STUDIES ON IMPERFECT PRODUCTION INVENTORY SYSTEM UNDER DIFFERENT ENVIRONMENTS

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BY

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This thesis is dedicated to

my respected teachers

Dr. Jayanta Kumar Dey

and

Prof. Shyamal Kumar Mondal

for their endless support and encouragement.

CERTIFICATE

This is to certify that the thesis entitled **"STUDIES ON IMPERFECT PRODUCTION INVENTORY SYSTEM UNDER DIFFERENT ENVIRONMENTS"** being submitted to the **VIDYASAGAR UNIVERSITY** by **Sri Amalesh Kumar Manna** for the award of degree of **DOCTOR OF PHILOSOPHY in Mathematics** is a record of bona fide research work carried out by him under our guidance and supervision. **Sri Manna** has worked in the department of **Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University** as per the regulations of this University.

In our opinion, this thesis is of the standard required for the award of the degree of **DOCTOR OF PHILOSOPHY IN SCIENCE**.

The results, embodied in this thesis, have not been submitted to any University or Institution for the award of any degree or diploma.

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DECLARATION

I, Amalesh Kumar Manna, do hereby declare that, I have not submitted the work presented in my thesis – "**STUDIES ON IMPERFECT PRODUCTION INVENTORY SYSTEM UNDER DIFFERENT ENVIRONMENTS**"or a part of it for any degree/ diploma or any other academic award elsewhere.

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List of Published Papers

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2. An EPQ model with promotional demand in random planning horizon: population varying genetic algorithm approach. ***Journal of Intelligent Manufacturing, 27(1) (2016) 1-17***, **Springer**.

3. Three-layer supply chain in an imperfect production inventory model with two storage facilities under fuzzy rough environment. ***Journal of Uncertainty Analysis and Applications, 2(1) (2014) 1-31***, **Springer**.

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7. Two layers supply chain in an imperfect production inventory model with two storage facilities under reliability consideration. *Communicated.*

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9. Two layers supply chain imperfect production inventory model with fuzzy credit period, time and production rate dependent imperfectness. *Communicated.*

Abbreviation

The following abbreviated forms have been used in the thesis.

ANOVA	Analysis of Variance
AUD	All Unit Discount
Cr	Credibility
DM	Decision Maker
EOQ	Economic Order Quantity
EPL	Economic Production Lot-size
EPQ	Economic Production Quantity
FPT	Fuzzy Programming Technique
$FSGA$	Fuzzy Simulation Based Genetic Algorithm
GA	Genetic Algorithm
$GAVP$	Genetic Algorithm with Varying Population
GFN	General Fuzzy Number
GHG	Green House Gas
GRG	Generalized Reduced Gradient
HM	Hessian Matrix
ISC	Integrated Supply Chain
$MAXGEN$	Maximum number of generation
MF	Membership Function
$MCDM$	Multi-criteria Decision Making
Nec	Necessity
NLP	Non-Linear Programming
OR	Operations Research
$p.d.f.$	Probability density function
PFN	Parabolic Fuzzy Number
Pos	Possibility
SC	Supply Chain
SCM	Supply Chain Model
$s.d.$	Standard deviation
TFN	Triangular Fuzzy Number
$TrFN$	Trapezoidal Fuzzy Number
w.r.t	With respect to

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General Introduction, Basic Concepts and Solution Methodologies

Chapter 1

General Introduction

1.1 Inventory Control System in Operations Research

Operations Research (OR) was introduced during the Second World-War. Basically, it is the collection of modern methods on the problems arising in the management of large systems of men, machines, materials and money related to industry, business and defence. During this war, the military management in England called upon a team of scientist to study the strategic and tactical problems related to air and land defence of the country with very limited military resources, it was necessary to decide upon the most effective utilization of them, e.g., the efficient ocean transport, effective bombing, etc. As the team was dealing with research on (military) operations, the work of this team of scientists was named as ‘Operational Research’ in England.

