crisp set $A_t = \{x : m_A(x) > t\}$ is the t-cut level set of A.*

1.3 Fuzzy graphs

Due to the haziness or uncertainty of the system parameters, graphs do not accurately represent all the systems. When account relationships are to be evaluated as good or bad based on the communication frequency between accounts, fuzziness should be added to the representation. These and several other issues contribute to the concept of fuzzy graphs(FGs). In 1975, Rosenfeld [108] first presented the FGs considering fuzzy relations on fuzzy sets. In the evaluation and optimisation of networks, this FG definition was used by Koczy [68]. So, there are many areas of study in FG theory. Applications of fuzzy graphs include scheduling, planning, communication, networking, image capturing, clustering, image segmentation, data mining, etc.

The following definition is given for a FG.

Definition 1.3.1. (Fuzzy graph) *[108] A fuzzy graph(FG) $\xi = (V, \mu, \rho)$ is a non-empty set V together with a pair of functions $\mu : V \to [0,1]$ and $\rho : V \times V \to [0,1]$ s.t. for all $s, t \in V$, $\rho(s,t) \leq \min\{\mu(s), \mu(t)\}$, where $\mu(s)$ and $\rho(s,t)$ represent the membership values of the vertex s and of the edge (s,t) in ξ respectively.*

$\rho(s,s) \neq 0$ is a loop at a vertex s in a fuzzy graph. If $\rho(s,t) \neq 0$ then an edge is called non-trivial.

Some terminologies of fuzzy graphs

Fuzzy subgraph is also a fuzzy graph (FG) and the vertex set of these fuzzy subgraph is a subset of the vertex set of the given fuzzy graph. The formal definition is given below.

Definition 1.3.2. (Fuzzy subgraph) *[90] The FG $\xi' = (V', \tau, \nu)$ is called a fuzzy subgraph of ξ if $\tau(s) \leq \mu(s)\ \forall\ s \in V'$ and $\nu(s,t) \leq \rho(x,y)\ \forall\ s,t \in V'$ where $V' \subset V$.*

For the FG $\xi = (V, \mu, \rho)$, if

$$\frac{1}{2}\{\mu(s) \wedge \mu(t)\} \leq \rho(s,t)$$

then the edge (s,t) is called strong [48] and (s,t) is a weak edge, otherwise. The strength of an edge (s,t) is denoted by

$$I_{(s,t)} = \frac{\rho(s,t)}{\mu(s) \wedge \mu(t)}.$$

Definition 1.3.3. *A fuzzy graph $\xi = (V, \mu, \rho)$ is called bipartite if the set V can be divided into two nonempty sets V_1 and V_2 s.t. $\rho(s_1, s_2) = 0$ if $s_1, s_2 \in V_1$ or $s_1, s_2 \in V_2$. Further, if $\rho(s_1, s_2) = \min\{\mu(s_1), \mu(s_2)\}\ \forall\ s_1 \in V_1$ and $s_2 \in V_2$, then ξ is called a complete bipartite fuzzy graph.*

In a FG, if an edge (s, t) satisfies the relation $\rho(s, t) = \min\{\mu(s), \mu(t)\}$, then the (s, t) edge is effective edge [93]. If any one end vertex is fuzzy pendant vertex then a fuzzy edge is called a fuzzy *pendant edge* [118]. The minimum membership value between the membership values of the end vertices is the membership value of the pendant edge.

A FG ξ is a called *regular* [92] if $d(s) = k$, $\forall\ s \in V$ and where k is a positive real number. A FG ξ is called a *totally regular* fuzzy graph if every vertex of ξ has the same total degree k. It is said that a fuzzy graph is *irregular* [95], if there is a vertex which is adjacent to vertices with distinct degrees.

Like complete graph, the definition of complete fuzzy graph is given below.

Definition 1.3.4. (Complete fuzzy graph) *[10] Let $\xi = (V, \mu, \rho)$ be a FG. If $\rho(s, t) = \min\{\mu(s), \mu(t)\}$ $\forall\ s, t \in V$ then ξ is called complete, where (s, t) is the arc from s to t.*

The *complement* of a FG $\xi = (V, \mu, \rho)$ [90] is the fuzzy graph $\xi' = (V, \mu', \rho')$ where $\mu'(s) = \mu(s)$ $\forall\ u \in V$ and

$$\rho'(s, t) = \begin{cases} 0, & \text{if } \rho(s, t) > 0 \\ \mu(s) \wedge \mu(t), & \text{otherwise.} \end{cases}$$

There are many variations in fuzzy graphs, like (i) Fuzzy intersection graph, (ii) Fuzzy hypergraph, (iii) Fuzzy planar graph, (iv) Bipolar fuzzy graph(BFG), (v) m-polar fuzzy graph(mPFG), etc.

