2018

M.Sc.

Part-I Examination

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

PAPER—I

Full Marks : 100

Time : 4 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answers wherever necessary.

Write the answer to questions of each group in Separate answer booklet.

Group—A

(*Real Analysis*)

[Marks : 40]

Answer Q. No. 1 and any *three* from Q. No. 2 to Q. No. 6.

1. Answer any *one* question : 1×1

   (a) Define positive and negative variations of a function of bounded variation f on $[a, b]$.

   (b) Define null set.

(Turn Over)

2. (a) Establish a necessary and sufficient condition for a function $f:[a,b]\to\mathbb{R}$ to be of bounded variation on $[a, b]$. 7

(b) Show that the function $f(x)=|x-1|, x\in[0,2]$ is a function of bounded variation on [0, 2]. Find the positive and negative variation function on [0, 2]. Hence, express f as the difference of two monotone increasing functions on [0, 2]. 6

3. (a) If $f(x)$ is continuous and $g(x)$ is monotonic increasing on $[a, b]$ then prove that the Riemann-Stieltjges integral $\int_a^b f(x)dg(x)$ exists. 5

(b) If f is monotonic increasing and g is continuous on $[a, b]$, then prove that there exists a number ξ in $[a, b]$, such that 5

$$\int_a^b f(x)dg(x)=f(a)\int_a^{\xi} dg(x)+f(b)\int_{\xi}^b dg(x)$$

(c) Find the value of $\int_{-1}^{3}(2x+3)d\alpha(x)$ where 3

$$\alpha(x)=\begin{cases}-8, & -1\le x<0\\ 3, & 0\le x<1\\ 4, & x=1\\ -2, & 1<x<2\\ 6, & 2\le x<3\\ 5, & x=3\end{cases}$$

4. (a) Let A_1, A_2 be subsets of [a, b]. Prove that following :

$$m^*(A_1)+m^*(A_2)\ge m^*(A_1\cup A_2)+m^*(A_1\cap A_2)$$

and $m_*(A_1)+m_*(A_2)\le m_*(A_1\cup A_2)+m_*(A_1\cap A_2)$ 4

(b) Let $f:[a,b]\to\mathbb{R}$ be a measurable function. Then show that the following are equivalent— 6

(i) $\{x: f(x)>\alpha\}$ is a measurable set for every real α,

(ii) $\{x: f(x)\ge\alpha\}$ is a measurable set for every real α,

(iii) $\{x: f(x)<\alpha\}$ is a measurable set of every real α,

(iv) $\{x: f(x)\le\alpha\}$ is a measurable set for every real α.

(c) Let $\{f_n\}_{n\ge1}$ be a sequence of measurable function on [a, b]. Show that $\limsup_{n\to\infty} f_n$ and $\liminf_{n\to\infty} f_n$ are also measurable functions on $[a, b]$. 3

5. (a) State the following theorem :
Lusin's theorem, Egoroff's theorem. 3

(b) Show that the function

$$f(x)=\begin{cases}4 & \text{if } x\in[1,8]\cap Q\\ -3 & \text{if } x\in[1,8]\cap Q^c\end{cases}$$

is a measurable function on [1, 8]. 4

(c) Let f be a bounded measurable function on $[a, b]$. Then show that f is Lebesgue integrable on [a, b]. 6

6. (a) Let f and g be bounded Lebesgue integrable function on $[a, b]$. If $f(x) \geq g(x)$ a.e. on $[a, b]$, then show that

$$L\int_a^b f(x)dx \geq L\int_a^b g(x)dx$$ 4

(b) State the Bounded convergence theorem. Verify the Bounded convergence theorem for the sequence of functions 6

$$f_n(x) = \frac{5}{\left(2+\frac{3x}{n}\right)^n}, 0 \leq x \leq 1, n = 1, 2, 3, \ldots$$

(c) Show that the function f(x) defined by

$$f_n(x) = \begin{cases} \frac{1}{x^{2/3}}, & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}$$

is Labesgue integrable on [0, 1]. Find $L\int_0^1 f(x)dx$.

Group—B

(Complex Analysis)

[Marks : 30]

Answer *all* questions.