After the end of war, the success of military teams attracted the ‘Industrial managers’, who were seeking solutions to their complex executive-type problems. In this field, the first mathematical technique called the Simplex Method of linear programming was developed in 1947 by American mathematician, George B. Dantzig (Dantzig (1963). Since then, new techniques and applications have been developed through the efforts and cooperations of interested individuals in academic institutions and industries both. According to Churchman et al. (1957), OR is defined as the application of scientific methods, techniques and tools to decision making problems (DMP) involving the operations of systems so as to provide these in the control of the operations with optimum solutions to the problem.

Today, the impact of OR can be felt in many areas. Apart from military and business applications, the OR activities include transportation system, library, hospital, city planning, financial institution, etc. For example, in real life, it is observed that a small retailer knows roughly the demand of his/her customer in a month or a week and accordingly places order on the wholesaler to meet the demand of his/her customer. But, this is not the case with a manager of a big departmental store or a big retailer, because in such cases the stocking depends on several factors, such as demand, time of ordering, time lag between the order and actual receipt, deterioration, amelioration, time value of money, inflation, trade credit period etc. and the impreciseness of these factors. So, the problem for managers/retailers is to have a compromise between over-stocking and under-stocking. The study of such type of problem is known by the term '*Inventory Control*'.

Inventory control is concerned with the flow of materials from supplier to production and the subsequent flow of products through distribution centers to the customers. It is responsible for the planning, acquisition, storage, movement and control of materials and final products. It attempts to get the right goods at the right price at the right time to maintain desired service level at minimum cost.

1.2 Basic Concepts and Terminologies

The production inventory systems depend on several parameters such as objectives, objective functions, constraints, demand, production, replenishment, resources, various type of costs, shortages, deterioration, defectiveness, trade credit period, inflation etc. Though detailed descriptions on these parameters are available in the literature on inventory Control problems (cf., Hadley and Whitin [87], Naddor [158], Tersine [206], Silver and Peterson [194], etc.), a short overview has been followed.

Objectives

An *objective* is something which is to be pursued to its fullest extent. An objective generally indicates the direction desired. For example, in the problem of making development plans of a developing country, the 'objectives' of the government may be to maximize the national welfare, to minimize the dependence on foreign aid, the unemployment rate to be minimized, to minimize the complexity etc.

Constraints

Constraints in inventory system deal with various restrictions such as storage space limitations, number of replenishment, etc. Constraints may also be imposed on the amount of investment i.e., budget constraints, resources and finance, the amount of inventory held, average inventory expenditure, etc. These constraints can also be fuzzy in nature i.e., data for constraints, goals for the objectives, resources, etc. may be imprecise and vague. Beside there may be some probabilistic, possibility or necessity type constraints too in inventory system.

Inventory

In broad sense, *inventory* is defined as an idle resource of an enterprize / company /

manufacturing firm. It can be defined as a stock of physical goods, commodities or other economic resources which are used to meet the customer's demand or requirement of production. The inventories or stock of goods are classified into the following forms.

(A) Direct Inventories: The items which play a direct role in the manufacturing and become an integral part of finished goods are included in the category of direct inventories. These may be further classified into following four main groups:

(i) **Raw materials inventory:** This type of inventory consists of the purchased items or extracted materials that are transformed into components or products. This is provided (i) for economical bulk purchasing, (ii) to enable production rate changes, (iii) to provide production buffer against delays transportation and (iv) for seasonal fluctuations.

(ii) **Work-in-process inventory (WIP):** This consists of any item that is in some stage of completion in the manufacturing process. This is provided for (i) to enable economical lot production, (ii) to cater to the variety of products, (iii) for replacements of wastage and (iv) to maintain uniform production even if amount of sales may vary.

(iii) **Finished goods inventory:** This consists of completed products that will be delivered to customers. This is provided (i) for maintaining off-self delivery, (ii) to allow stabilization of the production level and (iii) for sales promotion.

(iv) **Components inventory:** This type consists of parts or sub-assemblies used in building the final product.

