NEW

2018

Part I

MATHEMATICS

PAPER—I

(General)

Full Marks : 90

Time : 3 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answers wherever necessary.

Group—A

(Classical Algebra)

[Marks : 25]

1. Answer any *one* question : 1×15

(a) (i) If α,β,γ are the roots of the equation $x^3+3x+1=0$, find the equation where roots are $\frac{\alpha}{\beta}+\frac{\beta}{\gamma},\frac{\beta}{\gamma}+\frac{\gamma}{\beta},\frac{\gamma}{\alpha}+\frac{\alpha}{\gamma}$ and hence find the value of $\sum\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)$. 4+1

(ii) Show that the equation $\tan\left(i\log\left(\frac{x-iy}{x+iy}\right)\right)=2$ represents the rectangular hyperbola $x^2-y^2=xy$. 5

(iii) Solve by Cardan's method $x^3-3x+1=0$. 5

(b) (i) If $x=\cos\alpha+i\sin\alpha, y=\cos\beta+i\sin\beta$, $z=\cos\gamma+i\sin\gamma$ and $x+y+z=xyz$, prove that $\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)=-1$. 5

(ii) Solve by Cramer's rule :

$$x+y+z=6$$
$$x-y+2z=5$$
$$3x+5y-7z=-8$$

5

(iii) If $A=\begin{pmatrix}5 & 4 & -2\\ 4 & 5 & -2\\ -2 & -2 & 2\end{pmatrix}$, then show that $A^2-11A+10I_3=0$, where I_3 is the 3×3 unit matrix. Hence find A^{-1}. 3+2

2. Answer any *one* question : 1×8

(a) (i) Find the sum of the 33th powers of the roots of the equation $x^5-1=0$. 4

(ii) Prove that every skew-symmetric determinant of even order is a perfect square. 4

(b) (i) Investigate for what values of λ and μ the following equations $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$ have (I) no solution, (II) a unique solution and (III) an infinite number of solutions. 4

(ii) Find the range of the values of k for which the equation $x^4+4x^3-2x^2-12x+k=0$ has four real and unequal roofs. 4

3. Answer any *one* question : 1×2

(a) Show that all the values of i^i are real, where $i=\sqrt{-1}$. 2

(b) If A be a skew-symmetric matrix, then show that the matrix A^2 is symmetric. 2

Group—B

(Modern Algebra)

[Marks : 20]

4. Answer any *two* questions : 2×8

(a) (i) Let $G=\{1,w,w^2\}$ be the set of the three cube roofs of unity. Show that G is a group under usual multiplication of complex numbers and G has no proper subgroup. 4+1

(ii) Let $f : R \to R$ and $g : R \to R$ be given by the formula : $g(x) = x^2, f(x) = \sin x \quad (x \in R)$. Then prove that $f \circ g \neq g \circ f$ where $(f \circ g)$ and $(g \circ f)$ are the composite maps. 3

(b) (i) Let $M = \left\{ \begin{bmatrix} a & b \\ 2b & a \end{bmatrix} : a, b \text{ are rational numbers} \right\}$.

Prove that under usual matrix addition and matrix multiplication, M is a field. 5

(ii) If R is a ring such that $a^2 = a \forall a \in R$ prove that (i) $a + a = 0 \ \forall a \in R$ and (ii) $a + b = 0$ implies $a = b, \ b \in R$. 2+1

(c) (i) Find the eigen values and the corresponding eigen vectors of the matrix

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$ 5

(ii) For three non-empty sets A, B, C, prove that

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$ 3

5. Answer any *one* question : 1×4

(a) If H_1 and H_2 are two subgroups of a group G, then prove that $H_1 \cap H_2$ is also a subgroup of G. 4

(b) Prove that a ring R is commutative,

if $(a+b)^2 = a^2 + 2ab + b^2$ for every $a, b \in R$. 4

Group—C

(Analytic Geometry)

[Marks : 30]

6. Answer any *one* questions : 1×15

(a) (i) Reduce the equation

$x^2 + 4xy + y^2 - 2x + 2y + 6 = 0$ to its canonical form. Hence find the nature of the curve. 7+1

(ii) Prove that the straight line $lx + my + n = 0$ touches the parabola $y^2 - 4px + 4pq = 0$ if $l^2q + ln - pm^2 = 0$. 7

