3–Remainder Cordial Labeling of Cycle Related Graphs

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ABSTRACT

Let G be a (p, q) graph. Let f be a function from V (G) to the set \{1, 2, \ldots, k\} where k is an integer 2 < k ≤ |V (G)|. For each edge uv assign the label r where r is the remainder when f(u) is divided by f(v) (or) f(v) is divided by f(u) according as f(u) ≥ f(v) or f(v) ≥ f(u). Then the function f is called a k-remainder cordial labeling of G if |v_f(i) − v_f(j)| ≤ 1, i, j ∈ \{1, \ldots, k\} where v_f(x) denote the number of vertices labeled with x and |η_e − η_o| ≤ 1 where η_e and η_o respectively denote the number of edges labeled with an even integers and number of edges labeled with an odd integers. A graph admits a k-remainder cordial labeling is called a k-remainder cordial graph. In this paper we investigate the 3-remainder cordial labeling behavior of the Web graph, Umbrella graph, Dragon graph, Butterfly graph, etc.,

Keywords: Web graph, Umbrella graph, Dragon graph, Butterfly graph

Mathematical Subject Classification (2010): 05C78

1. Introduction

Graphs considered here are finite and simple. Graph labeling is used in several areas of science and technology like coding theory, astronomy, circuit design etc. For more details refer Gallian [1]. The origin of graph labeling is graceful labeling which was introduced by Rosa (1967). Ponraj et al. [3,5], introduced remainder cordial labeling of graphs and investigate the remainder cordial labeling behavior of certain graphs, and also the concept of k-remainder cordial labeling introduced [4,6,7] and investigate the 4-remainder cordial labeling behavior of Grid, Subdivision of crown, Subdivision of bistar, Book, Jelly fish, Subdivision of Jelly fish, Mongolian tent, Flower graph, Sunflower graph and Subdivision of Ladder graph, L_n O K_1, L_n O 2K_1, L_n O K_2. Recently, Ponraj et al. [8,9], further introduced the 3-remainder cordial labeling behavior of certain graphs. we prove that path, cycle, star, comb, crown, wheel, fan, square of path, subdivision of wheel, subdivision of star, subdivision of comb, armed crown, K_{i,n} O K_2 are 3-remainder cordial.
In this paper we investigate the 3-remainder cordial labeling behavior of Web graph, Umbrella graph, Dragon graph, Butterfly graph, etc. Terms are not defined here follows from Harary [2] and Gallian [1].

2. Preliminary results

Definition 2.1. The butterfly graph \( B_{m,n} \) is a two even cycles of the same order say \( C_n \), sharing a common vertex with \( m \) pendant edges attached at the common vertex is called a butterfly graph.

Definition 2.2. The umbrella graph \( U_{n,m} \) is obtained from a fan \( F_n = P_n + K_1 \) where \( P_n = u_1, u_2, \ldots, u_n \) and \( V(K_1) = \{u\} \) by pasting the end vertex of the path \( P_m = v_1, v_2, \ldots, v_m \) to the vertex of \( K_1 \) of the fan \( F_n \).

Definition 2.3. The web graph \( W_{2,n} \) is the graph obtained from a closed Helm \( CH_n \) by adding a single pendent edge to each vertex of the outer cycle.

Definition 2.4. A dragon graph is a graph formed by joining an end vertex of a path \( P_n \) to a vertex of the cycle \( C_m \). It is denoted as \( C_m \oplus P_n \).

2. k-remainder cordial labeling

Definition 2.1. Let \( G \) be a \((p, q)\) graph. Let \( f \) be a function from \( V(G) \) to the set \{1, 2, \ldots, k\} where \( k \) is an integer \( 2 < k \leq |V(G)| \). For each edge assign the label \( r \) where \( r \) is the remainder when \( f(u) \) is divided by \( f(v) \) or \( f(v) \) is divided by \( f(u) \) according as \( f(u) \geq f(v) \) or \( f(v) \geq f(u) \). The function \( f \) is called a \( k \)-remainder cordial labeling of \( G \) if \( |v_f(i) - v_f(j)| \leq 1 \), \( i, j \in \{1, \ldots, k\} \) where \( v_f(x) \) denote the number of vertices labeled with \( x \) and \( |\eta_e - \eta_o| \leq 1 \) where \( \eta_e \) and \( \eta_o \) respectively denote the number of edges labeled with an even integers and number of edges labeled with an odd integers. A graph with a \( k \)-remainder cordial labeling is called a \( k \)-remainder cordial graph.

