

Labelling of Cactus Graphs

Kalyani Das

Department of Mathematics
Belda Collge,Vidyasagar University, Midnapore – 721 453, India.
e-mail: kalyanid380@gmail.com

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ABSTRACT

Communication network signal interference can be modelled using distance labelling where the labels assigned to each vertex depend on the distance between vertices and the strength of the network signal. This paper assumes three levels of signal interference within a graph, G . In the graph, if the distance between any two vertices, v_x , and v_y is 1 i.e., ($D(v_x, v_y) = 1$) then the difference of their labels must be at least d i.e., ($|f(v_x) - f(v_y)| \geq d$). Similarly, if $D(v_x, v_y) = 2$, then $|f(v_x) - f(v_y)| \geq 2$ and if $D(v_x, v_y) = 3$, then $|f(v_x) - f(v_y)| \geq 1$. The $L(d, 2, 1)$ -labelling number $k_d(G)$ of G is the smallest positive integer k_d such that G has an $L(d, 2, 1)$ -labelling with k_d as the maximum label. This paper presents a general k_d -value for paths, bipartite graphs, complete graphs, cycles as well as Cactus Graphs that is d -dependent for any $d \geq 3$.

Keywords: distance labelling, distance labelling, network labelling

1. Introduction

Let $G = (V, E)$ be a finite, connected, undirected, simple graph of n vertices and m edges, where V is the set of vertices and E is the set of edges. A *cactus graph* is a connected graph in which every block is either an edge or a cycle.

A vertex v is called a *cut-vertex* if removal of v and all edges incident to v disconnect the graph. A *non-separable graph* is a connected graph which has no cut-vertex and a *block* means a maximum non-separable sub-graph. A block is a *cyclic block* or simply *cycle* in which every vertex is of degree two.

The channel assignment problem is an engineering problem in which the task is to assign a channel (non negative integer) to each FM radio station in a set of given stations such that there is no interference between stations and the span of the assigned channels is minimized. The level of interference between any two FM radio stations correlates with the geographic locations of the stations. Closer stations have a stronger interference, and thus there must be a greater difference between their assigned channels. In 1980, Hale introduced a graph theory model of the channel assignment problem where the problem was represented using the idea of vertex coloring.[19]. Vertices on the graph correspond to the radio stations and the edges show the proximity of the stations. In 1991, Roberts proposed a variation of the channel assignment problem in which the FM radio stations were considered either "close" or "very close." "Close" stations were vertices of distance

two apart on the graph and were assigned channels that differed by two; stations that were considered "very close" were adjacent vertices on the graph and were assigned distinct channels [29]. More precisely, Griggs and Yeh [18] defined the $L(2,1)$ -labelling of a graph as a function f which assigns to every vertex a label from the set of positive integers such that the following conditions are satisfied: $|f(v_x) - f(v_y)| \geq 2$ if the distance between v_x, v_y , $D(v_x, v_y) = 1$ and $|f(v_x) - f(v_y)| \geq 1$ if $D(v_x, v_y) = 2$. $L(2,1)$ -labelling has been studied in recent years.

In 2001, Chartrand et al. introduced the radio-labelling of graphs; this was motivated by the regulations for the channel assignments in the channel assignment problem [4]. Radio-labelling takes into consideration the diameter of the graph, and as a result, every vertex is related.

