 UG/4th Sem/MATH./19

2019

4th Semister Examination

MATHEMATICS

Subject Code - GE4T

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable. Illustrate the answers wherever necessary.

Numerical Method

Full Marks : 40 Time : 2 Hours

1. Answer any ***five*** questions : 2×5

(a) What are the sources of errors in numerical computation ?

(b) Write down the sufficient condition for the convergence of the Gauss - Seidel - iteration method.

(c) Write the advantages and disadvantages of fixed point iteration method.

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(d) Prove that $\Delta\nabla f(x) = \Delta f(x) - \nabla f(x)$, where the symbol's Δ and ∇ carry their usual meaning.

(e) Write the formula of Runge - Kutta method of order four to solve the initial value problem $y' = f(x, y)$ with $y(x_0) = y_0$.

(f) Define 'Degree of precision' of a numerical integration formulae.

(g) Why relative error is a better indicator of the accuracy of a computation than the absolute error?

(h) Show by an example that the Simpson's $\frac{1}{3}$ rule is exact for integrating a polynomial of degree 3.

2. Answer any ***four*** questions $5\times4=20$

(a) Describe Newton - Raphson method for computing a simple root of an equation $f(x) = 0$. What is the sufficient condition for convergent of Newton - Raphson method?

(b) Derive the Simpson's one third integration formula in the form

$$\int_a^b f(x)dx = \frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]$$

$$-\frac{(b-a)^5}{2^5 \times 90} f^{iv}(z) \text{ where } a < z < b.$$

(c) Solve the following system of equation by Gauss - Seidal method correct to three significant figures

$3x + y + z = 3$

$2x + y + 5z = 5$

$x + 4y + z = 2$

(d) Apply Runge Kutta method of order 4, find the values of y at 0.1 where $y' = x^2 + y^2$ with $x = 0, y = 1$.

(e) Show that the *n-th* order divided difference of a polynomial of degree n is constant.

3. Answer any ***one*** question [1×10]

(a) (i) Establish Newton's forward difference interpolation formula for the equispaced interpolating points.

(ii) Construct Lagrange's interpolation

polynomial for the function $y = \sin \pi x$, choosing the points $x_0 = 0, x_1 = \frac{1}{6}, x_2 = \frac{1}{2}$. 5+5

(b) (i) Establish Newton - cotes quadrature formula for numerical integration of f(x) in [a, b] whose functional values are unknown at (n + 1) equispaced distinct points.

(ii) Write down the modified Eule formula to solve the differential equation $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ and state why it is better than Euler method. [6+(2+2)]

Partial Differential Equation and Applications

1. Answer any ***ten*** questions out of following fifteen questions : [10×2]

(a) What is the order and degree of the following partial differential equation

$$\left(\frac{\delta z}{\delta x}\right)^2 + \frac{\delta^3 z}{\delta y^3} = 2x\left(\frac{\delta z}{\delta x}\right)$$

(b) Eliminate arbitrary constants a and b from

$z = (x - a)^2 + (y - b)^2$ to form the partial differential equation.

(c) Show that characteristics equation of the partial differential equation $x^2r + 2xys + y^2t = 0$ represents a family of striaght lines passing through origin.

(d) Define semi-linear and Quasi linear partial differential equation.

(e) The equation $\frac{\delta^2 u}{\delta t^2} = c^2 \frac{\delta^2 u}{\delta x^2}$ is

(i) Parabolic,

(ii) hyperbolic,

(iii) elliptic,

(iv) none of these

(f) Find the complete integral of $yp + xq = pq$.

(g) Solve : $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} + z\frac{\delta u}{\delta z} = xyz$.

(h) Classify heat equation and Laplace equation.

(i) Classify the following partial differential equation

$$\frac{\delta^2 u}{\delta x^2} + 4\left(\frac{\delta^2 u}{\delta x \delta y}\right) + 4\frac{\delta^2 u}{\delta y^2} = 0$$

[Turn Over]

(j) Find complete integral of $z = px + qy + p^2 + q^2$.