First we investigate the 3-remainder cordial labeling behavior of the web graph \( W_{2,n} \).

Theorem 2.2. The web graph \( W_{2,n} \) is 3-remainder cordial for all even values of \( n \).
Proof: Let \( V(W_{2,n}) = V(C_n \times P_2) \cup \{w_i : 1 \leq i \leq n\} \) and \( E(W_{2,n}) = E(C_n \times P_2) \cup \{v_iw_i : 1 \leq i \leq n\} \). Then it is easy to verify that \( W_{2,n} \) has 3n vertices and 4n edges. First we consider the vertices \( u_1, v_1 \) and \( w_1 \). Assign the labels 1, 2 and 3 respectively to the vertices \( u_1, v_1 \) and \( w_1 \). Next consider the vertices \( u_2, v_2 \) and \( w_2 \). Assign the labels 1, 3 and 2 to the vertices \( u_2, v_2 \) and \( w_2 \) respectively. Next we move to the vertices \( u_3, v_3 \) and \( w_3 \) and assign the labels 1, 2 and 3 respectively to the vertices \( u_3, v_3 \) and \( w_3 \). Next assign the labels 1, 3 and 2 to the vertices \( u_4, v_4 \) and \( w_4 \). That is assign the labels 1, 2, 3; 1, 1, 3; 2, 1, 3; respectively to the vertices \( u_1, v_1, w_1; u_2, v_2, w_2; \ldots; u_n, v_n, w_n \). Note that the vertex condition and edge condition are \( v_f(1) = v_f(2) = v_f(3) = n \) and \( \eta_e = \eta_o = 2n \) respectively. Hence the function \( f \) is 3-remainder cordial labeling behavior of the web graph \( W_{2,n} \) all even values of \( n \).
Next we investigate the dragon $C_m \circ P_n$.

**Theorem 2.3.** The dragon graph $C_m \circ P_n$ is $3$-remainder cordial for all $m \geq 3$ and $n=m$.

**Proof:** Let $C_m = u_1u_2 \ldots \ldots u_nu_1$ be the cycle and $P_n = v_1, v_2 \ldots \ldots v_n$ be the path. Without loss of generality, unify the vertices $u_1$ with $v_1$. Clearly the order and size of the dragon $C_m \circ P_n$ are $2n-1$ and $2n-1$ respectively.

**Case(1):** $n \equiv 0 \pmod{3}$ and $n$ is odd.

First consider the vertices $v_1, v_2, \ldots \ldots v_n$ of the path. Assign the label $2$ to the vertices $v_1, v_3, \ldots \ldots v_{n-1}$ and $3$ to the vertices $v_2, v_4, \ldots \ldots v_{n-1}$. Next consider the vertices $u_i$ for $2 \leq i \leq n$. Fix the labels $1, and 1$ to the first two vertices $u_2$ and $u_3$. Next assign the labels $1,2,$ and $1$ respectively to the next three vertices $u_4,u_5$, and $u_6$. Then assign the labels $1,3,$ and $1$ respectively to the next three vertices $u_7,u_8$, and $u_9$. Proceeding like this until we reach the vertices $u_{n-2},u_{n-1},$ and $u_n$. Clearly the vertices $u_{n-2},u_{n-1},$ and $u_n$ received the labels $1,3,$ and $1$ for this pattern.

**Case(2):** $n \equiv 0 \pmod{3}$ and $n$ is even.

First consider the vertices $v_i$ of the path. Assign the label $2$ to the vertices $v_1, v_3, \ldots \ldots v_{n-1}$ and $3$ to the vertices $v_2, v_4, \ldots \ldots v_{n}$. Next consider the vertices $u_i$ for $2 \leq i \leq n$. As in case(1), assign the labels to the vertices $u_i$ for $2 \leq i \leq n-3$. Finally assign the labels $1,2,$ and $1$ respectively to the last three vertices $u_{n-2},u_{n-1},$ and $u_n$.

