

Chapter 6

Holling-Tanner model with Beddington-DeAngelis functional response and time delay introducing harvesting*

The chapter is designed with the Holling-Tanner prey-predator model with Beddington-DeAngelis functional response including prey harvesting. Gestational time delay of predator and the dynamic stability of time delay preventing system are incorporated into the system in this chapter. The equilibria of the proposed system are determined and the existence of interior equilibrium point for the proposed system is described. Local stability of the system with the magnitude of time delay at the interior equilibrium point is discussed. Thereafter, the direction and the stability of Hopf bifurcation are established with the help of normal theory and center manifold theorem. Furthermore, profit function is calculated with the help of bionomic equilibrium and it is optimized using optimal control. Finally, some numerical simulations are introduced to verify the validity of analytic results of the proposed model.

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6.1 Introduction

The relationship between prey and predator is natural phenomena for universal existence in our ecological system. Interactions of prey and predator are one of the common and well known factor in ecological systems. This ecological system is one of the important fields in the study of mathematical biology. In the 1920, Vito Volterra described whether it would be possible to clarify the fluctuations that had been noticed in the fish population of the Adriatic sea-fluctuations that were of great concern to fishermen in time of low fish populations. Time delay has an important role in biological population. Differential equation with time delay has received great attention in research of recent years among theoretical and mathematical ecologists. Although, study of time delay can have very complex impact on the dynamics of a system, for example, it can cause the loss of stability, can induce various oscillations and periodic solutions. In different ecological systems, interaction between predator and prey is also different. Due to this reason functional response is changed. Beddington-DeAngelis functional response is most important in biological interaction. This functional response was constructed by Beddington (7) and DeAngelis *et al.*(22).

In ecology, predation is a biological interaction where predator feeds on its prey. In particular case, the prey population is so large than the predator population, then a particular type of prey-predator model has been considered which plays a special role in view of interesting dynamics and it possesses the Holling-Tanner predator-prey system (110). Then the Holling-Tanner prey-predator model is defined as follows:

$$\left. \begin{aligned} \frac{dx}{dt} &= rx \left(1 - \frac{x}{K}\right) - \psi(x, y)y \\ \frac{dy}{dt} &= \beta y \left(1 - \frac{y}{\gamma x}\right) \end{aligned} \right\} \quad (6.1)$$

with initial conditions $x(0) > 0, y(0) \geq 0$ and $\gamma > 0$. The function $\psi(x, y)$ denotes the predator response function.

In Southern Ocean, one species, the Antarctic krill (*Euphausia superba*) makes up an estimated biomass. Of this, over half is eaten by whales, seals, penguins,

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squid and fish each year, and is replaced by growth and reproduction. Considering this phenomenon, the Holling-Tanner prey-predator model has been designed with Beddington-DeAngelis functional response in this chapter and consider $\psi(x, y) = \frac{\alpha x}{a+bx+my}$ where α denotes the maximal predator per capita consumptions rate, i.e., the maximum number of prey population can be eaten by a predator in each time unit and $a, b, m > 0$. Since most of the krill is used for aquaculture and aquarium feeds, as bait in sport fishing, or in the pharmaceutical industry, for this reason, we introduce the harvesting effort $h(t)$ on prey population. Then the system of equations (6.1) reduces to

$$\left. \begin{aligned} \frac{dx}{dt} &= rx \left(1 - \frac{x}{K}\right) - \frac{\alpha xy}{a + bx + my} - h(t) \\ \frac{dy}{dt} &= y \left(\beta - n \frac{y}{x}\right) \end{aligned} \right\} \quad (6.2)$$

where $n = \frac{\beta}{\gamma}, x(0) > 0, y(0) \geq 0$.

Considering that the harvest rate $h(t)$ in the form $h(t) = qEx$. Thus the system (6.2) becomes as follows:

$$\left. \begin{aligned} \frac{dx}{dt} &= rx \left(1 - \frac{x}{K}\right) - \frac{\alpha xy}{a + bx + my} - qEx \\ \frac{dy}{dt} &= y \left(\beta - n \frac{y}{x}\right) \end{aligned} \right\} \quad (6.3)$$

with initial conditions $x(0) > 0, y(0) \geq 0$.

Gestation is an important factor in predator population, for that reason, here the time delay is introduced. To increase the predator population, all the metabolic energy of predator obtains through its food is inherently assumed that it is used for growth. It is also considered that the predator population consumes the prey population at a constant rate, but the reproduction of predators after predating the prey population is not instantaneous. Hence it will be incorporated by some time lag required for the gestation of predators. Let τ be the time delay in time interval between the moments when an individual prey is killed and the corresponding biomass is added to the predator

population. Therefore, the system (6.3) finally stated as below:

$$\left. \begin{aligned} \frac{dx}{dt} &= rx \left(1 - \frac{x}{K}\right) - \frac{\alpha xy}{a + bx + my} - qEx \\ \frac{dy}{dt} &= y \left(\beta - n \frac{y(t - \tau)}{x(t - \tau)}\right) \end{aligned} \right\} \quad (6.4)$$

with initial conditions $x(0) > 0, y(0) \geq 0$ and $(x(t), y(t)) \in C_+ = C([- \tau, 0], R_+^2)$.

6.2 Notation

Table-6.2.1: Description of the parameters.

Parameter	Description of the parameters
x	Population of prey at time t
y	Population of predator at time t
r	Intrinsic growth rate of prey
K	Environmental carrying capacity of the prey
β	Biotic potential of the predator population
$\frac{1}{\gamma}$	Amount of prey required to support a predator at equilibrium
q	Catchability co-efficient
E	Fishing effort for harvesting the generalist predator population

6.3 Equilibria and their existence criteria

The steady states of the system (6.4) are obtained by solving the equations.