(B) Indirect Inventories: Indirect inventories include those items which are necessarily required for manufacturing but do not become the components of finished production, like : oil, grease, petrol, lubricants, maintenance materials, office materials, etc. This is also known as maintenance, repair and operational (MRO) inventory.

Demand

Demand is one of the most important factor in production inventory systems. It usually depends upon the decisions of retailer/customer outside the organization which has the inventory problem. When the demand size is the same from period to period, we say that it is constant. Otherwise, it is variable. In reality, demand depends on several factors such as follows:

- constant,
- depend on time,
- depend on inventory level/displayed stock-level,
- depended on the selling price of an item,
- depend on the discount of selling price of an item,
- depend on the advertisement,
- depend on the warranty period of an item, etc.

Sometimes when demand size is not known, it is possible to ascertain its probability distribution, in this case it is termed probabilistic demand. In some cases, demand may be represented by vague, imprecise and uncertain data. This type of demand is termed as fuzzy demand.

Production

Production systems are characterized by resources and the products. These resources are deigned to manufacture the products in response to market demand.

Imperfect Production

In real world manufacturing system, it is seen that every produced item may not be 100% perfect due to different factors involved in the system such as machine, ray-materials, labour etc. In any production system, it is seen that initially the production process is in an in-control state, because every factors in the system are fresh and the items produced perfect quality items. Generally, increasing production-run-time increases the probability of components of machine failure and impatience of labor staff, and thus accelerates the deterioration of the quality of the product. However, the production process starts in an in-control state by producing perfect items, and then it may become out-of-control state by producing mixture of perfect and imperfect item due to deterioration of machinery system as well as other factors. The rate of imperfectness may be constant or variable and also considered as crisp, stochastic or fuzzy in nature.

Inspection/Screening Process

In any imperfect production system, there is necessary to inspect each item after production to check whether the item is perfect or not. After inspection, the manufacturer meets the demand of retailers/customers.

Inspection Error

In real situation, the inspection process also may not be 100% perfect and error-free due to machine error and human factor. After inspection there may exist some possibility that a perfect item is falsely treated as an imperfect item and an imperfect item may be falsely considered as a perfect item which are known as type-I error and type-II error respectively.

Rework Policy

In real-life imperfect production systems, a portion of imperfect quality items may be reworked by replacing parts for assemblage of new products or the entire (upgraded) product can be sold again. The rework option plays an important role in eliminating waste and affecting the cost of manufacturing.

Discount Policy

In order to introduce larger purchases, a manufacturer often offers a reduced price if amounts greater than some minimum amount are ordered. This means the price per unit is lower if a large order is placed. Normally, two types of discount are considered: (i) All Unit Discount (AUD) and (ii) Incremental Quantity Discount (IQD).

Warranty Policy

In practice, the manufacturer and retailer usually offer a warranty for all selling items for a specific warranty period due to increasing the selling rate and reliability of product. Warranty period of a product is a duration in which a purchased product provides satisfactory performance to the customer. If any purchased product failed to work within its warranty period, then the servicing center replaces it with a new item or repair the product by replacing owing one or more parts. Due to this reason, a manufacturer considers warranty cost if there exists free-warranty on selling items in the warranty period.

Learning Effect

Reduction in labour time or cost or defective units, as a result of workers' proficiency in performing a repetitive task due to the gain in previous work experience, is well known as the 'Learning effect'. Its reverse effect is known as Forgetting effect.

Supply Chain Model (SCM)

A supply chain model (SCM) is a network of supplier(s), producer, distributor(s) and customers which synchronizes a series of inter-related business process in order to have:
(i) Optimal procurement of raw materials from nature;
(ii) Smooth transportation of raw-materials into warehouse;
(iii) Production of goods in the production center and Distribution of finished goods to retailer for sale to customers.