We now briefly describe these one by one as follows.

1.3.1 Fuzzy intersection graph

McAllister [83] first introduced the fuzzy intersection graph. The following definition is for the fuzzy intersection graph.

Definition 1.3.5. *Let a finite family of fuzzy sets $\mathcal{F}= \{B_1 = (X, m_1), B_2 = (X, m_2), \ldots,$ $B_n = (X, m_n)\}$ be defined on X and take $\mathcal{F}$ as vertex set $V = \{s_1, s_2, \ldots, s_n\}$. The FG Int($\mathcal{F}$)= (V, μ, ρ) is the fuzzy intersection graph of $\mathcal{F}$ where $\mu : V \to [0, 1]$ is given by $\mu(s_k) = h(B_k)$ and $\rho : V \times V \to [0, 1]$ is defined by*

$$\rho(s_k, s_l) = \begin{cases} h(B_k \cap B_l), & \textit{if } k \neq l \\ 0, & \textit{if } k = 1 \end{cases}$$

1.3.2 Fuzzy hypergraphs

Goetschel [57] introduced fuzzy hypergraphs. The following definition is given for the fuzzy hypergraph

Definition 1.3.6. *Let $\mathcal{E}$ be a finite family of nontrivial fuzzy sets on a finite set X (or subsets of X) s.t. $X = \bigcup\{suppA|A \in \mathcal{E}\}$. Then the pair $\mathcal{H}= (X, \mathcal{E})$ is a fuzzy hypergraph on X.*

In a fuzzy hypergraph $\mathcal{H}$, X is the vertex set and $\mathcal{E}$ is the fuzzy edge set. $h(\mathcal{H}) = max\{h(A)|A \in \mathcal{E}\}$ is the height of $\mathcal{H}$. A fuzzy hypergraph is *simple* if $\mathcal{E}$ does not have multiple fuzzy edges and if $A \subseteq B$ and $A, B \in \mathcal{E}$, then $A = B$. A fuzzy hypergraph $\mathcal{H}= (X, \mathcal{E})$ is *support simple* if whenever $A, B \in \mathcal{E}$, $supp(A) = supp(B)$ and $A \subseteq B$, then $A = B$. Suppose $A = (X_1, \mu) \in \xi, X_1 \subseteq X$ and $c \in (0, 1]$. The $c-$cut of A is defined by $A^c = \{x \in X|\mu(x) \geq c\}$. If $\mathcal{E}^c = \{A^c| \in \mathcal{E}/\{\phi\}\}$ and $X^c = \bigcup\{A^c|A \in \mathcal{E}\}$. If $\mathcal{E}^c \neq \phi$, then the (crisp) hypergraph $H^c = (X^c, \mathcal{E}^c)$ is the $c-$ *level hypergraph* of $\mathcal{H}$.

1.3.3 Fuzzy planar graph

Due to human demand, there is a rise in the need of pipelines, flyovers, subway tunnels and subway lines day by day. The number of road crossings raises the risk of an accident. The costs of underground metro crossings are also high. Yet traffic congestion is minimized on the subway routes. It is true that two congested crossing of routes is safer than a congested and non-congested road crossing. The word "congested" has no specific meaning or definition. To clarify the exact load of a route, we generally use terms like "congested", "very congested", "highly congested" routes etc. for roads. These linguistic terms can be dealt in mathematics by giving some positive membership values and negative membership values in a fuzzy sense. Strong route and weak route means highly congested and low congested route respectively in a mathematical sense. It is also easier to cross between a strong route and a weak route than to cross between two strong routes. That is, in town planning, it is possible to allow crossing between weak routes and strong routes. And the permission to use the crossing between weak and strong arcs leads to a bipolar fuzzy planar graph concept. The definition of the fuzzy planar graph was introduced by Abdul-jabbar et al. [1] and Nirmala and Dhanabal [101]. Recently, Samanta and Pal [119,120] introduced a fuzzy planar graph in a different way where crossing of edges is allowed and studied different properties of it.