Let $E_3^*(x^*, y^*)$ be the interior equilibrium point, then

$$y^* = \frac{\beta}{n}x^*$$

and x^* is the positive root of the following equation

$$(nb + m\beta)\frac{rx^2}{K} + \left\{ \frac{r}{K}na + \alpha\beta - (r - qE)(nb + m\beta) \right\} x - na(r - qE) = 0 \quad (6.5)$$

From above equation (6.5), it is seen that when $r > qE$ then there exists a positive root of the equation. Hence the intrinsic growth rate is greater than harvest rate per unit population, which is the sufficient condition of the proposed system for existing a positive interior equilibrium point.

6.4 Boundedness

Here the boundedness of the system has been discussed by stating a theorem including the proof.

Theorem 6.4.1. *The solution of the system (6.4) with positive initial condition satisfies:*

$$\limsup_{t \rightarrow \infty} x(t) \leq \max\{x(0), K\}$$

and

$$\limsup_{t \rightarrow \infty} y(t) \leq g_1$$

where $g_1 = \beta \frac{e^{\beta\tau}}{n} \max\{x(0), K\}$.

Proof: Assuming that $(x(t), y(t))$ be an arbitrary positive solution of the system with positive initial condition, we have

$$x(t) = x(0) \exp \left[\int_0^t F(x(u), y(u)) du \right] \quad (6.6)$$

where $F(x(u), y(u)) = r \left(1 - \frac{x(u)}{K} \right) - \frac{\alpha y(u)}{a + bx(u) + my(u)} - qE$.

Now two cases occur,

Case 6.4.1.1: Assuming $x(0) \leq K$ and our claim is $x(t) \leq K \quad \forall t \geq 0$. Otherwise there exist two positive real numbers t_1 and t_2 with $t_1 < t_2$ such that $x(t_1) = K$ and $x(t) > K \quad \forall t \in (t_1, t_2)$. Then $\forall t \in (t_1, t_2)$,

$$\begin{aligned} x(t) &= x(0) \exp \left[\int_0^t F(x(u), y(u)) du \right] \\ &= x(0) \exp \left[\int_0^{t_1} F(x(u), y(u)) du \right] \exp \left[\int_{t_1}^t F(x(u), y(u)) du \right] \\ &= x(t_1) \exp \left[\int_{t_1}^t F(x(u), y(u)) du \right] \text{ [Using equation (6.6)]} \\ &< x(t_1) \end{aligned}$$

as $F(x(u), y(u)) < 0 \quad \forall t \in (t_1, t_2)$, which contradicts of our hypothesis. Hence our assumption is right, i.e., $x(t) \leq K \quad \forall t \geq 0$.

Case 6.4.1.2: Here we consider $x(0) > K$. Then as long as $x(t) \geq K, t \geq 0$.

$x(t) = x(0)\exp\left[\int_0^t F(x(u), y(u))du\right] < x(0)$, as $F(x(u), y(u)) < 0 \quad \forall x(t) > K$.

Combining cases 6.4.1.1 and 6.4.1.2, we get for any positive solution which satisfies $x(t) \leq \max\{x(0), K\}$ for $t \geq 0$.

Again from (6.4), we have

$$\frac{dy(t)}{dt} < \beta y(t).$$

Hence for $t > \tau$, $y(t) \leq y(t - \tau)e^{\beta\tau}$, i.e., $y(t) \geq y(t)e^{-\beta\tau}$.

Substituting this value in system of equations (6.4), we get

$$\frac{dy(t)}{dt} \leq y \left(\beta - \frac{e^{-\beta\tau}}{n} y(t) \max\{x(0), K\} \right)$$

which implies that

$$\limsup_{t \rightarrow \infty} y(t) \leq \beta \frac{e^{\beta\tau}}{n} \max\{x(0), K\} = g_1, \text{ for } t > \tau.$$

6.5 Local stability and Hopf bifurcation analysis

In this Section, the stability has been described at the interior equilibrium point. Then two cases arise:

Case 6.5.1: For $\tau = 0$

Here the system has been discussed without delay around its interior equilibrium point (x^*, y^*) . The characteristic equation of the system in equation (6.4) is

$$\lambda^2 + h_1\lambda + h_2 = 0$$

where $h_1 = \frac{r}{K}x^* + \beta - \frac{bn\alpha\beta x^{*2}}{\{an+(nb+m\beta)x^*\}^2}$ and $h_2 = \frac{r\beta}{K}x^* + \frac{an\alpha\beta x^*}{\{an+(bn+\beta m)x^*\}^2}$. Now using Routh-Hurwitz criteria around the interior equilibrium point, we get the system (6.4) will be locally asymptotically stable in (x^*, y^*) if the system follows the following condition $\frac{r}{K} + \beta > \frac{bn\alpha\beta x^*}{\{an+(nb+m\beta)x^*\}^2}$.

Case 6.5.2: For $\tau \neq 0$

Here, we discuss the system in presence of delay around its interior equilibrium point (x^*, y^*) , For simplicity, we consider $v_1 = x - x^*$ and $v_2 = y - y^*$. Then by

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Taylor series expansion of system in equations (6.4) about the point (x^*, y^*) as follows:

$$\left. \begin{aligned} \frac{dv_1(t)}{dt} &= a_1 v_1(t) + a_2 v_2(t) + \sum_{i+j \geq 2} a_{ij} v_1^i(t) v_2^j(t) \\ \frac{dv_2(t)}{dt} &= b_1 v_1(t - \tau) + b_2 v_2(t - \tau) + \sum_{i+j+k \geq 2} b_{ijk} v_1^i(t - \tau) v_2^j(t - \tau) v_2^k(t - \tau) \end{aligned} \right\} (6.7)$$

where

$$\begin{aligned} a_1 &= r - 2\frac{rx^*}{K} - qE - \frac{(a + my^*)\alpha y^*}{(a + bx^* + my^*)^2}, & a_2 &= -\frac{(a + bx^*)\alpha x^*}{(a + bx^* + my^*)^2} \\ b_1 &= n\frac{y^{*2}}{x^{*2}}, & b_2 &= -n\frac{y^*}{x^*} \\ a_{ij} &= \frac{1}{i!j!} \frac{\partial^{i+j} f_1}{\partial x^i(t) \partial y^j(t)} \Big|_{(x^*, y^*)} \\ b_{ijk} &= \frac{1}{i!j!k!} \frac{\partial^{i+j+k} f_2}{\partial x^i(t - \tau) \partial y^j(t - \tau) \partial y^k(t - \tau)} \Big|_{(x^*, y^*)} \\ f_1 &= rx \left(1 - \frac{x}{K}\right) - \frac{\alpha xy}{a + bx + my} - qEx \\ f_2 &= y \left(\beta - n\frac{y(t - \tau)}{x(t - \tau)}\right) \end{aligned}$$