Green Supply Chain (GSC)

Green Supply Chain (GSC) tries to reduce the undesirable environmental impacts of supply chain processes within the participating organizations and the whole supply chain as well. But during recent years, SC managers tend more to consider environmental aspects in their decision making process. Green supply chain (GSC) is not just about considering environment aspect in SC decision making processes, but also about productivity and making more profit. Srivastava [197] defined GSCM as integrating environmental thinking into supply-chain management, including product design, material sourcing and selection, manufacturing processes, delivery of the final product to the consumers as well as end-of-life management of the product after its useful life. GSCM has emerged from two origins. First, environmental managers started to use life cycle assessment (LCA) approach for assessing products environmental impacts. In addition to usual product design and manufacturing processes, this approach considers many logistical activities such as material handling, packaging, distribution and by production to reduce disposal.

Two Warehouse

In a busy market place like super market, corporation market, municipality market etc. the storage space of a showroom is limited. When an attractive price discount for bulk purchase is available or the cost of procuring goods is higher than the inventory related other costs or there are some problems in frequent procurement's or the demand of items is very high or the bulk transport facility is available etc. then management decides to purchase a huge quantity of items at a time. All these units cannot be stored in the existing storage house at the market place (showroom) due to limited capacity. Then for storing the excess units, one (sometimes more than one) additional warehouse (called Secondary Warehouse (SW)) is hired on a rental basis. These SW may be located near the showroom or a little away from it. The products are first stored in showroom and then the excess amounts are stored in SW. The actual service to the customer is offered at Showroom only. The units of SW are transferred to showroom in a continuous/bulk release pattern to meet up the demand at showroom until the stocks in SW are emptied and lastly the units at showroom are released.

Inventory Cost

Inventory costs are the costs related to storing and maintaining its inventory over a certain period of time. Different types of cost related to inventory are calculated such as production cost, purchase cost, screening/ inspection cost, rework cost, setup cost, holding cost, idle cost, advertisement cost, warranty cost, shortage or stock-out cost, disposal cost, etc.

Production Cost

It is the unit production price to produce the item at the production center. Unit production cost is also may be depend upon the production, labour, maintenance when production is done in large quantities as it results in reduction of production cost per unit.

Purchase Cost

It is the unit purchase price to obtain the item from an external source. Also, when quantity discounts are allowed for bulk orders, unit price is reduced.

Screening/Inspection Cost

It is a cost for separate imperfect (defective items) and perfect item from produced item.

Rework Cost

A cost for reworking of defective items to make the products as new as the perfect in a manufacturing system. This cost to reduce the consumption of fresh raw materials, to reduce energy usage and to reduce air pollution in production inventory system.

Advertisement Cost

The advertisement cost is the cost associated with advertisement of an item in popular media like News paper, Magazine, TV, Radio, etc. and through the sales representative to increase the sale of that item.

Warranty Cost
The warranty cost is the cost associated with warranty policy of an item.

Idle Cost
It is the cost per unit time due to idle period of each supplier, manufacturer and retailers in SCM model. In this thesis, constant or imprecise idle cost is considered.

Setup Cost
It is the cost associated with the expense of issuing a purchase order to an out-side supplier or setting up machines before internal production. These costs also include clerical and administrative costs, telephone charges, telegram, transportation costs, loading and unloading costs, watch and ward costs, etc.

Holding Cost
It is the cost associated with the storage of the inventory until its use or sale. It is directly proportional to the quantity in inventory and the time for which the stocks are held. This cost generally includes the costs such as rent for storage space, interest on the money locked-up, insurance, taxes, handling, etc.

Shortage Cost
It is the penalty incurred when the stock proves inadequate to meet the demand of the customers. This cost parameter does not depend upon the source of replenishment of stock but upon the amount of inventory not supplied to the customer.

Disposal Cost
When an amount of some units of an item remains excess at the end of inventory cycle and if this amount is sold at a lower price in the next cycle to derive some advantages like clearing the stock, winding up the business etc. the revenue earned through such a process is called the disposal cost.

Time/Planning Horizon
The time period over which the inventory level will be controlled is called the time horizon. It may be finite or infinite depending upon the nature of the inventory system of the commodity.