1.3.4 Bipolar fuzzy graph

In 1994, Zhang [149] created the concepts of a bipolar fuzzy set as a generalisation of a fuzzy set. A bipolar fuzzy set is a generalization of Zadeh's fuzzy set.

A *bipolar fuzzy set* [148, 149] B in a non-empty set X is characterized by $B = \{(s, \mu_B^P(s), \mu_B^N(s)) | s \in X\}$, where $\mu_B^P : X \to [0,1]$ and $\mu_B^N : X \to [-1,0]$ are positive and negative membership functions respectively. $\mu_B^P(s)$ indicates the element s satisfies the property corresponding to a bipolar fuzzy set B, and $\mu_B^N(s)$ indicates the element s satisfies the implicit counter-property to some extent corresponding to a bipolar fuzzy set B. For every two bipolar fuzzy sets $A = (\mu_A^P, \mu_A^N)$ and $B = (\mu_B^P, \mu_B^N)$ on X,

$(A \cap B)(s) = (min(\mu_A^P(x), \mu_B^P(s)), max(\mu_A^N(s), \mu_B^N(s)))$.

$(A \cup B)(s) = (max(\mu_A^P(s), \mu_B^P(s)), min(\mu_A^N(s), \mu_B^N(s)))$.

Akram [3,5,6] introduced BFGs, regular BFGs and investigated some properties of it. Later on, Yang et al. [142] modified their definition of bipolar fuzzy graphs and introduced generalized bipolar fuzzy graphs.

1.3.5 m-polar fuzzy graph

The notion of mPF set was launched by Chen et al. [38] in 2014 as a generalisation of BFS. The theory behind this concept is that "multipolar information" (not only bipolar knowledge that represents the two-valued logic) exists, since real world problems are often received from multiple agents. For example, mankind's accurate telecom security level is one point in $[0,1]^n$ ($n \approx 7 \times 10^9$) because different persons have been monitored at different times. There are some other examples, such as logic formula degrees of $n(n \geq 2)$, degrees of similarity in two mathematical formulas dependent on the mathematical implication of n operators ($n \geq 2$), magazine performance, the ordering of university results and the inclusion of degrees of a rough set (accuracy measures, rough measures, approximations qualities, fuzziness measure and avoidance of decisions).

Here $[0,1]^m$ (m-power of [0,1]) is considered to be a poset with point-wise order $\leq$, where m is a natural number. $\leq$ is defined by $s \leq t \Leftrightarrow$ for each $i = 1, 2, \ldots, m$; $p_i(s) \leq p_i(t)$ where $s, t \in [0,1]^m$ and $p_i : [0,1]^m \to [0,1]$ is the i-th projection mapping.

Definition 1.3.7. *[38]* **(mPF set)** *An mPF set on X is a mapping $A : X \to [0,1]^m$. The mPF set on X is labelled as $m(X)$.*

Definition 1.3.8. (Operations) *[51] Let A and B are two mPF sets in X. Then $A \cup B$ and $A \cap B$ be also mPF sets in X defined by the following $\forall\ i = 1, 2, 3, \ldots, m$ and $s \in X$,*

$p_i \circ (A \cup B)(s) = max\{p_i \circ A(s), p_i \circ B(s)\}$,

$p_i \circ (A \cap B)(s) = min\{p_i \circ A(s), p_i \circ B(s)\}$,

$A \subseteq B$ *iff* $p_i \circ A(s) \leq p_i \circ B(s)$ *and*

$A = B$ *iff* $p_i \circ A(s) = p_i \circ B(s)$.

