M.Sc. 2nd Semester Examination, 2012

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

(Functional Analysis)

PAPER — MTM-205

Full Marks : 50

Time : 2 hours

Answer Q.No.1 and 2 and any four from Q.No.3 to 8

The figures in the right hand margin indicate marks

1. Answer any two questions : 2 x 2

(a) What do you mean by totally bounded subset of a metric space?

(b) Define Banach space.

(c) Is the set \( \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\} \) compact in \( \mathbb{R}^3 \)?

(Turn Over)
2. Answer any one:

(a) What do you understand by equivalence of norms? Show that any two norms on $\mathbb{R}^n$ are equivalent. $2 + 6$

(b) Let $X$ be a compact metric space. Let $C(X)$ be the set of all real valued continuous function on $X$ and is endowed with sup norm metric. Show that a subset $B \subseteq C(X)$ is compact iff $B$ is closed, bounded and equicontinuous. $8$

3. (a) What do you mean by a contraction mapping of a metric space $(X, d)$?

(b) Show that every contraction mapping on a complete metric space has a unique fixed point. $2 + 5$

4. (a) What do you mean by bounded linear operator?

(b) Let $X$ and $Y$ be normed linear spaces over $\mathbb{R}$ and $T : X \rightarrow Y$ be a linear operator. Show that $T$ is continuous if and only if $T$ is bounded. $2 + 5$
5. (a) Consider the Banach space \((C [0,1], \| \cdot \|_{\infty})\). Assume that the function \(k : [0, 1] \times [0, 1] \rightarrow \mathbb{R}\) is continuous. Define \(T : C [0,1] \rightarrow C [0,1]\) by

\[
(T_x)(t) = \int_{0}^{1} k(t, \tau)x(\tau)d\tau, \; x \in C[0,1]
\]

Is \(T\) a linear operator?

(b) Let \(X\) and \(Y\) be normed linear spaces over the same field \(F\) and \(T : X \rightarrow Y\) be a continuous linear operator. Show that the null space \(N(T)\) is closed.

6. (a) Define dual space of a normed linear space.

(b) Let \(f : \mathbb{R}^3 \rightarrow \mathbb{R}\) be given by

\[
f(\vec{x}) = x_1 + x_2 + x_3 \text{ when } \vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3.
\]

Show that \(f\) is a bounded linear functional. Also, find the distance of the origin from the hyperplane \(x_1 + x_2 + x_3 = 1\).

(c) Define Hilbert space.
7. (a) Let $X$ be an inner product space and $C (\neq \emptyset)$ be a convex subset of $X$ which is complete in the metric induced by the inner product on $X$. Show that for every $x \in X$, there exists a unique $y_0 \in C$ such that

$$\inf_{y \in C} ||x - y|| = ||x - y_0||$$

(b) If in an inner product space $<x, u> = <x, v>$ for all $x$ in the space, show that $u = v$. 5 + 2

8. Let $X$ be a Banach space and $Y$ be a normed linear space over the same field $F$ (IR or $\mathbb{C}$). Show that a set $B$ of bounded linear operators from $X$ to $Y$ is uniformly bounded if and only if it is pointwise bounded. 7

[Internal Assessment – 10 Marks ]