Inflation
Inflation is a persistent increase in the level of consumer prices or a persistent decline in the

purchasing power of money, caused by an increase in available currency and credit beyond the proportion of available goods and services. It is the rate at which the general level of prices for goods and services is rising, and subsequently, purchasing power is falling. As inflation rises, every dollar will buy a smaller percentage of a good.

Time Value of Money

The basic principle that money can earn interest, so something that is worth 1 today will be worth more in the future if invested. This is also referred to as future value. For example, today, invested for one year at 5% return, would be worth 1.05 in one year. The time value of money (TVM) or the present discounted value is one of the basic concepts of finance.

1.3 Different Environments

The parameters, like inventory cost (viz., unit production cost, holding cost, set-up cost, shortage cost, transportation cost, advertisement cost, etc.), demand, available resources, etc. involved in the inventory system may be deterministic (crisp/precise) or some of these may be non-deterministic (i.e., stochastic or imprecise or both stochastic and imprecise). Depending on the nature of such parameters, the environment in which inventory models are developed can be classified as follows:

Crisp Environment: When all the system parameters and the resources, etc. are deterministic and precisely defined, the environment is known as crisp environment.

Stochastic Environment: In this environment, it may happen that the demand or any factor of a commodity in the society is uncertain, not precisely known, but some past data about it is available. From the available records, the probability distribution of demand or any other factor of the commodity can be determined and with that distribution the inventory control problem can be analyzed and solved.

Fuzzy Environment: In this system, some parameters and /or resources are fuzzy in nature. For example, when management launches a new product then they have no knowledge about demand and the other factors related to the product. Then management needs to collect the demand and the others information from experts. If the expert's opinion is imprecise then demand or other factors related to the expert opinion to be taken as a fuzzy and the corresponding environment is known as fuzzy environment.

Fuzzy-Stochastic Environment: It is an environment, which is the combination of both stochastic and fuzzy environments. Here, some parameters are fuzzy and some others are random. Some constraints / resources may be imprecise.

1.4 Historical Literature Reviews on Imperfect Production Inventory System in Different Environment

In developing the models, researchers have adopted many of the usual assumptions in imperfect production inventory, viz. (i) finite/infinite planning horizon, (ii) constant/varying production rate, (iii) constant/varying defective rate, screening rate and reworked rate, (iv) constant/varying demand rate, (v) allowing shortage or without shortage, (vi) constant/known distribution production time, (viii) single/multi-item production inventory system, (ix) single/multi-retailer, (x) inventory problems with delay in payment, etc. The available models can be grouped in the following ways:

(i) Imperfect Production Inventory Models in Crisp Environment

(ii) Imperfect Production Inventory Models in Fuzzy Environment

(iii) Imperfect Production Inventory Models in Stochastic Environment

(iv) Imperfect Production Inventory Models in Fuzzy Stochastic Environment

1.4.1 Historical Literature Reviews on Imperfect Production Inventory Models in Crisp Environment

In this environment, many parameters in imperfect production inventory system such as demand, production, production run time, inventory costs etc. are crisp nature. The earliest analysis of an inventory system was developed by Ford Harris [89] of Westinghouse Corporation, USA, in 1915. He derived the classical lot size formula. The same formula was also developed independently by R. H. Wilson [225], after few years and it has been named as Harris-Wilson model or Wilson's model. Since then, the above mentioned model has been modified and extended by several researchers changing the assumptions with the objective to make it more and more realistic. Later full length books devoted to the mathematical properties of inventory systems were published by Hadley and Whitin [87], Naddor [158], Tersine [206], Silver and Paterson [194] and others. The literature survey has been made separately on imperfect production, screening process, advertisement and time dependent demand and production cost in crisp environment which are discussed below.