(k) Write down D'Alemberts formula for the non-homogeneous wave equation.

(l) Eliminate arbitrary functions f and F from $y = f(x - at) + F(x + at)$ to form the partial differential equation.

(m) State basic existance theorem for caucly problem.

(n) Write the heat conduction and Laplace equation.

(o) If the Partial differential equation $(x - 1)^2u_{xx} - (y - 2)^2u_{yy} + 2xu_x + 2yu_y + 2xyu = 0$ is parabolic in $S \subseteq R^2$ but not in $R^2 \setminus S$, then S is

(i) $\{(x, y) \in \mathbb{R}^2 : x = 0 \text{ or } y = 2\}$,

(ii) $\{(x, y) \in \mathbb{R}^2 : x = 1 \text{ and } y = 2\}$

(iii) $\{(x, y) \in \mathbb{R}^2 : x = 1\}$

(iv) $\{(x, y) \in \mathbb{R}^2 : y = 2\}$

Choose the correct answer.

2. Answer any ***four*** questions out of six questions :

[5×4]

(a) Using Lagrange's method solve the partial differential equation $z(x + y)p + z(x - y)q = x^2 + y^2$.

(b) Write down the canonical form of one-dimensional wave equation : $\frac{\delta^2 z}{\delta x^2} - \frac{\delta^2 z}{\delta y^2} = 0$.

(c) Find the integral surface of the linear partial differential equation $x(y^2 + z)\, p - y(x^2 + z)q = (x^2 - y^2)z$ which contains the straight line $x + y = 0$, $z = 1$.

(d) A particle moves in teh curve $y = a\log_e \sec\left(\frac{x}{a}\right)$ in such a way that the tangent to the cuve rotaties uniformly. Prove that the resultant acceleration of the particle varies as the square of curvature.

(e) Establish the formula : $\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}$

for the motion of a particle describing a central orbit under an attractive force P per unit mass, the symbols having usual meaning.

(f) Obtain the POE which has its general solution $z = xf\left(\frac{y}{x}\right)$, where f is an arbitrary function.

3. Answer any ***two*** questions out of four questions [10×2]

(a) Use the method of separation of variables to defermine the solution u(x, y) of the problem which consists of Laplace equation $\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$ and the boundary conditions : $u(x, 0) = f(x)$, $0 \le x \le \Pi$, $u(x, \Pi) = 0$, $0 \le x \le \Pi$, $u(0, y) = u(\Pi, y) = 0, 0 \le y \le \Pi$.

(b) Find the solution of the initial boundary value problem

$$u_{tt} = u_{xx},\ 0 < x < 2,\ t > 0$$

$$u(x, 0) = \sin\left(\frac{\pi x}{2}\right),\ 0 \le x \le 2$$

$$u_t(x, 0) = 0,\ 0 \le x \le 2$$

$$u(0, t) = 0,\ u(2, t) = 0,\ t \ge 0$$

(c) Reduce the equation $\frac{\delta^2 z}{\delta x^2}+2\frac{\delta^2 z}{\delta x \delta y}+\frac{\delta^2 z}{\delta y^2}=0$ to cononical form and hence solve it.

(d) Find the solution of one - dimensional diffusion equation $k\frac{\delta^2 u}{\delta x^2}=\frac{\delta u}{\delta t}$ Satisfying the following boundary

Condition :

(i) u is bounded as $t \to \propto$

(ii) $u_x(0, t) = 0, u_x(a, t) = 0 \ \forall t$

(iii) $u(x, 0) = x(a - x), 0 < x < a$

Ring Theory and Linear Algebra I

Full Marks : 60 Time : 3 Hours

1. Answer any *ten* questions [10×2]

(a) Prove that the vectors (2, 3, 1), (2, 1, 3), (1, 1, 1) are linearly dependent.

(b) Prove that $T: R^3 \to R^3$ defined by $T(x_1, x_2, x_3) = (x_1 + 1, x_2 + 1, x_3 + 1), (x_1, x_2, x_3) \in R^3$

[Turn Over]

is not a linear mapping.