**Case(3):** $n \equiv 1 \pmod{3}$ and $n$ is odd.

As in case(1), assign the labels to the vertices $v_i$ of the path for $1 \leq i \leq n$. Next consider the vertices $u_i$ for $2 \leq i \leq n$. Fix the labels $1,3,$ and $1$ to the first three vertices $u_2,u_3,$ and $u_4$. Next assign the labels $1,3,$ and $1$ respectively to the next three vertices $u_{i},u_{i+1},$ and $u_{i+2}$. Then assign the labels $1,2,$ and $1$ respectively to the next three vertices $u_{i+3},u_{i+4},$ and $u_{i+5}$. Continuing like this until we reach the vertices $u_{n-2},u_{n-3},$ and $u_n$. Then clearly the vertices $u_{n-2},u_{n-1},$ and $u_n$ received the labels $1,3,$ and $1$ for this pattern.

**Case(4):** $n \equiv 1 \pmod{3}$ and $n$ is even.

As in case(2), assign the labels to the vertices $v_i$ of the path for $1 \leq i \leq n$. Next consider the vertices $u_i$ for $2 \leq i \leq n$. As in case(3), assign the labels to the vertices $u_i$ for $2 \leq i \leq n-3$. Finally assign the labels $1,2,$ and $1$ respectively to the last three vertices $u_{n-2},u_{n-1},$ and $u_n$.

**Case(5):** $n \equiv 2 \pmod{3}$ and $n$ is odd.

As in case(1), assign the labels to the vertices $v_i$ of the path for $1 \leq i \leq n$. Next consider the vertices $u_i$ for $2 \leq i \leq n$. Fix the labels $1,1,3,$ and $1$ to the first four vertices $u_2,u_3,$ $u_4,$ and $u_5$. Next assign the labels $1,2,$ and $1$ respectively to the next three vertices $u_{i},u_{i+1},$ and $u_{i+2}$. Then assign the labels $1,3,$ and $1$ respectively to the next three vertices $u_{i+3},u_{i+4},$ and $u_{i+5}$. Proceeding like this until we reach the vertices $u_{n-2},u_{n-1},$ and $u_n$. Then clearly the vertices $u_{n-2},u_{n-1},$ and $u_n$ received the labels $1,3,$ and $1$ for this pattern.

**Case(6):** $n \equiv 2 \pmod{3}$ and $n$ is even.
As in case(2), assign the labels to the vertices $v_i$ of the path $P_n$ for $1 \leq i \leq n$. Next consider the vertices $u_i$ for $2 \leq i \leq n$. As in case(5), assign the labels to the vertices $u_i$ for $2 \leq i \leq n -3$. Finally assign the labels 1, 2, and 1 to the last three vertices $u_{n-2}, u_{n-1}$, and $u_n$ respectively. The table -1, establish that the function $f$ is 3- remainder cordial labeling of the dragon graph $C_m \circ P_n$.

<table>
<thead>
<tr>
<th>Nature of $n$</th>
<th>$v_i(1)$</th>
<th>$v_i(2)$</th>
<th>$v_i(3)$</th>
<th>$\eta_e$</th>
<th>$\eta_o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \equiv 0 \ (mod\ 3)$</td>
<td>$\frac{2n}{3}$</td>
<td>$\frac{2n}{3}$</td>
<td>$\frac{2n-3}{3}$</td>
<td>$n$</td>
<td>$n-1$</td>
</tr>
<tr>
<td>$n \equiv 1 \ (mod\ 3)$</td>
<td>$\frac{2n-2}{3}$</td>
<td>$\frac{2n-2}{3}$</td>
<td>$\frac{2n+1}{3}$</td>
<td>$n$</td>
<td>$n-1$</td>
</tr>
<tr>
<td>$n \equiv 2 \ (mod\ 3)$</td>
<td>$\frac{2n-1}{3}$</td>
<td>$\frac{2n-1}{3}$</td>
<td>$\frac{2n-1}{3}$</td>
<td>$n$</td>
<td>$n-1$</td>
</tr>
</tbody>
</table>

Table 1:

Next we investigate the 3-remainder cordial labeling behavior of the butterfly graph $BF_{m,n}$.