To investigate the stability at the interior equilibrium point of system of equations (6.4), we linearize the system (6.7) as follows:

$$\left. \begin{aligned} \frac{dv_1(t)}{dt} &= a_1 v_1(t) + a_2 v_2(t) \\ \frac{dv_2(t)}{dt} &= b_1 v_1(t - \tau) + b_2 v_2(t - \tau) \end{aligned} \right\} (6.8)$$

Then the characteristic equation of the system will become as:

$$\begin{aligned} &\begin{vmatrix} a_1 - \lambda & a_2 \\ b_1 e^{-\lambda\tau} & b_2 e^{-\lambda\tau} - \lambda \end{vmatrix} = 0 \\ \text{i.e., } &\lambda^2 - P\lambda - Q\lambda e^{-\lambda\tau} + R e^{-\lambda\tau} = 0 \end{aligned} \quad (6.9)$$

where $P = a_1$, $Q = b_2$ and $R = a_1 b_2 - b_1 a_2$.

The eigen values of the system (6.4) around its interior equilibrium point (x^*, y^*) are denoted as $\lambda(\tau) = \phi(\tau) \pm i\omega(\tau)$, ($\omega > 0$). But the change of stability around interior equilibrium point will occur for $Real(\lambda) = 0$. Hence

it is found that the position of stability for $\lambda(\tau) = \pm i\omega(\tau)$. So, from equation (6.9), we have

$$\omega^2 + Q\omega \sin(\omega\tau) - R \cos(\omega\tau) + i(P\omega + Q\omega \cos(\omega\tau) + R \sin(\omega\tau)) = 0 \quad (6.10)$$

Separating real and imaginary parts from equation (6.10), we have

$$\omega^2 + Q\omega \sin(\omega\tau) - R \cos(\omega\tau) = 0 \quad (6.11)$$

$$P\omega + Q\omega \cos(\omega\tau) + R \sin(\omega\tau) = 0 \quad (6.12)$$

Solving equations (6.11) and (6.12), we get

$$\cos(\omega\tau) = \omega^2 \frac{R - PQ}{R^2 + Q^2\omega^2} \quad (6.13)$$

$$\sin(\omega\tau) = -\omega \frac{PR + Q\omega^2}{R^2 + Q^2\omega^2} \quad (6.14)$$

Eliminating trigonometric functions from equations (6.13) and (6.14), we get

$$\omega^4 + \omega^2(P^2 - Q^2) - R^2 = 0 \quad (6.15)$$

Hence, the unique positive root (say ω_0) of equation (6.15) is

$$\omega_0 = \sqrt{\frac{-(P^2 - Q^2) + \sqrt{(P^2 - Q^2)^2 + 4R^2}}{2}} \quad (6.16)$$

Since ω_0 is the root of equation (6.15), then from equation (6.14), we get the corresponding critical value of time delay τ_k as

$$\tau_k = \frac{1}{\omega_0} \left\{ \arcsin \left(-\omega_0 \frac{PR + Q\omega_0^2}{R^2 + Q^2\omega_0^2} \right) + 2k\pi \right\}, \quad k = 0, 1, 2, \dots \quad (6.17)$$

Now we test whether the roots of (6.10) pass the imaginary axis of the complex plane as τ varies. Assuming that $\phi(\tau_k) = 0$ and $\omega(\tau_k) = \omega_0$ when $\lambda(\tau) = \phi(\tau) + i\omega(\tau)$ be the root of equation (6.10).

Theorem 6.5.1. *The following transversality conditions are satisfied*

$$\left[\frac{d(R(\lambda(\tau)))}{d\tau} \right]_{\tau=\tau_k} > 0.$$

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Proof: Applying $\lambda(\tau) = \phi(\tau) + i\omega(\tau)$ in equation (6.9) and taking the derivative with respect to τ , we get

$$\left[\frac{d\lambda}{d\tau} \right]^{-1} = \frac{e^{\lambda\tau}(2\lambda - P)}{\lambda(R - Q\lambda)} - \frac{Q}{\lambda(R - Q\lambda)} - \frac{\tau}{\lambda} \quad (6.18)$$

Using (6.9) in (6.18), we get

$$\begin{aligned} R \left[\frac{d\lambda}{d\tau} \right]_{\lambda=i\omega_0}^{-1} &= R \left[\frac{e^{\lambda\tau}(2\lambda - P)}{\lambda(R - Q\lambda)} - \frac{Q}{\lambda(R - Q\lambda)} \right]_{\lambda=i\omega_0} \\ &= \left[\frac{(-PR - 2Q\omega_0^2) \sin(\omega_0\tau) + (-PQ + 2C)\omega_0 \cos(\omega_0\tau) - Q^2\omega_0}{(R^2 + Q^2\omega_0^2)\omega_0} \right] \\ &= \frac{R^2}{(R^2 + Q^2\omega_0^2)\omega_0^2} > 0 \end{aligned}$$

For each $\tau = \tau_k$, the transversality condition and Hopf-bifurcation are valid. When $\tau = \tau_k$, the biquadratic equation (6.15) has only one positive root and therefore, there exists no interval for the time delay τ for which the switching of stability from instability to stability of the interior equilibrium is possible [cf. (56),(60),(90)].

