# 2019

# MATHEMATICS

## [ General ]

## PAPER – I

*Full Marks* : 90

*Time* : 3 hours

*The figures in the right hand margin indicate marks*

## GROUP – A

## (*Classical Algebra*)

[*Marks* : 25]

**1.** Answer any *one* question : $15 \times 1$

(*a*) (*i*) If

$$(1+x)^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots,$$

then show that

$$a_0 - a_2 + a_4 - \dots = 2^{n/2} \cos \frac{n\pi}{4}$$

and $a_1 - a_3 + a_5 - \dots = 2^{n/2} \sin \frac{n\pi}{4}$ 5 (*ii*) If $\alpha, \beta, \gamma$ be the roots of the equation $x^3 - px^2 + 9x - x = 0$, then form the equation whose roots are

$$\beta\gamma + \frac{1}{\alpha},\ \gamma\alpha + \frac{1}{\beta},\ \alpha\beta + \frac{1}{\gamma}$$ 5

(*iii*) Prove that

$$\sin\left(i \log \frac{a - ib}{a + ib}\right) = \frac{2ab}{a^2 + b^2},$$

where $a$, $b$ are real numbers. 5

(*b*) (*i*) Find the general value and principal value of $(1 + i)^{1+i}$. 5

(*ii*) Solve by Cardan's method 5

$$x^3 - 15x^2 - 33x + 847 = 0.$$

(*iii*) Find the rank of the matrix

$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}$$ 5

2. Answer any *one* question : $8 \times 1$

(*a*) (*i*) Prove that any square matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrix. 4

(*ii*) Investigate for what values of $\lambda$ and $\mu$ the following equations

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

has (*i*) no solution,

(*ii*) a unique solution,

and (*iii*) an infinite number of solutions. 4

(*b*) (*i*) Show that the roots of the equation

$$\frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c} = \frac{1}{x},$$

where $a > b > c > 0$ are real. 4

(*ii*) Apply Descartes' rule of signs to examine the nature of the roots of the equation

$$x^6 + x^4 + x^2 + x + 3 = 0.$$ 4

**3.** Answer any *one* question : $2 \times 1$

(*a*) Show that the equation

$$1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} = 0$$

Cannot have a multiple root. 2

(*b*) If $A$ be a skew-symmetric matrix, then show that the matrix $A^2$ is symmetric. 2

## GROUP – B

### (*Modern Algebra*)

[*Marks* : 20]

**4.** Answer any *two* questions : $8 \times 2$

(*a*) (*i*) Show that the set $G = \{1, -1, i, -i\}$ of the fourth roots of unity is a group with multiplicative composition. 4

(*ii*) Prove that in a group $(G, 0)$, $(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$ for all $a, b \in G$. 4

(*b*) (*i*) Prove that the set $S$ of all real matrices of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, where $a \neq 0$ forms a commutative group with respect to matrix multiplication. 4

(*ii*) Prove that every group of order less than 6 is commutative. 4

(*c*) (*i*) Prove that the set of all real numbers of the form $(a+b\sqrt{2})$, where $a$ and $b$ are rational numbers is a field under usual addition and multiplication. 4

(*ii*) If $R$ be a ring such that $a^2 = a$, $\forall a \in R$, then prove that 4

1. $a + a = 0$, for all $a \in R$
2. $a+b=0 \Rightarrow a=b$
3. $ab = ba$, for all $a, b \in R$.

5. Answer any *one* question : $4 \times 1$

(*a*) Prove that every group of prime order is cyclic. 4

(*b*) Show that the mapping $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos x, x \in \mathbb{R}$, is neither one-one nor onto. $\mathbb{R}$ is the set of all real numbers. 4

## GROUP – C

(*Analytical Geometry*)

[*Marks* : 30]

6. Answer any *one* question : 15 × 1

(*a*) (*i*) If one of the straight lines $ax^2 + 2hxy + by^2 = 0$ coincides with one of the straight lines $a'x^2 + 2h'xy + b'y^2 = 0$ and the remaining two straight lines are at right angles, then show that

$$h\left(\frac{1}{b}-\frac{1}{a}\right)=h'\left(\frac{1}{b'}-\frac{1}{a'}\right).$$ 7

(*ii*) Reduce the equation

$$3x^2 + 10xy + 3y^2 - 12x - 12y + 4 = 0$$

to its canonical form and determine the type of the conic represented by it. 8

(*b*) (*i*) Find the polar equation of the straight line joining two points on the conic $\frac{l}{r} = 1 - e\cdot\cos\theta$ whose vectorial angles are $\alpha$ and $\beta$. 8

