M.Sc. 1st Semester Examination, 2014

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

(*Complex Analysis*)

PAPER— MTM - 102

Full Marks : 50

Time : 2 hours

Answer Q. No.1 and any two from the rest

The figures in the right-hand margin indicate marks

1. Answer any *four* questions : 2 × 4

(*i*) Prove that

$$f(z)=\overline{z}$$

is nowhere differentiable.

(*ii*) Construct the analytic function $w=f(z)$ if its real part is $e^x \cos y$ and if $f(0)=1$.

(Turn Over)

(*iii*) If C is the curve $y = x^3 - 3x^2 + 4x - 1$ joining point(1, 1) and (2, 3), find the value of

$$\int_C (12z^2 - 4iz)\,dz$$

(*iv*) Discuss the nature of singularities of the function

$$f(z) = \frac{\sin z}{(z - \pi)^2}$$

(*v*) Show that the function

$$f(z) = \frac{z}{e^z - 1}$$

has a removable singularity at the origin.

(*vi*) Evaluate

$$\int_{|z|=1} z\bar{z}\,dz$$

2. (*a*) If $z = re^{i\theta}$ and $f(z) = u(r, \theta) + iv(r, \theta)$ obtain the Cauchy-Riemann relation in terms of z. 4

(*b*) Given $f(z)$ to be analytic, show that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$ 4

(*c*) If $f(z) = u + iv$, where

$$f(z)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, \quad z \neq 0$$

$$= 0 \qquad\qquad , \quad z = 0$$ 4

(*d*) If $u = x^3 - 3xy^2$, find v so that $u + iv$ may be analytic. Find $f(z) = u + iv$. 4

3. (*a*) Show that, under suitable conditions, to be stated by you

$$f'(a)=\frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^2}$$

where C is a closed contour surrourding the point $z = a$. 4

(*b*) Expand the function

$$f(z)=\frac{1}{z^2-3z+2}$$

in series in the region $0<|z|<2$ and $|z|>2$. 4

(*c*) Investigate the nature of singularities of the function

$$f(z)=\frac{z-2}{(z^2-1)^3}\cot\left(\frac{1}{1-z}\right).$$ 4

(*d*) If *c* is a closed contour arround origin, prove that

$$\left(\frac{a^n}{n!}\right)^2=\frac{1}{2\pi i}\int_C \frac{a^n e^{az}}{n!\,z^{n+1}}\,dz.$$

Hence deduce

$$\sum_{n=0}^{\infty}\left(\frac{a^n}{n!}\right)^2=\frac{1}{2\pi}\int_0^{2\pi} e^{2a\cos\theta}\,d\theta.$$ 4

4. (*a*) State and prove Rouche's theorem. 4

(*b*) Show that

$$w = \frac{5-4z}{4z-2}$$

transform $|z| = 1$ into a circle in the z-plane, find the centre and radius of the circle. 4

(*c*) Evaluate the following by the method of contour integration (any *two*) : 4 + 4

(*i*) $$\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$$

(*ii*) $$\int_0^{\infty} \frac{dx}{x^2+x+1},$$

(*iii*) $$\int_{-\infty}^{\infty} \frac{x\cos x}{x^2+1}\, dx$$

(*iv*) $\int_{0}^{\infty} \frac{\sin x}{x(1+x^2)} \, dx.$

[*Internal Assessment* : 10 Marks]