Considering $\tau^* = \min_k \{\tau_k\}$ i.e., τ_* is the smallest positive value of τ_k , $k = 1, 2, 3, \dots$ is given by equation (6.17). Now from case 6.5.1 and case 6.5.2 at the positive equilibrium point E_3^* with Hopf bifurcation of the proposed model, we consider the following lemma.

Lemma 5. *The interior equilibrium point E_3^* of the system (6.4) exists and locally asymptotically stable when $\tau \in [0, \tau_*)$ and unstable when $\tau > \tau^*$ if $\frac{r}{K} + \beta > \frac{bn\alpha\beta x^*}{\{an+(nb+m\beta)x^*\}^2}$. Also, when $\tau = \tau^*$ then the system (6.4) undergoes a Hopf bifurcation at E_3^* .*

6.6 Direction and stability of Hopf bifurcation

Here, we study the direction and stability of Hopf bifurcation of the bifurcating periodic solutions at positive interior equilibrium point $E_3^*(x^*, y^*)$ by using normal theory and the center manifold theorem introduced by Hassard et al. (40). We derive the sufficient condition that the system (6.4) undergoes the

Hopf bifurcation at the critical value $\tau = \tau^*$. Considering $\tau = \tau^* + \mu, \mu \in \mathbb{R}$.

That implies Hopf bifurcation occurs for $\mu = 0$ of the system (6.4).

Let $v_1(t) = x(t) - x^*$ and $v_2(t) = y(t) - y^*$. We normalize the delay with time scaling $t \rightarrow \frac{t}{\tau}$. Then the system of equations (6.4) can be written as

$$\dot{v}(t) = L_\mu(v_t) + F(\mu, v_t) \quad (6.19)$$

where $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $L_\mu : C \rightarrow \mathbb{R}, F : \mathbb{R} \times C \rightarrow \mathbb{R}$ are given respectively by

$$L_\mu(\phi) = (\tau^* + \mu)B_1 \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + (\tau^* + \mu)B_2 \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix} \quad (6.20)$$

where $B_1 = \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 0 \\ b_1 & b_2 \end{pmatrix}$ and

$$F(\mu, \phi) = (\tau^* + \mu) \begin{pmatrix} \sum_{i+j \geq 2} a_{ij} \phi_1^i(0) \phi_2^j(0) & \cdots \\ \sum_{i+j+k \geq 2} b_{ijk} \phi_1^i(-1) \phi_2^j(0) \phi_2^k(-1) & \cdots \end{pmatrix} \quad (6.21)$$

Where the values of a_1, a_2, b_1 and b_2 are same as in equation (6.7) and $a_{20} = -\frac{r}{K} + b\alpha \frac{(a+my^*)y^*}{(a+bx^*+my^*)^3}, a_{11} = -\frac{\alpha a}{(a+bx^*+my^*)^2} - 2\alpha b m \frac{x^*y^*}{(a+bx^*+my^*)^3}, a_{02} = \frac{m\alpha(a+bx^*)x^*}{(a+bx^*+my^*)^3}, b_{200} = -n \frac{y^{*2}}{x^{*3}}, b_{210} = -n \frac{y^*}{x^{*3}}, b_{300} = n \frac{y^{*2}}{x^{*4}}, b_{011} = -\frac{n}{x^*}, b_{110} = n \frac{y^*}{x^{*2}}, b_{101} = n \frac{y^*}{x^{*2}}$. By Riesz representation theorem, there exists a 2×2 matrix function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$, such that

$$L_\mu \phi = \int_{-1}^0 d\eta(\theta, \mu) \phi(\theta), \quad \phi \in C \quad (6.22)$$

Assuming

$$\eta(\theta, \mu) = (\tau^* + \mu)(B_1 \delta(\theta) + B_2 \delta(\theta + 1)) \quad (6.23)$$

where $\delta(\theta)$ is the Kronecker delta function defined by

$$\delta(\theta) = \begin{cases} 0, & \theta \neq 0, \\ 1, & \theta = 0. \end{cases}$$

For $\phi \in C([-1, 0], \mathbb{R}^2)$, we define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^0 d\eta(\mu, s) \phi(s), & \theta = 0, \end{cases}$$

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and

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ F(\mu, \phi), & \theta = 0, \end{cases}$$

Then the system of equation (6.19) can be transformed into the following equation

$$\dot{v}(t) = A(\mu)v_t + R(\mu)v_t \quad (6.24)$$

where $v_t = v(t + \theta) = (v_1(t + \theta) + v_2(t + \theta))$.

For $\psi \in C^1([0, 1], (\mathbb{R}^2)^*)$, where $(\mathbb{R}^2)^*$ is the 2-dimensional space of row vectors, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in [-1, 0), \\ \int_{-1}^0 d\eta^T(t, 0)\psi(-t), & s = 0, \end{cases}$$

and a bilinear inner product:

$$\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0)\phi(0) - \int_{\theta=-1}^0 \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi,$$

where $\eta(\theta) = \eta(\theta, 0)$ and $A(0)$, A^* are adjoint operators. From Section 6.5, we have $\pm i\omega_0\tau^*$ are eigenvalues of $A(0)$ for the time scaling $t \rightarrow \frac{t}{\tau}$, thus, these are the eigenvalues of A^* . Let $q(\theta) = (1, \rho)^T e^{i\omega_0\tau^*\theta}$ is the eigenvector of $A(0)$ corresponding to eigenvalue $i\omega_0\tau^*$.