(c) What do you mean by factor rings?

(d) Prove that a ring R is commutative, if $(a + b)^2 = a^2 + 2ab + b^2$ for al a, $b \in R$ holds.

(e) Show that matrix ring over a field contains divisor of zero.

(f) Define prime and maximal ideals.

(g) Define homomorphism in ring. Check whether $\phi : R \to R$ defined by $\phi(x) = 2x$ is homomorphism or not, where $R = (\square, +, .)$.

(h) Prove that intersection of two subspaces of a vector space is a subspace.

(i) What do you mean by null space and quotient space?

(j) Define rank and nullity of a linear transformation.

(k) Prove that kernel of a liner transformation $T : V \to W$ is a subspace of V.

(l) If $T : R^3 \to R^3$ and $T(x_1, x_2, x_3) = (x_1, x_2, 0)$, then show that T is a linear transformation.

(m) Define prime ideal and maximal ideals.

(n) Prove that the set of real matrices of the form $\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$ is a left ideal of the ring of all square matrices of order 2.

(o) Write first isomorphism theorem of rings.

2. Answer any ***fouir*** questions [4×5]

(a) Find a basis and dimension of the subspace w of R^3, where

$$W = \{(x, y, z) \in R^3 \mid x + y + z = 0\}$$

(b) Prove that any two bases of a fi9nite dimensional vector space V have the same number of vectors.

(c) Prove that intersection of two ideals is an ideal.

(d) A ring R has no divisors of zero, if and only if the cancellation laws hold in R.

(e) Let $M_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R \right\}$. Then

[Turn Over]

$M_2(R)$ forms a ring with respect to matrix addition '+' and matrix multiplication '.'. Let $S = \left\{ \begin{pmatrix} a & o \\ b & c \end{pmatrix} : a, b, c \in z \right\} \subseteq M_2(R)$. Prove that S is a subring of $M_2(R)$, but it is neither a left nor a right ideal of $M_2(R)$.

(f) Let R and R' be two rings and $\phi : R \to R'$ be an onto homomophism. Then ϕ is an isomorphism if and only if ker $\phi = \{0\}$.

3. Answer any ***two*** questions [2×10]

(a) A mapping $T : R^3 \to R^3$ is defined by $T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, 2x_1 + x_2, + 2x_3, x_1 + 2x_2 + x_3)$, $(x_1, x_2, x_3) \in R^3$. Show that T is a linear mapping. Find ker T and the dimensionof ker T.

(b) Let S be a set of all square matrices of the form $\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$, where a, b are integers. Show that S is a ring, but not a field.

(c) Let R be ring and $\phi : R \to R$ be a homomorphism. Prove that

(i) $\phi(0) = 0'$

(ii) $\phi(-a) = -\phi(a)$

(iii) $\phi(1) = 1'$

(iv) $\phi(a^{-1}) = [\phi(a)]^{-1}$

(d) Let $\{d_1, d_2, ..., d_n\}$ be a basis of a vector space v over a field F and a non zero vector is of v can be expressed as a linear combination of these vectors as

$\beta = a_1\alpha_1 + a_2\alpha_2 + ... + a_{n\alpha_n}$, $a_i \in F$ If $a_i \neq 0$, then β can be replaced by α_i in the basis of V.

Multivariate Calculus

Full Marks : 60 Time : 3 Hours

1. Answer any ***one*** question [2×1]

(a) Show that $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$ does not exist.

(b) State the necessary condition for the existance

[Turn Over]

of extreme value of a function of three variables.

(c) If $u = \tan^{-1} \frac{x^3 + y^3}{x - y}$, then prove that

$x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} = \sin 2u$.

(d) Evaluate $\iint_R xy(x^2 + y^2)dxdy$ over R : [0, a; 0, b].

(e) Change the order of integration in

$\int_0^1 dy \int_0^{\sqrt{y}} f(x, y)dx$.

(f) Prove that curl grad $\phi = \vec{0}$.

(g) What do you mean by conservative vector Field?