Models on Imperfect Production:
In the existing literature of Economic Production Quantity (EPQ) models, most of the works are based on imperfect units. Lin et al. [138] formulated an integrated production inventory models with imperfect production process. Hsu and Hsu [95] developed an integrated vendor-buyer model with defective items which are treated as a single batch and returned to the vendor after a 100% screening process. For inventory problems, many studies considered the items produced as perfect in their models. However, imperfect items are produced due to

non-ideal production processes. Wee et al. [219] have investigated the effect of imperfect items on the EOQ model. Taleizadeh et al. [204] presented an EPQ model with rework process for a single stage production with one machine.

Inventory Models with Screening Process:
Now from the literature survey on imperfect production inventory models, it is seen that there exist two classes of the models on the basis of inspection methods to sort out the defective units from the perfect one. In one class of research, it has been seen that over the period of time the produced items deteriorate in manufacturing system. In this field, the different researchers (Lee and Rosenblatt [181], Lee and Park [131], Kim et al. [119], Jaber et al. [101, 102], Lin et al. [138]) examined an inspection method on the produced items based on deteriorating production process. Sana [185, 186] extended the EOQ model with assumptions that a known proportion of defective units was included in delivering lots and that the fixed and variable inspection costs were incurred in finding and removing these units. In this field, there are also many research papers in which different inspection methods had been used by Hsu and Hsu [94], Cheng [32], Salameh and Jaber [184], Chiu [35], Tripathy and Pattnaik [208], Zhang and Gerchak [241] used in EOQ/EPQ model.

Models on Advertisement and Time Dependent Demand:
In Harris and Wilson [225] lot-size model, demand was assumed as constant, but, in reality, it depends on several factors like time, initial/ on-hand displayed stock-level, selling price of an item, advertisement etc. Silver and Meal [193] published a lot size model taking time varying demand. After that, the research on the models with time dependent demand rate have been gradually studied by several researchers such as Donaldson [59], Goswami and Choudhuri [76], Lin et al. [138], Wang and Chen [210], Zhou et al. [243] and others. According to the market research, it is observed that time-to-time advertisement of an item also changes its demand. For this reason, Cho [41] developed an optimal production and advertising policies in crisp environment. In this area, Bhunia and Maiti [11] and others developed different types of inventory or production models considering different real life situations.

Models with Different Types of Production Cost:
In many research articles, production cost is assumed to be constant in EPQ models. But in reality, it depends on the many factors such as raw materials, labours engaged, wear and tear of machineries and rate of production. Khouja and Mehrez [114] assumed a unit production cost involving all these costs. After that, several authors (cf. Khouja [117], Chakraborty and Giri [17]) have implemented this in their EPQ models. Roy et al. [182], Das and Maiti [48] used this type of production cost in terms of volume flexibility. So, unit production cost varies directly with the products quality.

1.4.2 Historical Literature Reviews on Imperfect Production Inventory Models in Stochastic Environment

In this environment, many parameters in imperfect production inventory system such as demand, production, production run time, inventory costs etc. are random in nature and specified by probability distributions. The literature survey has been made separately on imperfect production, various type of demand, two warehouse, inflation, carbon-emission in stochastic environment which are discussed below.

Models on Imperfect Production:
In the literature, few Economic Production Lot-size (EPL) models are available for imperfect units in stochastic environment. Rosenblatt and Lee [181] studied the effects of an imperfect production process on the optimal production run time by assuming that time to out-of-control state is exponentially distributed. Hayek and Salameh [97] derived an optimal operating policy for the finite production model under the effect of reworking of imperfect quality items. They assumed that all defective units are repairable and allowed back-orders. Chiu [35] extended the work of Hayek and Salameh [97] and examined an EPQ model with defective items reworking the repairable units immediately. Sana [187] presented an EPL model with random imperfect production process and defective units were repaired immediately when they were produced. Tripathy and Pattnaik [208] obtained the optimal reliability for an EPL model connecting process reliability with imperfect production system. Here, some imperfect products were reworked and others were sold at a reduced price. Krishnamoorthi and Panayappan [124] studied an EPQ model that incorporated imperfect production quality, not screening out proportion of defects and thereby passing them on to customers and resulting in sales returns. Recently, Wang and Sheu [212] investigated a problem with production preventive maintenance, inspection and inventory for an imperfect production process. Annadurai and Uthayakumar [6] formulated an EPQ model with imperfect production process and stochastic demand. Yao et al. [234] considered a three-stage supply chain co-ordination model under imperfect production process with fuzzy-random demand. Taleizadeh et al. [202] revisited the EPQ model with rework process at a single stage manufacturing system with planned back-orders. The production system was assumed to be imperfect having random defective rates.

