M.Sc. 2nd Semester Examination, 2012

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

(*Functional Analysis*)

PAPER — MTM-205

Full Marks : 50

Time : 2 hours

Answer **Q.No.1** and **2** and any **four** from **Q.No.3** to **8**

The figures in the right hand margin indicate marks

1. Answer any *two* questions : 2×2

(*a*) What do you mean by totally bounded subset of a metric space ?

(*b*) Define Banach space.

(*c*) Is the set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}$ compact in $\mathbb{R}^3$?

(*Turn Over*)

2. Answer any *one* :

(*a*) What do you understand by equivalence of norms ? Show that any two norms on $\mathbb{R}^n$ are equivalent. 2 + 6

(*b*) Let X be a compact metric space. Let $C(X)$ be the set of all real valued continuous function on X and is endowed with sup norm metric. Show that a subset $B \subseteq C(X)$ is compact iff B is closed, bounded and equicontinuous. 8

3. (*a*) What do you mean by a contraction mapping of a metric space (X, d) ?

(*b*) Show that every contraction mapping on a complete metric space has a unique fixed point. 2 + 5

4. (*a*) What do you mean by bounded linear operator ?

(*b*) Let X and Y be normed linear spaces over $\mathbb{R}$ and $T : X \rightarrow Y$ be a linear operator. Show that T is continuous if and only if T is bounded. 2 + 5

5. (*a*) Consider the Banach space $(C[0,1], \|\cdot\|_\infty)$. Assume that the function $k : [0, 1] \times [0, 1] \to \mathbb{R}$ is continuous. Define $T : C[0,1] \to C[0,1]$ by

$$(T_x)(t) = \int_0^1 k(t,\tau)x(\tau)d\tau, \ x \in C[0,1]$$

Is T a linear operator ?

(*b*) Let X and Y be normed linear spaces over the same field F and $T : X \to Y$ be a continuous linear operator. Show that the null space $N(T)$ is closed. 4 + 3

6. (*a*) Define dual space of a normed linear space.

(*b*) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by

$f(\tilde{x}) = x_1 + x_2 + x_3$ when $\tilde{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$.

Show that f is a bounded linear functional. Also, find the distance of the origin from the hyperplane $x_1 + x_2 + x_3 = 1$.

(*c*) Define Hilbert space. 1 + 4 + 2

7. (*a*) Let X be an inner product space and $C\ (\neq \phi)$ be a convex subset of X which is complete in the metric induced by the inner product on X. Show that for every $x \in X$, there exists a unique $y_0 \in C$ such that

$$\inf_{y \in C} \| x - y \| = \| x - y_0 \|$$

(*b*) If in an inner product space $< x, u > = < x, v >$ for all x in the space, show that $u = v$. 5 + 2

8. Let X be a Banach space and Y be a normed linear space over the same field F (IR or ₵). Show that a set B of bounded linear operators from X to Y is uniformly bounded if and only if it is pointwise bounded. 7

[*Internal Assessment* – 10 Marks]