(*ii*) Prove that the lines $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ and $\frac{x-3}{4}=\frac{y-4}{4}=\frac{z-5}{6}$ are coplanar. Find also the equation of the plane in which they lie. 4

(*iii*) Show that the condition that the straight line $\frac{l}{r}=a\cos\theta+b\sin\theta$ may touch the circle $r=2k\cos\theta$ is $b^2k^2+2ak=1$. 3

7. Answer any *one* question : 8 × 1

(*a*) (*i*) Find the equation of the plane which passes through the point $(2, 1, -1)$ and is orthogonal to each of the planes $x-y+z=1$ and $3x+4y-2z=0$. 4

(*ii*) A plane passing through a fixed point $(a, b, c)$ cuts the axes in $A, B, C$. Show that the locus of the centre of the sphere $OABC$ is $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$. 4

(*b*) (*i*) Find the equation of the right circular cone with vertex at the point (3, 2, 1 ), semi-vertical angle $30^\circ$ and axis

$$\frac{x-3}{1}=\frac{y-2}{4}=\frac{z-1}{3}.$$ 4

(*ii*) Find equation of the circular cylinder whose guiding circle is $x^2+y^2+z^2=9$, $x-y+z=3$. 4

8. Answer any *one* question : 4 × 1

(*a*) Find the value of $K$ for which the plane $x+y+z=k$ touches the sphere

$$x^2+y^2+z^2-2x-2y-2z-6=0$$ 4

(*b*) Find the shortest distance between the straight lines $\frac{x-3}{-3}=\frac{y-8}{1}=\frac{z-3}{-1}$ and $\frac{x+3}{3}=\frac{y+7}{-2}=\frac{z-6}{-4}$ and the equations of the line of shortest distance. 4

9. Answer any *one* question : 3 × 1

(*a*) Show that the condition that two of the straight lines represented by the equation

$$ax^3 + bx^2y + cxy^2 + dy^3 = 0$$

may be at right angles is $a^2 + ac + bd + d^2 = 0$. 3

(*b*) Find the point of intersection of the two tangents at $\alpha$ and $\beta$ to the conic $\frac{l}{r} = 1 + e \cdot \cos\theta$. 3

GROUP – D

(*Vector Algebra*)

[*Marks* : 15]

**10.** Answer any *one* question : $8 \times 1$

(*a*) (*i*) $ABCD$ is a parallelogram and $E$ is the mid-point of $CD$ and $F$ is a point on $AE$ such that $AF = \frac{2}{3} AE$; show that $F$ lies on the diagonal $BD$ and $BF = \frac{2}{3} BD$. 5

(*ii*) Find the unit vector perpendicular to both $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$. 3

(*b*) (*i*) Given two vectors $\vec{\alpha} = 3\hat{i} - \hat{j}$, $\vec{\beta} = 2\hat{i} + \hat{j} - 3\hat{k}$, express $\vec{\beta}$ in the form $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$. 4

(*ii*) If $G$ be the centroid of the triangle $ABC$, then show that

$$AB^2 + BC^2 + CA^2 = 3(AG^2 + BG^2 + CG^2).$$ 4

**11.** Answer any *one* question : $4 \times 1$

(*a*) If $\vec{\alpha} \times \vec{\beta} + \vec{\beta} \times \vec{\gamma} + \vec{\gamma} \times \vec{\alpha} = 0$, then show that the vectors $\vec{\alpha}, \vec{\beta}, \vec{\gamma}$ are coplanar. 4

(*b*) Show that the centroid of the vertices of a triangle trisects the medians. 4

**12.** Answer any *one* question : $3 \times 1$

(*a*) Find the vector equation of the plane through the point $(8\hat{i} + 2\hat{j} - 3\hat{k})$ and perpendicular to each of the planes $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=0$ and $\vec{r}\cdot(\hat{i}+3\hat{j}-5\hat{k})+5=0$. 3

(*b*) Find, in terms of $K$, the shortest distance between the lines $\rho=\alpha+t\beta$ and $\rho=\gamma+s\delta$, where $\alpha=(1, 2, 3)$, $\beta=(2, 3, 4)$, $\gamma=(K, 3, 4)$ and $\delta=(3, 4, 5)$. For what value of $K$ are the lines coplanar ? 3

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