UG/5th Sem/Math(H)/T/19

2019

B.Sc. (Honours)

5th Semester Examination

MATHEMATICS

Paper - C12T

(Group Theory II)

Full Marks : 60 Time : 3 Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Unit - I

(Automorphism Groups)

1. Answer any *three* questions : 2×3

(a) Define inner automorphism on a group.

(b) Show that characteristic subgroups are normal.

(c) Is $\mathbb{Z}\oplus\mathbb{Z}$ a cyclic group ? Justify your answer.

[Turn Over]

(d) Let G be a finite group, ϕ an automorphism of G with $\phi(x)=x$ for $x \in G$ if and only if $x=e$. Prove that every $g \in G$ can be represented as $g=x^{-1}\phi(x)$ for some $x \in G$.

(e) Give an example to show that a normal subgroup of a group is not a characteristic subgroup of the group.

2. Answer any *two* questions : 5×2

(a) Let G be a group. Then show that

$G/Z(G) \simeq Inn(G)$, where $Inn(G)$ denotes the group of all inner automorphisms of G.

(b) Find the number of inner automorphisms of the Symmetric group S_3.

(c) Define the commutator subgroup of a group. Prove that a commutator subgroup of a group is a characteristic subgroup of the group.

Unit - II

(Direct Products)

3. Answer any *three* questions : 2×3

(a) Find the number of elements of order 5 in the group $\mathbb{Z}_{15} \times \mathbb{Z}_{10}$.

(b) Let H and K be two finite cyclic groups of order m and n respectively. Show that the group $H \times K$ is cyclic if and only if $\gcd(m, n) = 1$.

(c) State the fundamental theorem of finite abelian groups. Use it to classify all abelian groups of order 540.

(d) Prove that the direct product of two groups A and B is abelian if and only if both A, B are commutative.

(e) Express U_{12} as external direct product of cyclic groups.

[*Turn Over*]

4. Answer any *one* question : 5×1

(a) Let G be an internal direct product of its normal subgroups $N_1, N_2, ..., N_n$.

Show that $G \simeq N_1 \times N_2 \times ... \times N_n$ (external direct product)

(b) Let G be a group and H, K be two subgroups of G. Prove that G is an internal direct product of H and K if and only if the following conditions are satisfied.

(i) $G = HK$;

(ii) H, K are normal in G;

(iii) $H \cap K = \{e\}$.

Unit - III

(Group Actions)

5. Answer any *two* questions : 2×2

(a) Show that the kernel of the group action is a subgroup.

(b) Let G be a finite group acting on a set S, and let $x \in S$. Then show that

$$|G| = |Orb_G(x)|\,|Stab_G(x)|.$$

(c) Let G be a group acting on a non-empty set S. Define orbits of G on S and stabilizer of a in G where $a \in S$.

6. Answer any *one* question : 10×1

(a) (i) Let G be a group and S be a G-Set. Then show that the left action of G on S induces a homomorphism from G onto $A(S)$, where $A(S)$ is the group of all permutations of S. 6

(ii) Let $X = \{1, 2, 3, 4, 5, 6\}$ and suppose that G is the permutation group given by the permutations {(1), (12)(3456), (35)(46), (12)(3654)}. Find the stabilizer subgroups. 4

(b) (i) Let G be a group acting on a non-empty set S. Prove that $[G : G_a] = |[a]|$ where

[*Turn Over*]

$[a]$ denotes the orbit of a, G_a denotes the stabilizer of a. 5

(ii) Let G be a group acting on a non-empty set S. Then show that this action of G on S induces a homomorphism from G to $A(S)$. 5

Unit - IV

(Class Equation and Sylow's Theorem)

7. Answer any *two* questions : 2×2

(a) State Sylow's third theorem.

(b) Give example of an infinite p-group, p is a prime.

(c) Let H be a normal subgroup of a group G. If H and G/H are both p-groups, then show that G is also a p-group.

8. Answer any *one* question : 5×1

(a) Let G be a finite group and H be a Sylow-p-subgroup of G. Then prove that H is a unique Sylow p-subgroup if and only if H is normal in G.

(b) Find the conjugacy classes in the dihedral group D_4 and write down the class equation.

9. Answer any *one* question : 10×1

(a) (i) Let G be a group of order pq where p, q are primes. Then prove that G can't be simple.

(ii) Deduce the class equation of S_3. 5+5

(b) (i) Classify all the groups of order 99 upto isomorphism.

(ii) Prove that in a finite group G, the number of elements in the conjugacy class of $a(\in G)$ is a divisor of $0(G)$. 5+5