M.Sc. 1st Semester Examination, 2014

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

(*Real Analysis*)

PAPER—MTM-101

Full Marks : 50

Time : 2 hours

Answer **Q. No. 1** and any **four** from **Q. No. 2 to Q. No. 7**

The figures in the right-hand margin indicate marks

1. Answer any *four* questions : 2 × 4

   (*a*) In a metric space (X, d), if $A, B \subseteq X$ such that $d(A, B) > 0$, then prove that A and B are separated.

   (*b*) Define RS-integral with the help of limit.

(Turn Over)

(*c*) If the outer measure of a set is zero, prove that the set is measurable.

(*d*) Define the following with examples :

(*i*) Non-negative simple measurable function

(*ii*) Measure.

(*e*) If $E \subset X$ and

$$X_E(x) = \begin{cases} 1, \text{ if } x \in E \\ 0, \text{ if } x \notin E \end{cases}$$

is a measurable function, then show that E is a measurable set in X.

2. (*a*) Is the set

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}$$

compact in $\mathbb{R}^3$? Justify. 2

(*b*) Let A and B be compact subsets of a metric space X. Is $A \cup B$ compact ? Justify. 2

(*c*) Prove that the union of connected sets in a mertic space, no two of which are separated, is a conected set. 4

3. (*a*) Let $f : [a, b] \rightarrow \mathbb{R}$ be a function of bounded variation on $[a, c]$ and $[c, b]$ where $c \in (a, b)$. Then show that

$$V_f[a, c] + V_f[c, b] = V_f[a, b]$$

Hence deduce that if $c_1, c_2 \in (a, b)$ with $c_1 < c_2$ then $V_f[a, c_1] + V_f[c_1, c_2] + V_f[c_2, b] = V_f[a, b]$. 5

(*b*) Let $x_1, x_2, .., x_n, ..$ be an enumeration of all rationals in $[0, 2]$ and let $f : [0, 2] \rightarrow \mathbb{R}$ be defined by

$$f(x_n) = \frac{1}{n^4}, n = 1, 2, 3, \ldots \text{ and } 0 \text{ elsewhere}$$

Prove that f is a function of bounded variation on $[0, 2]$. 3

4. (*a*) Let $\alpha(x)$ be monotonically increasing function on $[a, b]$. Prove that the following statements are equivalent

(*i*) $f \in R(\alpha)$ on $[a,b]$ (limit definition)

(*ii*) $\underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha$ 5 (*b*) Evaluate the RS-integral

$$\int_{-1}^{2}(2x+5)\,d\,(|x|+3)$$ 3

5. **State the following theorems and prove any *one* of them :** 4 + 4

Fatou's Lemma, Lebesgue's Dominated Convergence theorem, Monotone Convergence theorem.

6. (*a*) **Define sigma algebra (σ-algebra) in a set X with example.**

Suppose m is a σ-algebra in $X, f : X \to [-\infty, \infty]$ be a function. If $f^{-1}((\alpha, \infty]) \in m$ for every real α, then show that f is a measurable function. 1 + 4

(*b*) Let $f_n : X \to [-\infty, \infty]$ be measurable, for $n = 1, 2, 3, \ldots$

Define $\limsup_{n \to \infty} f_n$, $\liminf_{n \to \infty} f_n$.

Also, show that $\limsup_{n \to \infty} f_n$, $\liminf_{n \to \infty} f_n$ are measurable. 3

7. (*a*) Give an example of a function which is Lebesugue Integrable but not Riemann Integrable. 3

(*b*) Show that a bounded function f is lebesgue Integrable on $[a, b]$ if and only if for every $\in > 0$ there is a measurable partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \in$. 3

(*c*) Define Lebesgue Integral for unbounded measurable function on $[a, b]$. 2

[*Internal Assessment* : **10 Marks**]