Definition 1.3.9. *[51]* **(mPF relation)** *An mPF relation on an mPF set A is an mPF set B of $X \times X$ s.t. $B(s,t) \leq min\{A(s), A(t)\}\ \forall\ s,t \in X$, i.e. $p_i \circ B(s,t) \leq min\{p_i \circ A(s), p_i \circ A(t)\}\ \forall\ s,t \in X$, $i = 1, 2, \ldots, m$. An mPF relation B on X is called symmetric if $B(s,t) = B(t,s)\ \forall\ s,t \in X$.*

Chen et al. [38] defined mPFG in the following way:

Definition 1.3.10. *[38]* **(mPFG)** *An mPFG with an underlying pair (V, E) is defined to be a pair $G = (A, B)$, where $A : V \to [0,1]^m$ and $B : E \to [0,1]^m$ fulfilling $B(st) \leq min\{A(s), A(t)\}\ \forall\ st \in E$.*

1.4 Review of literature

Several research projects were conducted after the introduction of fuzzy graphs. The fuzzy intersection graphs was characterized by McAllister [83]. Fuzzy interval graphs were characterized after that by Craine [42]. Then, as an extension of crisp hypergraph, Goetschel [57] implemented fuzzy hypergraphs. In his paper "Fuzzy colorings of fuzzy hypergraphs" [58], he also defined another major branch of fuzzy hypergraph theory. Goetschel and Voxman [59] discussed the intersection in fuzzy hypergraphs in a separate article. Somasundaram et al. [128] discussed domination in FGs. Mordeson and Nair [88] represented the successor and source of (fuzzy) directed graphs and (fuzzy) finite state machines. The descriptions of the fuzzy graphs and hypergraphs were given by Mordeson and Nair [90]. Following that, Mordeson and Peng [85–87] introduced fuzzy line graphs defining the operations on FGs and cycles and cocyles of FGs. Nair et al. [96–98] introduced triangle and parallelogram laws on FGs, cliques and fuzzy cliques in FGs, perfect and precisely perfect FGs. Bhutani and Battou [20] gave the concept of M- strong FGs. Bhutani and Rosenfeld [18] introduced strong arcs in fuzzy graphs. Mathew and Sunitha [80, 82] defined different types of arcs and investigated Menger's theorem for FGs. They examined the node connectivity and arc connectivity of a FG in another paper [81]. Bhutani et al. [22] provided some findings on degrees of end nodes and cut nodes in fuzzy graphs. Eslahchi and Onaghe [48] introduced vertex strength of fuzzy graphs. In FGs, they also described strong fuzzy edges. Then, regular and irregular FGs were introduced by Nagoorgani and Radha [92,95]. Nagoorgani et al. [93,94] also investigated isomorphism properties and fuzzy effective distance k-dominating sets of strong fuzzy graphs. The fuzzy dual graph was introduced by Jabbar et al. [1]. Special planar FG configurations were developed by Nirmala and Dhanabal [101]. Samanta and Pal [119, 120] analyzed fuzzy planar graph differently in order to emphasize the real issue.

Within Cohen's literature [43], there are many variations in the competition graphs. Some derivations of competition graphs have been found after Cohen. The m-step competition graph for the digraph was given by Cho et al. [39]. Kim et al. [65] described the p competition graph. The tolerance competition graphs were provided by Brigham et al. [27]. Fuzzy k-competition graphs and p-competition FGs is available in [117]. The competition hypergraphs have been found in Sonnatag et al. [129].

Tolerance graphs [60] are generalizations of interval graphs in which every node can be shown with an interval. After that, ϕ-tolerance competition graph was introduced by [27].

In a network of imprecise edge weight, Nayeem and Pal [100] have worked on the shortest path problem. Surveys of the major literature associated with the competition graph can be found in [40]. The bipolar fuzzy hypergraph is a hypergraph in which each vertex and edge are assigned bipolar fuzzy sets. Samanta and Pal [115] introduced the bipolar fuzzy hypergraphs which have emerged importance in complex networking systems.

Rosenfeld [108] introduced the concept of μ-distance in FGs. The concept of eccentricity and centre in FGs was introduced by Bhattacharya [16] using μ-distance. Sameena and Sunitha [122, 123] have further studied on the g-distance of FGs. Automorphism, fuzzy end nodes, geodesics in FGs are studied by Bhutani et al. [17, 19, 21].

The g-eccentric nodes, g-interior and g-boundary nodes of a FG were described by Linda and Sunitha [77].

Bershtein et al. [15] defined the cliques fuzzy set. Then, cliques and clique covers in FGs were introduced by Sun et al. [131]. Another variation of clique cover is edge clique cover which is studied by Javadi and Hajebi [61]. In most research work of clique cover, the main task is to find the clique cover number.