In 2013, Paul *et al.* investigated the problem of $L(0,1)$ -labeling on interval graph [24]. In [27] Paul *et al.* shows that $\lambda_{2,1}(G) \leq \Delta + w$ for interval graph and they shows that $\lambda_{2,1}(G) \leq \Delta + 3w$ for circular-arc graph, where w represents the size of the maximum clique [25]. Also, Paul *et al.* have shown that $\lambda_{2,1}(G) \leq \max\{4\Delta - 2, 5\Delta - 8\}$ and $\lambda_{0,1}(G) \leq \Delta - 1$ for permutation graph [26, 28]. Later, In [30], Amanathulla and Pal have showed that $\lambda_{0,1}(G) \leq \Delta$ and $\lambda_{1,1}(G) \leq 2\Delta$ for circular-arc graphs and they also shown that $\lambda_{3,2,1}(G) \leq 9\Delta - 6$ and $\lambda_{4,3,2,1}(G) \leq 16\Delta - 12$ for circular-arc graphs [31]. In 2017, Amanathulla and Pal shown that the upper bounds of $L(3,2,1)$ -labeling is $11\Delta - 8$ if $\Delta \leq 5$ and it is $13\Delta - 18$ if $\Delta > 5$ for permutation graphs [33], and they also proved that $\lambda_{3,2,1}(G) \leq 6\Delta - 3$ and $\lambda_{4,3,2,1}(G) \leq 10\Delta - 6$ for interval graphs [34]. By extending the idea of $L(3,2,1)$ - and $L(4,3,2,1)$ -labeling of interval graph and circular-arc graph they studied $L(h_1, h_2, \dots, h_m)$ -labeling for interval graphs [32] and for circular arc graphs [35] and obtained a tighter upper bound of $L(h_1, h_2, \dots, h_m)$ -labeling for both interval graph and circular-arc graph. Recently, they have studied about surjective $L(2,1)$ -labeling of cycles and circular-arc graphs and have showed a good results for it [36]. In 2018, Amanathulla *et al.* also have investigated $L(3,1,1)$ -labeling numbers of square of paths, complete graphs and complete bipartite graphs [37] and obtained exact result for each.

Practically, interference among channels may go beyond two levels. $L(3,2,1)$ -labelling naturally extends from $L(2,1)$ -labelling by taking into consideration vertices which are within a distance of three apart, but it remains less difficult than radio-labelling. In this paper the $L(d,2,1)$ -labelling number for paths, cycles, complete graphs, complete bipartite graphs and for Cactus Graphs is determined. The results of Clipperton et al [5] are used as a basis for the unknown value d .

2. Preliminaries and definitions

The definitions and some notations used in this section are adopted from those used by Clipperton, Gehrtz, Torkornoo, and Szanizslo [5].

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Definition 1. Let $G = (V, E)$ be a graph and f be a mapping $f : V \rightarrow N$. The distance between two such vertices is represented by $D(v_x, v_y)$ and the mapping of f is an $L(d, 2, 1)$ -labelling of G if for all vertices $v_x, v_y \in V$,

$$\begin{aligned} |f(v_x) - f(v_y)| &\geq d, \text{ if } D(v_x, v_y) = 1; \\ &\geq 2, \text{ if } D(v_x, v_y) = 2; \\ &\geq 1, \text{ if } D(v_x, v_y) = 3. \end{aligned}$$

Definition 2. The $L(d, 2, 1)$ -number, $k_d(G)$, of a graph G is the smallest natural number k_d such that G has an $L(d, 2, 1)$ -labelling with k_d as the maximum label. An $L(d, 2, 1)$ -labelling of a graph G is called a minimal $L(d, 2, 1)$ -labelling of G if, under the labelling, the highest label of any vertex is $k_d(G)$.

If 1 is not used as a vertex label in an $L(d, 2, 1)$ -labelling of a graph, then every vertex label can be decreased by one to obtain another $L(d, 2, 1)$ -labelling of the graph. Therefore in a minimal $L(d, 2, 1)$ -labelling, 1 will necessarily appear as a vertex label.

Definition 3. A graph G , where $G = (V, E)$, is called a complete graph on n vertices, denoted by K_n , if for all vertices $v_x, v_y \in V$, $(v_x, v_y) \in E$.

Definition 4. G is called a complete bipartite graph, denoted by $K_{m,n}$, if the following conditions are satisfied:

1. The set of vertices, V , can be partitioned into two disjoint sets of vertices, A and B , such that $|A| = m$, $|B| = n$, and $|V| = m + n$.
2. For all $a_i, a_j \in A$, (a_i, a_j) not belongs to E and for all $b_i, b_j \in B$, $(b_i, b_j) \notin E$
3. For all $a_i \in A$ and $b_j \in B$, $(a_i, b_j) \in E$.

A star, denoted by S_n , is the complete bipartite graph $K_{1,n-1}$. A star can also be denoted by $K_{1,\Delta}$, where Δ represents the degree of the graph.