Then $A(0)q(\theta) = i\omega_0\tau^*q(\theta)$. It follows from the definition of $A(0)$ and equations (6.20), (6.22), (6.23) that

$$\tau^* \begin{pmatrix} i\omega_0 - a_1 & -a_2 \\ -b_1 e^{-i\omega_0\tau^*} & i\omega_0 - b_2 e^{-i\omega_0\tau^*} \end{pmatrix} q(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus we can easily obtain $q(0) = (1, \rho)^T$, where $\rho = \frac{b_1 e^{-i\omega_0\tau^*}}{i\omega_0 - b_2 e^{-i\omega_0\tau^*}}$.

Similarly, let $q^*(s) = D(1, \rho^*)e^{i\omega_0\tau^*s}$ is the eigenvector of A^* corresponding to $-i\omega_0\tau^*$, then by definition of A^* and equations (6.20), (6.21), (6.22) that

$$q^*(s) = D(1, \rho^*)e^{i\omega_0\tau^*s} = D \left(1, -\frac{i\omega_0 + a_1}{b_1 e^{i\omega_0\tau^*}} \right) e^{i\omega_0\tau^*s}.$$

In order to assure that $\langle q^*(s), q(\theta) \rangle = 1$, we need to choose a suitable value of D . From equation (6.24), we have

$$D = \frac{1}{1 + b_1\tau^*\rho^*e^{i\omega_0\tau^*} + \rho^*\bar{\rho}(1 + b_2\tau^*e^{i\omega_0\tau^*s})}.$$

Recall that the Hopf-bifurcating periodic solutions $(x(t, \mu(\epsilon)), y(t, \mu(\epsilon)))$ of (6.4) have period $T(\epsilon)$ and a nonzero Floquet exponent $B(\epsilon)$, where μ, B and T have the following expression:

$$\begin{aligned}\mu &= \mu_2\epsilon^2 + \mu_4\epsilon^4 + \dots \\ B &= \beta_2\epsilon^2 + \beta_4\epsilon^4 + \dots \\ T &= 2\Pi \frac{(1 + T_2\epsilon^2 + T_4\epsilon^4 + \dots)}{\omega_0}\end{aligned}$$

ϵ is a small non-negative parameter. To determine the direction of Hopf-bifurcation, the stability of the periodic solutions and period of the bifurcating periodic solutions, we only investigate the coefficients μ_2, β_2 and T_2 . As per Pal and Mondal (73), we get the following results as stated below.

$$g_{ij}(\mu) = \frac{\partial^{r+s}}{\partial z^r \partial \bar{z}^s} (e_1(\mu), F(ze_2(\mu) + \bar{z}\bar{e}_2(\mu), \mu)) |_{z=0} \quad \text{when } r + s > 0,$$

$$i, j = 0, 1, 2, \dots$$

$$C_1(0) = \frac{i}{2\omega_0\tau^*} \left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3} \right) + \frac{g_{21}}{2} \quad (6.25)$$

$$\mu_2 = -\frac{Re\{C_1(0)\}}{Re\{d\lambda(\tau_0^*)/d\tau\}} \quad (6.26)$$

$$\beta_2 = 2Re\{C_1(0)\} \quad (6.27)$$

$$T_2 = -\frac{Im\{C_1(0)\} + \mu_2 Im\{d\lambda(\tau_0^*)/d\tau\}}{\omega_0\tau^*} \quad (6.28)$$

According to Hassard *et al.*(40), we can state the properties of Hopf bifurcation at the critical value of $\tau = \tau^*$ in the following Remark.

Remark: From the equations (6.25)-(6.28), we can conclude that

A. The direction of Hopf bifurcation is determined by the sign of μ_2 .

(i) If $\mu_2 > 0$; the Hopf bifurcation is supercritical.

(ii) If $\mu_2 < 0$; the Hopf bifurcation is subcritical.

B. The stability of the bifurcating periodic solution is determined by β_2 .

(i) If $\beta_2 > 0$; the bifurcated periodic solutions are unstable.

(ii) If $\beta_2 < 0$; the bifurcated periodic solutions are stable.

C. The period of the bifurcating periodic solution is determined by T_2 .

(i) If $T_2 > 0$; period of the bifurcating periodic solution increases.

(ii) If $T_2 < 0$; period of the bifurcating periodic solution decreases.

6.7 Bionomic equilibrium and optimal control

An amalgamation of the concept of economic equilibrium and biological equilibrium is called bionomic equilibrium.

The net economic revenue is obtained from the fishery is $P(x, y, E, t) =$ the total revenue obtained by selling the harvested biomass – total cost for the effort devoted to harvesting = $pqEx - cE$, where p is the constant price per unit biomass of the prey and c is the constant cost per unit effort, then we consider the present value J of a continuous time-stream of revenues as follows:

$$J = \int_0^{\infty} e^{-\delta t} P(x, y, E, t) dt \quad (6.29)$$

where $\delta (> 0)$ denotes the instantaneous annual rate of discount (82). Our aim is to maximize J subject to the state equations (6.4) using *Pontryagin's maximum principle* (80). The control variable $E(t)$ is subject to the constraint set $0 \leq E \leq E_{max}$. At first, we construct the corresponding Hamiltonian function

$$\begin{aligned} H = & e^{-\delta t} (pqx - c)E + \lambda_1 \left\{ rx \left(1 - \frac{x}{K} \right) - \frac{\alpha xy}{a + bx + my} - qEx \right\} \\ & + \lambda_2 y \left(\beta - n \frac{y}{x} \right) \end{aligned} \quad (6.30)$$

where λ_1 and λ_2 are called the adjoint variables.