Chvatal and Hammer [41] first introduced the threshold graph. In 1979, Manca [79] derived an efficient matrix method for testing a given graph to see whether or not it is a threshold graph. There is a great introduction to threshold graphs and their applications in [104]. Due to the importance of FGs, Samanta and Pal have introduced the fuzzy threshold graphs in [114].

The reader may found the works on various extensions of FGs in [28–33, 109–112]. For further studies on FGs and its variations the literatures [7–9, 11, 15, 24, 39, 49, 56, 65, 80, 116, 118–121] may be very helpful.

1.5 Motivation of the work

Sometimes data come from n agents ($n \geq 2$), i.e. "multipolar information" is present in several real world problems. These data can not be clearly expressed well by means of FGs or BFGs. Therefore the mPF set is used to define the relations between various individuals. In this direction, Chen et al. [38] first defined mPFG. The fundamental target of the Chapter 2 is to present the idea of superstrong and strong mPF vertex of mPFGs using the concept of strong mPFE, strength of connectedness of path, etc. Next we studied several properties on these paths. First, we have defined mPFP and connectedness. Next we defined strongest and strong mPFP. Then we studied several properties on these nodes. At the end, an application of strong path problems is presented. In this application, we consider a collection of cities and focus on finding the city which is best suitable to have a university or colleges. By suability we mean the city which is well connected with all the other cities of the collection and should be feasible to all in respect to communication, locality, ambiance, etc. We model this problem through a 3PF graph. The idea is to find out strong and superstrong vertices and finally concluding the superstrong vertices to be those cities most likely to have universities in them. And the strong vertices(cities) are suitable to have colleges.

In Chapter 3, at first mPFP, mPFC in an mPFG are defined. The strength of a connectedness of mPFP is introduced. Next, strongest and strong mPFP, mPFBs, mPFCNs, mPFT and mPFFs in an mPFG are considered. Also, it is proved that an arc of mPFT is strong mPFE iff it is an mPFB. Finally, mPFENs in an mPFG is defined and investigated with some properties on it. An application of the strongest path problem is also given at the end. Also we have presented the idea of α-strong mPFE, β-strong mPFE, δ-strong mPFE and δ^*-strong mPFE of mPFGs. Next we have studied several properties on these arcs. At the end, an application of strong path problems is given.

The genus of a graph is a well known notion in topological graph theory and has been studied for a variety of ordinary graphs. We have worked on embedding of mPFGs in Chapter 4. Also, we have introduced mPFGG with its genus value, strong and weak mPFGG. Important properties such as isomorphism of mPFGG and relation between genus value and planarity value of an mPFG have been discussed and established. In the concluding part, an useful application of mPFGG is also given.

In Chapter 5, first we have introduced mPF detour g-distance, mPF detour g-interior node, mPF detour g-boundary node and then explained their relations. Some properties of these parameters are investigated. Important property we established on mPF detour g graph. Here, we present an application of mPFG using a strong path. In modern days, if we go from one town to another town then we usually use car, train, bus, etc. The availability of buses or trains are not the same everywhere. Some people travel on their vacation to visit other states, cities or countries. If the communication system is good then the journey will be good. Again, if the economic system of a city is good then the condition of the road is generally good. This communication system depends not only on the economic condition but also on many other things such as for example infrastructure, environment, fire safety, security, etc.

In Chapter 6, we have described the connectivity index for mPFG. The lower and upper boundary of connectivity index for mPFG are discussed. If we delete an edge from a mPFG then the effects of the connectivity index in mPFG are studied in this chapter. The average connectivity index in mPFG is provided here.

In Chapter 7, we have defined five new operations on Dombi mPFG such as direct product, cartesian product and semi strong product, strong product, lexicographic product. It is proved that any of the products of Dombi mPFG are again an Dombi mPFG. Next we have defined the ring sum, union of two Dombi mPFGs. Then complement and self complement of Dombi mPFG is defined. Different properties on Dombi mPFGs are established.

Finally, Chapter 8 is devoted to the Conclusion of the thesis followed by the Bibliography.

1.6 Summary

This chapter introduces and discusses some preliminary notions used in the rest of the Thesis. Several types of graphs and fuzzy graphs are discussed. Some fuzzy set theoretic definitions and notations are also focused. Motivation of the work and survey of related works of the thesis are discussed in this chapter.

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