Definition 5. A graph G , where $G = (V, E)$, is called a path, denoted by P_n , if $V = \{v_1, v_2, \dots, v_n\}$ such that only $(v_i, v_{i+1}) \in E$ where $1 \leq i \leq n-1$. A graph G is called a cycle, denoted by C_n , if $V = \{v_1, v_2, \dots, v_n\}$ such that only $(v_i, v_{i+1}) \in E$ where $1 \leq i < n-1$ and $(v_1, v_n) \in E$.

Definition 6. A Cactus Graph G is a graph in which every blocks are either edge or a cycle *i.e.*, every edge contained only in one cycle.

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Shao [39] studied the $L(3,2,1)$ -labelling of Kneser graphs, extremely irregular graphs, Halin graphs, and gave bounds for the $L(3,2,1)$ -labelling numbers of these classes of graphs. Liu and Shao [38] studied the $L(3,2,1)$ -labelling of planar graphs, and showed that $\lambda_{3,2,1}(G) \leq 15(\Delta^2 - \Delta + 1)$ if G is a planar graph of maximum degree Δ . Clipperton et al. [5] determined the $L(3,2,1)$ -labelling numbers for paths, cycles, caterpillars, n -ary trees, complete graphs and complete bipartite graphs, and showed that $\lambda_{3,2,1}(G) \leq (\Delta^3 + \Delta^2 + 3\Delta)$ for any graph G with maximum degree Δ . Here we give the upper bound for the paths, cycles, trees and same for Cactus Graphs.

Theorem 1. [5] If G is a graph with maximum degree Δ , $k_3(G) \leq (\Delta^3 + \Delta^2 + 3\Delta)$

Corollary 1. [5] For $L(d,2,1)$ -Labelling

$$k_d(G) \leq (2d-1)\Delta + 3\Delta(\Delta-1) + \Delta(\Delta-1)^2 = \Delta^3 + \Delta^2 + (2d-3)\Delta.$$

3. Complete and complete bipartite graphs

In this section we will find the minimal $L(d,2,1)$ -number, $k_d(K_n)$ or $k_d(K_{a,b})$ for complete and complete bipartite graphs.

Theorem 2. For complete graphs, $k_d(K_n) = d(n-1) + 1$ where n is the number of vertices in the graph.

Proof: Let K_n be a complete graph with n vertices. One vertex in K_n must be labelled with 1. As all other vertices are adjacent to each other, no two vertex labels may have a difference less than d . Thus, $k_d(K_n) = d(n-1) + 1$. \square

Theorem 3. Let $K_{a,b}$ be a complete bipartite graph with partitions A and B , where $|A| = a$ and $|B| = b$. Then, $k_d(K_{a,b}) = d + 2(a+b) - 3$.

Proof: Every vertex in A is distance two from every other vertex in A and every vertex in B is distance two from every other vertex in B . Thus, the label on every vertex in partition A must differ by at least two from the label on every other vertex in partition A . Similarly, the vertex labels in partition B must also differ by at least two. Additionally, the difference between the maximum value of A and the minimum value of B must differ by at least d . Since the label 1 must be used,

$$k_d(K_{a,b}) = d + 2(a-1) + 2(b-1) + 1 = d + 2(a+b) - 3. \quad \square$$

Corollary 2. For a star, $K_{1,\Delta}$, $k_d(K_{1,\Delta}) = d + 2\Delta - 1$.

Proof: Since a star is a complete bipartite graph with $a = 1$ and $b = \Delta$, it follows that the maximum $k_d(K_{1,\Delta}) = d + 2\Delta - 1$. \square

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4. Paths

In this section, we will find repeatable labelling patterns for paths of length n that minimize $k_d(P_n)$. We will first look at cases of $k_3(P_n)$ before generalizing to $k_d(P_n)$. If $d \geq 6$, we will find that the length of repeatable patterns will be 4 vertices. Lemma 1 and Theorem 3 have been adapted from the work presented by Clipperton et al [5].