**Theorem 2.4.** The butterfly graph $BF_{m,n}$ is 3-remainder cordial for all $m \geq 3$ and $n=m$.

**Proof:** Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the two copies of the cycle $C_n$. Without loss of generality, unify the vertices $u_1$ and $v_1$. Let $w_1, w_2, \ldots, w_n$ be the $n$ pendent vertices. Clearly the order and size of the butterfly graph $BF_{m,n}$ are $3n-1$ and $3n$ respectively.

**Case(1):** $n$ is odd.

First we consider the vertices $u_i$ of the cycle $C_n$. Assign the label $1$ to the vertices $u_i$ for $2 \leq i \leq n$. Next we consider the vertices $v_i$ of the another cycle $C_n$. Assign the label $2$ to the vertices $v_1, v_3, \ldots, v_n$. Then next assign the label $3$ to the vertices $v_2, v_4, \ldots, v_{n-1}$. Now consider the pendent vertices $w_i$ for $1 \leq i \leq n$.

Assign the label $3$ to the first $\frac{n-1}{2}$ pendent vertices and assign the label $2$ to the remaining $\frac{n-1}{2}$ pendent vertices.

**Case(2):** $n$ is even.

As in case(i), assign the labels to the vertices $u_i$ for $2 \leq i \leq n$. Now we consider the vertices $v_i$ of the another cycle $C_n$. Assign the label $2$ to the vertices $v_1, v_3, \ldots, v_n$. Then next assign the label $3$ to the vertices $v_2, v_4, \ldots, v_{n-1}$. Now consider the pendent vertices $w_i$ for $1 \leq i \leq n$. Assign the label $3$ to the first $\frac{n}{2}$ pendent vertices and assign the label $2$ to the remaining $\frac{n}{2}$ pendent vertices. The table-2, shows that the function $f$ is 3- remainder cordial labeling of the butterfly graph $BF_{m,n}$.

<table>
<thead>
<tr>
<th>Nature of $n$</th>
<th>$v_i(1)$</th>
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<th>$\eta_e$</th>
<th>$\eta_o$</th>
</tr>
</thead>
<tbody>
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<td>$n$ is odd</td>
<td>$n-1$</td>
<td>$n$</td>
<td>$3n+1$</td>
<td>$\frac{3n-1}{2}$</td>
<td></td>
</tr>
<tr>
<td>$n$ is even</td>
<td>$n-1$</td>
<td>$n$</td>
<td>$3n$</td>
<td>$\frac{3n}{2}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 2:
3–Remainder Cordial Labeling of Cycle Related Graphs

Here we investigate the umbrella graph $U_{m,n}$.

**Theorem 2.5.** The umbrella graph $U_{m,n}$ is 3-remainder cordial for all $m \geq 3$ and $n = m$.

**Proof:** Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices of the umbrella graph $U_{m,n}$. Let $u_1, u_2, \ldots, u_n$ be the vertices of the path $P_n$ and $v_1, v_2, \ldots, v_n$ be the vertices of the fan $F_n$. Without loss of generality, unify the vertices $u_i$ and $v_i$. Then clearly the order and size of the umbrella graph $U_{m,n}$ are $2n$ and $3n-2$ respectively.

**Case(1):** $n \equiv 0 \pmod{3}$ and $n$ is even.

Assign the labels 2, 1 and 1 respectively to the vertices $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ of $U_{m,n}$. Next assign the labels 2, 1 and 1 respectively to the vertices $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ of $U_{m,n}$. First assign the labels 2, 3 and 1 to the vertices $u_1, u_2, u_3$, and $v_1$ respectively. Next assign the labels 2, 3 and 1 to the vertices $u_1, u_2, u_3$, and $v_1$ respectively. Proceeding like this until we reach the vertices $u_n, u_{n-1}$, and $u_n$ are received the labels 2, 1 and 1 respectively.