(h) For what value of x the vector field $\vec{F} = (x^2\hat{i} + 2y\hat{j} + 3z\hat{k})$ is solenoidal.

(i) Show that the directional derivative for the function $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$

(j) Evaluate $\iint_S \vec{r}.\hat{n}\, dS$, where S is a closed surface.

(k) Find the total work done in moving a particle in a force field given by $\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$.

(l) State Green's theorem in plane.

(m) If S be the surface with the indicated orientation $\vec{F} = y\hat{i} - x\hat{j} + e^{xz}\hat{k}$; $x^2 + y^2 = 1$, then evaluate

$\iint (\vec{\nabla} \times \vec{F}).\overrightarrow{dS}$.

(n) Define stationary point and saddle point.

(o) Show that $\vec{u} = (y^2 z)\hat{i} + (2xyz - z^2 \sin y)\hat{j} + (2z\cos y + y^2 x)\hat{k}$ is irrotational.

[Turn Over]

2. Answer any ***four*** questions : [4×5]

(a) Use Lagrange multiplier method to find the shortest distance from the point to straight line $12x - 5y + 71 = 0$.

(b) Evaluate $\iint_R \frac{\sqrt{a^2b^2 - b^2x^2 - a^2y^2}}{\sqrt{a^2b^2 + b^2x^2 + a^2y^2}} dxdy$ the field of integration being R, the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(c) Using the transformation $x = u$, $y = (1 - u)v$, $z = (1 - u)(1 - v)w$ show that $\iiint x^{a-a} y^{b-1} z^{c-1} (1-x-y-z)^{d-1} dxdydz$, $(a, b, c, d \geq 1)$ taken over the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y + z = 1$ is $\frac{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)}{\Gamma(a+b+c+d)}$.

(d) Show that $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force field. Also,

(i) Find the scalar potential for $\vec{F}$ and

(ii) Find the work done in moving an object in this field from the point (1, –2, 1) to the point (3, 1, 4).

(e) Verify the Green's theorem in plane for $\oint_c \{(y^2 + xy)dx + x^2 dy\}$, where c is the closed curve of the region bounded by the curve y = x and $y = x^2$.

(f) Proved that $\vec{\nabla}^2 f(r) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$. Hence find f(r) such that $\vec{\nabla}^2 f(r) = 0$.

3. Answer any ***two*** questions : [10×2]

(a) (i) State and prove Euler's theorem for the function of two variables.

(ii) If $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$ then prove that $\left(\frac{\delta u}{\delta x}\right)^2 + \left(\frac{\delta u}{\delta y}\right)^2 + \left(\frac{\delta u}{\delta z}\right)^2$

[Turn Over]

$$= 2\left(x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} + z\frac{\delta u}{\delta z} \right).$$ [5+5]

(b) (i) Show that a necessary and sufficient condition that $F_1dx + F_2dy + F_3dz$ to be an exact diferential is that $\vec{\nabla} \times \vec{F} = \vec{0}$ where $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$.

(ii) If $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$, Evaluate $\iint_S \vec{F}.\hat{n}dS$ where S is the surface of teh cube bounded by x = 0, x = 1, y = 0, y = 1, z = 0, z = 1. [5+5]

(c) (i) Find the area of the surface generated by revolving about the y-0axis the part of the astroid $x = a\cos^3\theta, y = a\sin^3\theta$ that lies in the first quadrant.

(ii) Evaluate the surface integral $\iint_S (x^3dydz + x^2ydzdx + x^2zdxdy)$ by transforming it to a volume integral by the divergence theorem, where S is the closed surface bounded by the plane z = 0, z = b and the cylinder $x^2 + y^2 = a^2$. [4+6]

(d) (i) Prove that

$$\int_0^1 dx \int_x^{1/x} \frac{y\,dy}{(1+xy)^2(1+y^2)} = \frac{\Pi - 1}{4}.$$

(ii) Show that the volume of the solid formed by revolving the ellipse $x = a\cos\theta$, $y = b\sin\theta$ about the line $x = 2a$ is $4\pi^2 a^2 b^2$. [5+5]

[Turn Over]