Lemma 1. For a path on n vertices, P_n , with $n \geq 8$, $k_3(P_n) \geq 8$.

$$k_3(P_n) = \begin{cases} 1, & \text{if } n = 1; \\ 4, & \text{if } n = 2; \\ 6, & \text{if } n = 3,4; \\ 7, & \text{if } n = 5,6,7; \\ 8, & \text{if } n \geq 8; \end{cases}$$

Theorem 4. For any path, P_n , when $d \geq 4$

$$k_d(P_n) = \begin{cases} 1, & \text{if } n = 1; \\ d+1, & \text{if } n = 2; \\ d+3, & \text{if } n = 3,4; \\ d+5, & \text{if } n \geq 5; \end{cases}$$

Thus from the above Lemmas and Theorems we conclude that

Theorem 5. If G is a path with maximum degree $\Delta (= 2)$ $k_d(G) \leq d\Delta + 2$, for any d .

Lemma 2. For a tree G with maximum degree Δ , $k_d(G) \leq 3d + 2\Delta - 5$, for any $d \geq 3$.

Proof: First we rearrange the tree in such a way that the root possesses the maximum degree Δ . Label the root by 1. Then the vertices adjacent to the are labelled by the numbers from the set $\{d+1, d+3, d+5, \dots, d+2\Delta-1\}$. The vertices adjacent to the vertices of level one are labelled with the numbers from the set

$$\{3, 5, \dots, 2d+1, 2d+2, \dots, 2d+2\Delta-3\}.$$

Vertices of next level are labelled from the set $\{2, 3, 4, \dots, 3d+1, 3d+2, \dots, 3d+2\Delta-5\}$

For labelling of the rest vertices of the rest levels can be selected from the set

$$\{1, 2, 3, \dots, d+1, d+2, d+3, \dots, d+2\Delta-1, 2d+1, 2d+2, \\ 2d+3, \dots, 2d+2\Delta-3, 3d+1, 3d+2, \dots, 3d+2\Delta-5\}.$$

Thus for any tree with maximum degree Δ , $k_d(G) \leq 3d + 2\Delta - 5$, $d \geq 3$. \square

5. Cycles

In this Section we label the vertices of cycle C_n in the following manner: The labelling pattern is $\{1, d+1, 2d+1\}$ for $n = 3$, $\{1, d+3, 3, d+1\}$ for $n = 4$

$\{1, d+3, 3, 2d+2, d+1\}$ for $n = 5$, $\{1, d+3, 3, d+5, 5, d+7\}$ for $n = 6$

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- $\{1, d+2, 2d+2, 2, 2d+4, 4, d+4\}$ for $n=7$
- $\{1, 2d, d, 2d+4, d+4, 2, d+2, 2d+2\}$ for $n=8$
- $\{1, d+1, 2d+2, 2, 2d+4, 5, d+5, 3, d+3\}$ for $n=9$
- $\{1, 2d, d, 2d+2, d+2, 2d+4, 3, d+3, 2d+3, d+1\}$ for $n=10$
- $\{1, d+1, 2d+2, 2, 2d+4, d+4, 2d+2, 1, 2d, d, 2d+3\}$ for $n=11$
- $\{1, 2d, d, 2d+2, d+2, 4, d+4, 2, d+2, 2d+2\}$ for $n=12$

and for cycle of any length can be labelled with maximum label $2d+4$. Thus we conclude the following theorem.

Theorem 6. *If G is a cycle with maximum degree $\Delta (= 2)$, $k_d(G) \leq d\Delta + 4$, for any $d \geq 3$.*

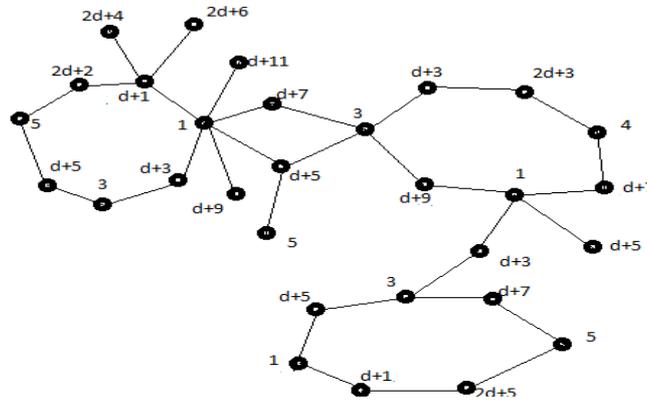


Figure 1:

6. Cactus graphs

Cactus graphs are those in which every blocks are either edge or cycle. Therefore an individual path, cycle, tree or combination of all of these is considered as cactus graph. The maximum label for the combined graph with maximum degree $\Delta (> 2)$, we get $k_d(G) \leq d\Delta$, for any $d > 3$. We illustrate this by an example given in Figure 1. Here $\Delta = 5$, and maximum label $k_d(G)$ obtained therefore $5d$ which is greater equal to $d+12$, $2d+9$, $3d+6$, $4d+3$.