By Pontryagin's maximum principle, the adjoint equations are as follows:

$$\begin{aligned} \frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x} = & - \left[e^{-\delta t} pqE + \lambda_1 \left\{ r - 2x \frac{r}{K} - \frac{(a + my)\alpha y}{(a + bx + my)^2} - qE \right\} \right. \\ & \left. + \lambda_2 n \frac{y}{x^2} \right] \end{aligned} \quad (6.31)$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial y} = \lambda_1 \frac{(a + bx)\alpha x}{(a + bx + my)^2} + \lambda_2 \left\{ 2n \frac{y}{x} - \beta \right\} \quad (6.32)$$

Now, we derive an optimal equilibrium solution of the problem at the interior equilibrium $E_3^*(x^*, y^*)$. Then from equations (6.31) and (6.32), we get

$$\frac{d\lambda_1}{dt} = -e^{-\delta t} pqE - \lambda_1 A_1 - \lambda_2 A_2 \quad (6.33)$$

$$\frac{d\lambda_2}{dt} = \lambda_1 A_3 + \lambda_2 A_4 \quad (6.34)$$

where $A_1 = -\frac{r}{K}x^* + \frac{b\alpha x^*y^*}{(a+bx^*+my^*)^2}$, $A_2 = \frac{\beta}{x^*}$, $A_3 = \frac{(a+bx^*)\alpha x^*}{(a+bx^*+my^*)}$, $A_4 = \beta$.
Combining equations (6.33) and (6.34), we have

$$\frac{d^2\lambda_2}{dt^2} + (A_1 - A_4)\frac{d\lambda_2}{dt} + (A_2A_3 - A_1A_4)\lambda_2 = -e^{-\delta t}A_3pqE \quad (6.35)$$

The particular solution of equation (6.35) is $\frac{A_3pqE}{\delta^2 - (A_1 - A_4)\delta + (A_2A_3 - A_1A_4)}e^{-\delta t}$. The auxiliary equation of (6.35) is

$$\eta^2 + (A_1 - A_4)\eta + (A_2A_3 - A_1A_4) = 0 \quad (6.36)$$

with two roots η_1 and η_2 . Here $\eta_1 + \eta_2 = A_4 - A_1$ and $\eta_1\eta_2 = (A_2A_3 - A_1A_4)$. Then the following cases arise.

Case 6.7.1: $(A_1 - A_4) > 0$ and $(A_2A_3 - A_1A_4) > 0$. In this case η_1 and η_2 are both either real and negative or complex conjugate with negative real part. Hence the solution of (6.35) is $\lambda_1 = C_1e^{\eta_1 t} + C_2e^{\eta_2 t} + \frac{A_3pqE}{\delta^2 - (A_1 - A_4)\delta + (A_2A_3 - A_1A_4)}e^{-\delta t}$.
Now $\lambda_1 \rightarrow 0$ as $t \rightarrow \infty$.

Case 6.7.2: $(A_1 - A_4) > 0$ and $(A_2A_3 - A_1A_4) < 0$. In this case η_1 and η_2 are both either real and unequal with negative root having greater magnitude. Hence the solution of (6.35) is $\lambda_1 = C_1e^{\eta_1 t} + C_2e^{\eta_2 t} + \frac{A_3pqE}{\delta^2 - (A_1 - A_4)\delta + (A_2A_3 - A_1A_4)}e^{-\delta t}$.
Now $\lambda_1 \rightarrow \infty$ as $t \rightarrow \infty$.

Case 6.7.3: $(A_1 - A_4) < 0$ and $(A_2A_3 - A_1A_4) > 0$. In this case η_1 and η_2 are both either real and positive or complex conjugate with positive real part. Hence the solution of (6.35) is $\lambda_1 = C_1e^{\eta_1 t} + C_2e^{\eta_2 t} + \frac{A_3pqE}{\delta^2 - (A_1 - A_4)\delta + (A_2A_3 - A_1A_4)}e^{-\delta t}$.
Now $\lambda_1 \rightarrow \infty$ as $t \rightarrow \infty$.

Case 6.7.4: $(A_1 - A_4) < 0$ and $(A_2A_3 - A_1A_4) < 0$. In this case η_1 and η_2 are both either real and unequal with positive root having greater magnitude. Hence the solution of (6.35) is $\lambda_1 = C_1e^{\eta_1 t} + C_2e^{\eta_2 t} + \frac{A_3pqE}{\delta^2 - (A_1 - A_4)\delta + (A_2A_3 - A_1A_4)}e^{-\delta t}$.
Now $\lambda_1 \rightarrow \infty$ as $t \rightarrow \infty$.

From above cases, it is seen that finite optimal equilibrium exists only for the first case, i.e., in Case 6.5.1. Hence we have the solution as follows:

$$\lambda_1 = C_1e^{\eta_1 t} + C_2e^{\eta_2 t} + \frac{A_3pqE}{\delta^2 - (A_1 - A_4)\delta + (A_2A_3 - A_1A_4)}e^{-\delta t}.$$

6.8 Numerical simulation

From Sections 6.3 and 6.5, we know that the stability of the trivial equilibrium and the boundary equilibrium are relatively simple. However, the stability of positive equilibrium (x^*, y^*) is complex. Roughly speaking, delay cannot switch only the stability but also leads to Hopf bifurcation. To analyze the sensitivity analysis of the experiments, we use MATLAB 7.10 software and some arbitrary data for describing the analytical results. Assuming that the parameters set taken as $r = 2.1; K = 50; \alpha = 0.86; a = 0.8; b = 0.1; m = 0.05; n = 0.5; q = 0.2; E = 0.76$; and the initial values $x(0) = 8, y(0) = 1$ for system (6.4).

For $\tau = 0$, two cases occur as follows:

Case 6.8.1.1: When $\beta (= 0.4) > \beta^*$,

From Figures 6.1 and 6.2, we conclude that system (6.4) will be stable after certain time.

Case 6.8.1.2: When $\beta (= 0.3) < \beta^*$,

From Figures 6.3 and 6.4, we conclude that system (6.4) will always unstable. Hence we conclude that, in absence of time delay of predator population, the system (6.4) will be stable when biotic potential of predator population is greater than or equal 0.4.