**Case(2):** $n \equiv 0 \pmod{3}$ and $n$ is odd.

As in case(1), assign the labels to the vertices $v_i$ for $1 \leq i \leq n$, and $u_i$ for $1 \leq i \leq n-3$. Finally assign the labels 2, 3 and 1 to the vertices $u_{n-2}, u_{n-1}$, and $u_n$ respectively.

**Case(3):** $n \equiv 1 \pmod{3}$ and $n$ is even.

Assign the label 1 to the vertices $v_1, v_2, \ldots, v_{n-1/3}$ and assign the label 3 to the vertices $v_{(n-1/3)+1}, v_{(n-1/3)+2}, \ldots, v_n$ and followed by assign the label 2 to the vertices $v_{(n-1/3)+4}, v_{(n-1/3)+5}, \ldots, v_n$. Next consider the vertices $u_i$ of the path $P_n$. First assign the labels 2, 3 and 1 to the vertices $u_1, u_2$, and $u_3$ respectively. Next assign the labels 2, 3 and 1 to the vertices $u_1, u_2$, and $u_3$ respectively. Proceeding like this until we reach the vertices $u_{n-3}, u_{n-2}, u_{n-1}$, and $u_n$ are received the labels 1, 2 and 3 respectively.

**Case(4):** $n \equiv 1 \pmod{3}$ and $n$ is odd.

As in case(3), assign the labels to the vertices $v_i$ for $1 \leq i \leq n$. Next assign the labels 2, 3 and 1 to the vertices $u_1, u_2$, and $u_3$ respectively and as in case(3), assign the labels to the vertices $u_i$ for $4 \leq i \leq n$.

**Case(5):** $n \equiv 2 \pmod{3}$ and $n$ is even.

Assign the label 1 to the vertices $v_1, v_2, \ldots, v_{n+1/3}$ and assign the label 3 to the vertices $v_{(n+1/3)+1}, v_{(n+1/3)+2}, \ldots, v_n$ and followed by assign the label 2 to the vertices $v_{(n+1/3)+4}, v_{(n+1/3)+5}, \ldots, v_n$. Next consider the vertices $u_i$ of the path $P_n$. Fix the labels 2, 3 and 1 to the vertices $u_1, u_2$, and $u_3$ respectively. Next assign the labels 2, 3 and 1 to the vertices $u_1, u_2$, and $u_3$ respectively. Assign the labels 2, 3 and 1 to the vertices $u_{n-3}, u_{n-2}, u_{n-1}$, and $u_n$ respectively. Proceeding like this until we reach the vertices $u_{n-3}, u_{n-2}, u_{n-1}$, and $u_n$ are received the labels 2, 3 and
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1 to the vertices \( u_{10}, u_{11}, \) and \( u_{12} \) respectively. Continuing like this until we reach the vertices \( u_{n-6}, u_{n-5}, \) and \( u_{n-4} \) are received the labels 2,3 and 1 respectively.

**Case (6):** \( n \equiv 2 \) (mod 3) and \( n \) is even.

As in case(5), assign the labels to the vertices \( v_i \) for \( 1 \leq i \leq n \), and assign the labels to the vertices \( u_i \) for \( 1 \leq i \leq n-2 \). Finally assign the labels 2 and 3 to the last two vertices \( u_{n-1} \) and \( u_n \) respectively. The table -3, establish that the function \( f \) is 3- remainder cordial labeling of the umbrella graph \( U_{m,n} \).

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<td>( \frac{3n-2}{2} )</td>
</tr>
<tr>
<td>( n \equiv 1 ) (mod 3) &amp; ( n ) is odd</td>
<td>( \frac{2n-1}{3} )</td>
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<td>( \frac{3n-2}{2} )</td>
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</tr>
<tr>
<td>( n \equiv 2 ) (mod 3) &amp; ( n ) is odd</td>
<td>( \frac{2n-1}{3} )</td>
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</tbody>
</table>

**Table 3:**

3. **Conclusion**

The main aim of this paper was to present blow-up results for a graph labeling problems subject to certain conditions. The possible generalization is plan to present the sufficient conditions which guarantee the occurrence of the blow-up.

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