(b) (i) If PQ is a variable chord of the conic $\frac{l}{r} = 1 - e.\cos\theta$, subtending a constant angle 2β at the focus S, where S is the pole, show that the locus of the foot of the perpendicular from S on PQ is the curve

$$r^2\left(e^2 - \sec^2\beta\right) + 2elr\cos\theta + l^2 = 0$$ 8

(ii) Show that the straight lines

$$\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8} \text{ and } \frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$$

intersect. Find their point of intersection and the equation of the plane in which they lie. 7

7. Answer any *one* question : 1×8

(a) (i) PSP' is a focal chord of the conic. Prove that the angle between the tangents at P and P' is $\tan^{-1}\frac{2e\sin\alpha}{1-e^2}$, where α is the angle between the chord and the major axis. 5

(ii) Show that if the planes $x=cy+bz$, $y=az+cx$, $z=bx+ay$ pass through one straight line, then $a^2+b^2+c^2+2abc=1$. 3

(b) (i) Find the image of the point $(1,3,4)$ on the plane $2x-y+z-9=0$. 3

(ii) A plane passing through a fixed point (a,b,c) cuts the axes in A,B,C. Show that the locus of the centre of the sphere $OABC$ is $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$. 5

8. Answer any *one* question : 4×1

(a) Find the equation of the sphere which touches the two planes $3x+2y-6z+7=0$ and $3x+2y-6z+35=0$ and whose centre lies on the straight line $x = 0$, $2y + z = 0$. 4

(b) Determine the angle between the lines of intersection of the plane $x-3y+z=0$ and the cone $x^2-5y^2+z^2=0$. 4

Answer any *one* question : 1×3

(a) Find the angle by which the axes should be rotated so that the equation $7x^2+4xy+3y^2$ becomes another equation in which the term xy is absent. 3

(b) Determine the nature of the conic $\frac{3}{r}=2+4\cos\theta$ and also find the length of its latus rectum. 3

Group—D

(Vector Algebra)

[Marks : 15]

10. Answer any *one* question : 1×8

(a) (i) If $\vec{\alpha},\vec{\beta},\vec{\gamma}$ are three vectors such that $\vec{\alpha}+\vec{\beta}+\vec{\gamma}=0$ and $|\vec{\alpha}|=3, |\vec{\beta}|=5, |\vec{\gamma}|=6$ then show that $\vec{\alpha}\cdot\vec{\beta}+\vec{\beta}\cdot\vec{\gamma}+\vec{\gamma}\cdot\vec{\alpha}=-35$. 4

(ii) Show that the diagonals of a rhombus are at right anyles. 4

(b) (i) A particle, acted on by constant forces $4\hat{i}+2\hat{j}+3\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$, is displaced from the point $\hat{i}+2\hat{j}$ to $2\hat{i}-\hat{j}+3\hat{k}$. Find the work done by the forces. 4

(ii) Given $\vec{\alpha}=3\hat{i}-\hat{j}$, $\vec{\beta}=2\hat{i}+\hat{j}-3\hat{k}$, express $\vec{\beta}$ in the form $\vec{\beta}=\vec{\beta_1}+\vec{\beta_2}$, where $\vec{\beta_2}$ is parallel to $\vec{\alpha}$ and $\vec{\beta_2}$ is perpendicular to $\vec{\alpha}$. 4

11. Answer any *one* question : 4×1

(a) Show that $\left|\vec{\alpha}\times\vec{\beta}\right|^2\left|\vec{\alpha}\times\vec{\gamma}\right|^2-\left\{\left(\vec{\alpha}\times\vec{\beta}\right)\cdot\left(\vec{\alpha}\times\vec{\gamma}\right)\right\}^2=\left|\vec{\alpha}\right|^2\left[\vec{\alpha}\,\vec{\beta}\,\vec{\gamma}\right]^2$. 4

(b) If the vectors $\vec{\alpha}$ and $\vec{\gamma}$ be perpendicular to each other, then show that the vectors $\vec{\alpha}\times\left(\vec{\beta}\times\vec{\gamma}\right)$ and $\left(\vec{\alpha}\times\vec{\beta}\right)\times\vec{\gamma}$ are also perpendicular to each other. 4

12. Answer any *one* question : 3×1

(a) Find the condition of intersection of two straight lines : $\vec{r}=\vec{a}+t\vec{b}$ and $\vec{r}=\vec{c}+s\vec{d}$ where t and s are scalars. 3

(b) Find the torque about the point $B(3, -1, 3)$ of a force $P(4, 2, 1)$ passing through the point $A(5, 2, 4)$. 3