For $\tau \neq 0$, two cases arise as follows:

Case 6.8.2.1: When $\tau (= 0.65) < \tau^*$,

From Figures 6.5 and 6.6, we conclude that system (6.4) will be stable after certain time.

Case 6.8.2.2: When $\tau (= 1) > \tau^*$,

From Figures 6.7 and 6.8, we conclude that system (6.4) will always unstable. It is well known that krill (prey) has an economical demand in marketing management. Due to this fact, we include the harvesting for prey population in the dynamical system. In Figures 1 and 2, we choose the parameter set $r = 2.1; K = 50; \alpha = 0.86; a = 0.8; b = 0.1; m = 0.05; n = 0.5; \beta = 0.4$ and $q = 0.2; E = 0.76$. Here we see that system with harvesting is stable after certain time. But when we consider same parameter set in absence of

harvesting i.e., $q = 0; E = 0$, then we draw Figures 6.9 and 6.10 and the system is unstable. Therefore from Figures 6.1 and 6.2, we observe that the system is stable with harvesting whereas the system is unstable at the same time without harvesting which is shown in Figures 6.9 and 6.10.

6.9 Chapter Summary

In this chapter an ecological food chain model has been presented based on Beddington-DeAngelis type functional response and using Holling-Tanner model for modeling the relationship between prey-predator populations. Harvesting effort on prey has been considered in the model with gestational delay of predator. The boundedness of the system, existence of an attracting set, existence of local stability of non-negative equilibrium point in delay preventing system have been established. The direction and stability of Hopf bifurcation have been established with the help of normal theory and center manifold theorem. The problem has been illustrated with a numerical example including many geometrical figures.

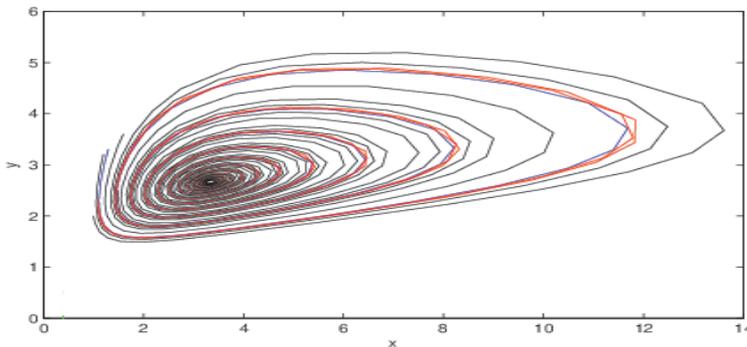


Figure 6.1: Phase space diagram of the system (6.4) for $\beta(= 0.4) > \beta^*$.

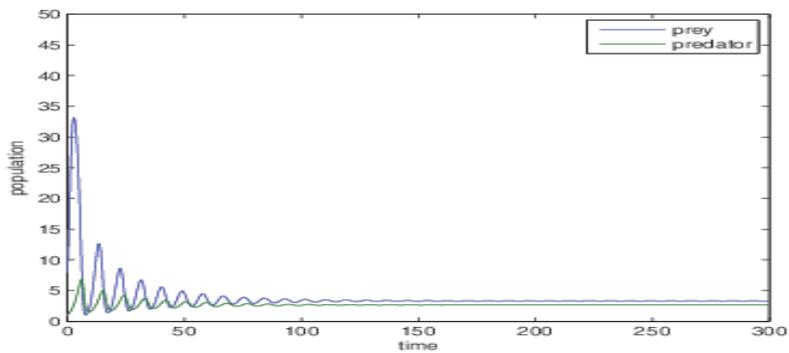


Figure 6.2: *Solution curve of the system (6.4) for $\beta(= 0.4) > \beta^*$.*

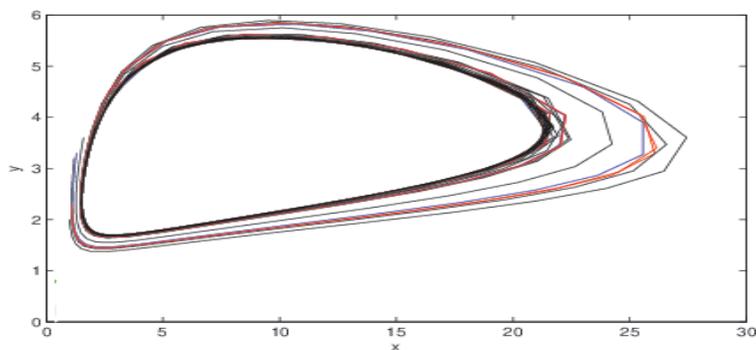


Figure 6.3: *Phase space diagram of the system (6.4) for $\beta(= 0.3) < \beta^*$.*

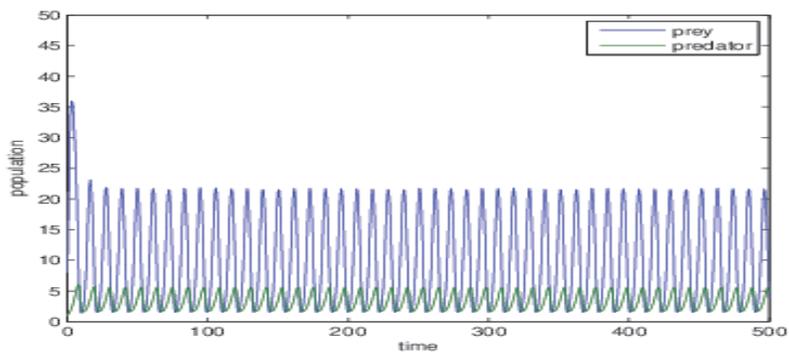


Figure 6.4: *Solution curve of the system (6.4) for $\beta(= 0.3) < \beta^*$.*

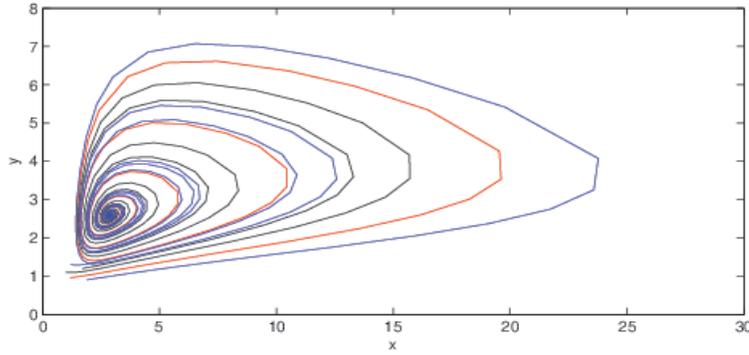


Figure 6.5: Phase space diagram of the system (6.4) for $\tau(= 0.65) < \tau^*$.