Theorem 7. *If G is cactus graph with maximum degree Δ , $k_d(G) \leq d\Delta + 4$, for any $d \geq 3$.*

Proof: From the previous sections we have obtained the maximum label $k_d(G)$ for path, tree, cycles individually which is less or equal to $d\Delta + 4$. Also for combined graph it is $d\Delta$. Hence we conclude that if G is a cactus graph with maximum degree Δ , $k_d(G) \leq d\Delta + 4$, for any $d \geq 3$. \square

REFERENCES

1. A.A.Bertossi, Total domination in interval graphs, Information Processing Letters, 23(1986) 131-134.

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2. D.Br  elaz, New methods to color the vertices of a graph. *Communications of the ACM*, 22(4) (1979) 251-256.
3. N.Barnier and P.Brisset, Graph coloring for air traffic flow management, In *CPAIOR'02: Foruth International Workshop on Integration of AI and OR Techniques in Constraints Programming for Combinatorial Optimization Problems*, Le Croisic, France, March 2002, pages 133-147.
4. G.Chartrand, D.Erwin, F.Harary and P.Zhang, Radio labeling of graphs, *Bull. Inst. Combin. Appl.*, 33 (2001), 77-85.
5. J.Clipperton, J.Gehertz, Z.Szaniszlo and D.Torkornoo, $L(3,2,1)$ -Labeling on Simple Graphs, *VERUM*, Valparaiso University (2006).
6. F.Comellas and J.Ozon, Graph coloring algorithms for assignment problems in radio networks, *Applications of Neural Networks to Telecommunications 2*. Edit. J. Alspector, R.Goodman and T.X.Brown, Lawrence Erlbaum Ass., 1995, pp. 49-56. <http://www-mat.upc.es/comellas/radio/radio.html>
7. D.Costa and A.Hertz, Ants can colour graphs, *Journal of the Operational Research Society* 48 (1998) 295-305.
8. M.Chiarandini and T.St  utzle, An application of iterated local search to graph coloring. In D.S.Johnson, A.Mehrotra, and M.Trick, editors, *Proceedings of the Computational Symposium on Graph Coloring and its Generalizations*, Ithaca, New York, USA, 2002 pages 112-125.
9. I.Dagan, M.C.Golumbic and R.Y.Pinter, Trapezoid graphs and their coloring, *Discrete Appl. Math.*, 21 (1988) 35-46.
10. K.Das and M.Pal, An optimal algorithm to find the maximum and minimum height spanning trees on cactus graphs, *Advanced Modelling and Optimization*, 10(1) (2008) 121-134.
11. K.Das and M.Pal, An optimal algorithm to find the maximum independent set and maximum 2-independent set on cactus graphs, *Advanced Modelling and Optimization*, 12(2) (2010) 239-248.
12. K.Das and M.Pal, An optimal algorithm to find the maximum weight 2-coloured set on cactus graphs, *Journal of Information and Computing Science*, 5(3) (2010) 211-223
13. K.Das and M.Pal, An optimal algorithm to find the minimum dominating set on cactus graph, *Mathematical Modelling and Applied Computing*, 3(1) (2012) 31-43.
14. K.Das and M.Pal, An optimal algorithm to find minimum 2-neighbourhood covering set on cactus graphs, *Annals of Pure and Applied Mathematics*, 2(1) (2012) 45-59.
15. K.Das and M.Pal, Some algorithms on cactus graphs, *Annals of Pure and Applied Mathematics*, 2(2) (2012) 114-121.
16. K.Das, Algorithms to find winner index of some graphs, *Annals of Pure and Applied Mathematics*, 7(1) (2014) 13-18.
17. F.Gavril and M.Yannakakis, The maximum k -colorable subgraph problem for chordal graphs, *Information Processing Letter*, 24 (1987) 133-137.
18. J.R.Griggs and R.K.Yeh, Labeling graphs with a condition at distance two, *SIAM J. Discrete Math.*, 5 (1992) 586-5995.
19. W.K.Hale, Frequency assignment: theory and application, *Proc. IEEE*, 68 (1980), 1497-1514.
20. K.Jonas, Graph coloring analogues with a condition at distance two: $L(2,1)$ -labelling and list λ -labellings, Ph.D. thesis, University of South Carolina (1993), 8-9.