1. Answer any *two* questions : 2×2

(a) If $f(z) = \frac{x^2y^5(x+iy)}{x^4+y^{10}}, z \neq 0$

$= 0, z = 0$

Verify whether Cauchy-Riemann equations are satisfied at the origin or not.

(b) State the Laurents' theorem.

(c) Let C be any simple closed contour' described the positive sense in the z-plane and write $g(w) = \int_C \frac{z^3+2z}{(z-w)^3}dz$. Then find $g(w)$ when w is inside C. 2

2. Answer any *four* questions : 4×5

(a) Let $f(z) = (x^3+2) + i(1-y)^2$. Find all the points in the complex plane where $f(z)$ is differentiable and compute $f'(z)$ at those points. Is $f(z)$ analytic at any point in the complex plane ? Justify. 5

(b) Find Taylor or Laurent series expansion of the function $f(z) = \frac{3}{z(z-i)}$ with center at c = – i, where region of convergence is $1 < |z+i| < 2$. 5

(c) Use calculus of residues to show that $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}d\theta = \frac{\pi}{6}$. 5

(d) If $z = x + iy$, x, y real and $i = (-1)^{1/2}$, obtain a set of sufficient conditions for f(z) to be analytic. 5

(e) Evaluate : $\int_C \frac{e^{3z}}{z-\pi i}dz$ where C is a circle $|z-1|=4$. 5

(f) Find all the Möbius transformation which transforms the half plane $I(z) \geq 0$ onto the unit circular disc $|w| \leq 1$. 5

3. Answer any *one* question : 1×6

(a) Suppose that $f(z) = u(x, y) + iv(x, y)$ and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$, then prove that the first order partial derivatives of u and v must exist at (x_0, y) and they satisfy the Cauchy Riemann equations : $u_x = v_y; u_y = -v_x$ at (x_0, y_0). Also, prove that $f'(z_0) = u_x + iv_x$ at (x_0, y_0). 6

(b) Evaluate $I = \int_{-\pi}^{\pi} \frac{dx}{x^2+x+1}$. 6

Group—C

(*Ordinary Differential Equations*)

[Marks : 30]

(The symbols have their usual meaning)

Answer any *two* questions :

10. (a) Let $W_1(z)$ and $W_2(z)$ be two solutions of $(1-z^2)w''(z) - 2zw'(z) + (\sec z)w = 0$ with Wronskian

$w(z)$. If $w_1(0) = 1, w'(0) = 0$ and $w\left(\frac{1}{2}\right) = \frac{1}{3}$, then find the value of $w_2'(z)$ at $z = 0$. 4

(b) Prove that if f(z) is continuous and has continuous derivative in [-1, 1] then f(z) has unique Legendre series expansion is given by $f(z) = \sum_{n=0}^{\infty} C_n P_n(z)$ where P_n's are Legendre Polynomials and $C_n = \frac{2n+1}{2}\int_{-1}^{1} f(z)P_n(z)dz,\ n = 1, 2, 3, \ldots$ 6

(c) Show that $J_0^2(z) + 2\sum_{n=1}^{\infty} J_n^2(z) = 1$ and prove that for real z, $|J_0(z)| < 1$, and $|J_n(z)| < \frac{1}{\sqrt{2}}$, for all $n \geq 1$. 5

11. (a) Show that $nP_n(z) = zP_n'(z) - P_{n-1}'(z)$, here $P_n(z)$ denotes the Legendre Polynomial of degree n. 3

(b) Prove that $\int_{-1}^{1} P_m(z)P_n(z)dz = \frac{2}{2n+1}\delta_{mn}$, where δ_{mn} and $Pn(z)$ are the Kroneker delta and Legendre's polynomial respectively. 5

(c) Show that Legendre differential equation reduces to hyper geometric differential equation by considering suitable transformation. Find the integral representation of hyper geometric function. 3+4

12. (a) Find the series solution near $z = 0$ of $(z + z^2 + z^3)\omega''(z)$ $+3z^2\omega'(z) - 2\omega(z) = 0$. 6

(b) Deduce Rodrigue's formula for Legendre's polynomial. 4

(c) Establish the generating function for Bessel's function $J_n(z)$. Use it, Prove the following 3+2

$$zJ_n'(z) = zJ_{n-1}(z) - nJ_n(z)$$