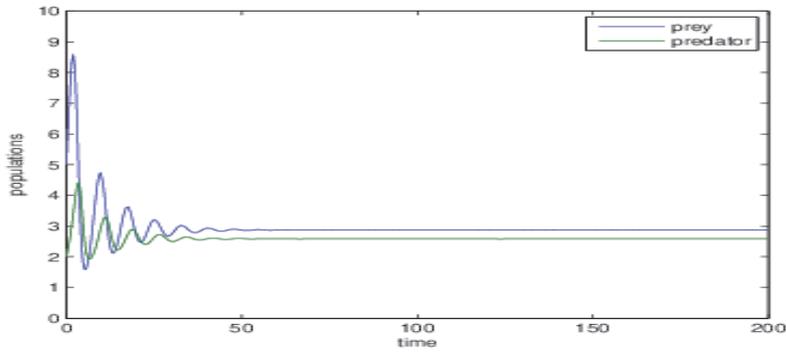


Figure 6.6: Solution curve of the system (6.4) for $\tau(= 0.65) < \tau^*$.

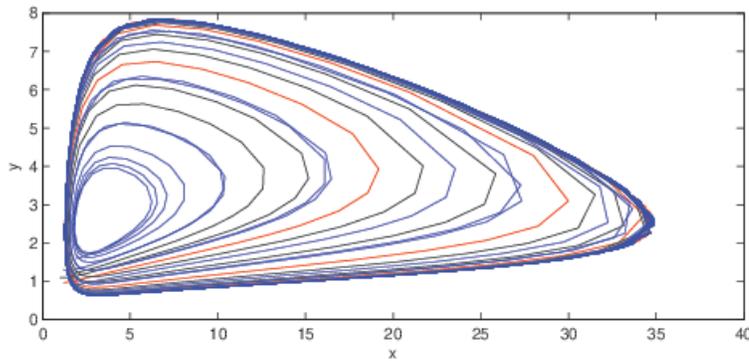


Figure 6.7: Phase space diagram of the system (6.4) for $\tau(= 1) > \tau^*$.

6.9. Chapter Summary

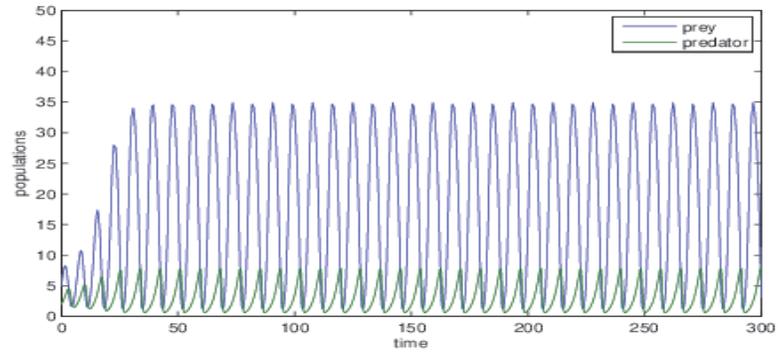


Figure 6.8: *Solution curve of the system (6.4) for $\tau(=1) > \tau^*$.*

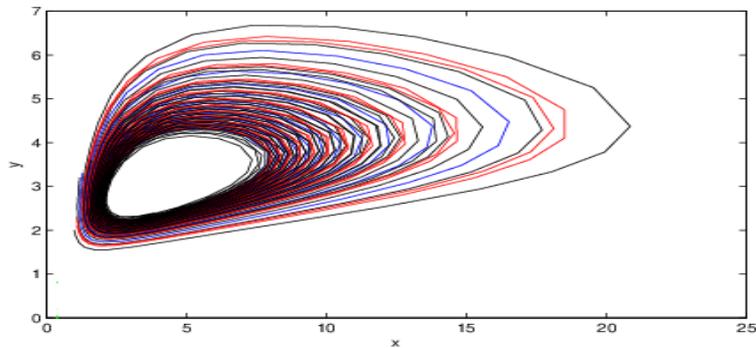


Figure 6.9: *Phase space diagram of the system (6.4) in absence of harvesting.*

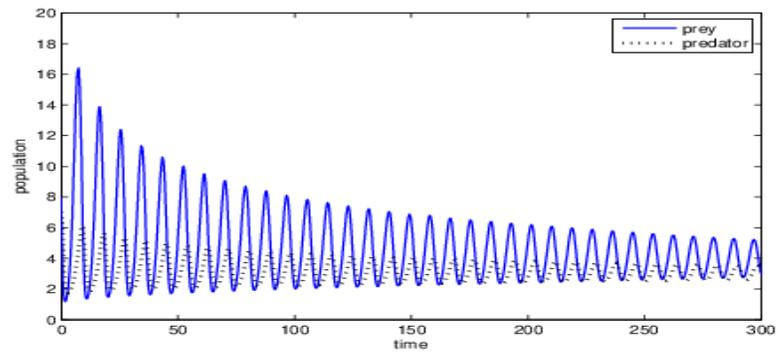


Figure 6.10: *Solution curve of the system (6.4) in absence of harvesting.*