21. G.Lewandowski and A.Condon, Experiments with parallel graph coloring heuristics and applications of graph coloring. In David S. Johnson and Michael A. Trick, editors, Cliques, coloring and Satisfiability: Second DIMACS Implementation Challenge, 1993, American Mathematical Society, 26 (1996) 309-334.
22. F.T.Leighton, A graph coloring algorithm for large scheduling problems. Journal of Research of the National Bureau of Standards, 85 (1979) 489-506.
23. M.Pal, Intersection graphs: an introduction, Annals of Pure and Applied Mathematics, 4(1) (2013) 41-93
24. S.Paul, M.Pal and A.Pal, An efficient algorithm to solve $L(0,1)$ -labeling problem on interval graphs, Advanced Modelling and Optimization, 15(1) (2013) 31-43.
25. S.Paul, M.Pal and A.Pal, $L(2,1)$ -labeling of circular-arc graph, Annals of Pure and Applied Mathematics, 5(2) (2014) 208-219.
26. S.Paul, M.Pal and A.Pal, $L(2,1)$ -labeling of permutation and bipartite permutation graphs, Mathematical in Computer Science, 9 (2015) 113-123.
27. S.Paul, M.Pal and A.Pal, $L(2,1)$ -labeling of interval graphs, Journal of Applied Mathematics and Computing, 49(1) (2015) 419-432.
28. S.Paul, M.Pal and A.Pal, $L(0,1)$ -labeling of permutation graphs, Journal of Mathematical Modelling and Algorithms, 14(4) (2015) 469-479.
29. F. S. Roberts, T-colorings of graphs: recent results and open problems, Discrete Math., 93 (1991), 229-245.
30. Sk Amanathulla and M.Pal, $L(0,1)$ - and $L(1,1)$ -labeling problems on circular-arc graphs, International Journal of Soft Computing, 11(6) (2016) 343-350.
31. Sk Amanathulla and M.Pal, $L(3,2,1)$ - and $L(4,3,2,1)$ -labeling problems on circular-arc graphs, International Journal of Control Theory and Applications, 9(34) (2016) 869-884.
32. Sk Amanathulla and M.Pal, $L(h_1, h_2, \dots, h_m)$ -labeling problems on interval graphs, International Journal of Control Theory and Applications, 10(1) (2017) 467-479.
33. Sk Amanathulla and M.Pal, $L(3,2,1)$ -labeling problems on permutation graphs, Transylvanian Review, 25(14) (2017) 3939-3953.
34. Sk Amanathulla and M.Pal, $L(3,2,1)$ - and $L(4,3,2,1)$ -labeling problems on interval graphs, AKCE International Journal of Graphs and Combinatorics, 14 (2017) 205-215.
35. Sk Amanathulla and M.Pal, $L(h_1, h_2, \dots, h_m)$ -labeling problems on circular-arc graphs, Far East Journal of Mathematical Sciences, 102(6) (2017) 1279-1300.
36. Sk Amanathulla and M.Pal, Surjective $L(2,1)$ -labeling of cycles and circular-arc graphs, Journal of Intelligent and Fuzzy Systems, 35 (2018) 739-748.
37. Sk Amanathulla, S.Sahoo and M.Pal, $L(3,1,1)$ -labeling numbers of square of paths, complete graphs and complete bipartite graphs, Journal of Intelligent and Fuzzy Systems, DOI:10.3233/JIFS-172195.
38. J.Z.Liu and Z.D.Shao, The $L(3, 2, 1)$ -labeling problem on graphs, Mathematica Applicata, 17(4) (2004), 596-602.
39. Z.D.Shao, The $L(3, 2, 1)$ -labeling problem on graphs, J. Qufu Normal University, 30(3) (2004), 24-28.