Models with Stock Dependent Demand:
It has been acknowledged that the displayed inventory has an effect on sales for many retail products-especially for style marchandise (Datta and Pal [52], Levin et al. [132]). This means that the demand rates of these items may be dependent on displayed stock level. Such type of demand in different forms was considered by Das et al. [49], Ray and Chaudhuri [173], Roy et al. [182], Yang et al. [232] and others. But, these propositions lead the retailer/manufacturer of the items of glass, ceramic, china clay etc. to a conflicting situation.

Models with Selling Price Dependent Demand:
It is observed that, the demand rate of an item is influenced by the selling price of an item, as, whenever the selling price of an item increases, the demand for that decreases and vice- verse. Generally, this type of demand is seen for different finished goods. Several authors like Guria et al. [85], Panda and Maiti [163], Goyal and Gunasekaran [77], Bhunia and Maiti [11], Abad and Jaggi [2], Karimi-Nasab et al. [109] have investigated this type of inventory models.

Models with Discount Dependent Demand:
In present situation, quantity discount is of growing interest due to its practical importance in purchasing and control of an item. Normally, one derives the better marginal cost of purchase/production availing the opportunities of cost savings through bulk purchase/production. Now-a-days, in the third world countries, with the introduction of open market system and advent of multi-nationals, there is a stiff competition amongst the companies to capture the maximum possible market. In the literature of discounted inventory problems, Chung and Lin [43] considered inventory replenishment models for deteriorating items in account of time discounting. Abad [1] determined an optimal policy for selling price and lot size when suppliers offer all unit quantity discount. Chakraborty and Martin [18] allowed discount price policies for inventory subject to declining demand. Weng [223] developed a channel coordination model with quantity discounts.

Models with Two Warehouse Facilities:
Inventory models with two warehouse facilities, one is existing storage maintained by the own management named as own warehouse (OW) and another is hired on rental basis named as rented warehouse (RW), have been discussed by Hartely [90], Sarma [189], Dave [53], Pakkala and Achary [161], Bhunia and Maiti [11], Benkherouf [10], Kar et al. [111], Dey et al. [57] and others.

Models with Inflation and Time Value of Money:
The initial attempt to analyze the effect of inflation and the time value of money on inventory control systems was made by Buzacott [12] in 1975. He dealt with an EOQ model with inflation subject to different types of pricing policies. Later on, Chandra and Bahner [23] showed the effects of inflation and the time-value of money on some inventory systems. Several authors then extended these works to make the more realistic inventory models under inflation and the time-value of money. Among these works, one can refer the work of Chen [31], Yang [231], Dey et al. [57] and others.

Models under Various Carbon Emissions Policies:
In reality, the green house effect and global warming have gained much attention due to strong and more frequent extreme change of climate. In every developing countries, there is a scope and regulation for measuring and maintaining such carbon-emission. Benjaafar et al. [9] first presented a model that illustrated how carbon emission can be incorporated to a decision-making problem. Dye and Yang [70] studied a deteriorating inventory system under various carbon emissions policies, like carbon taxes, carbon subsides etc.

1.4.3 Historical Literature Reviews on Imperfect Production Inventory Models in Fuzzy Environments

In this environment, one or more parameters in imperfect production inventory system such as demand, production, production run time, inventory costs etc. are fuzzy in nature.

In the world, reality is less or more uncertain, vague and ambiguous. In 1923, the philosopher Bertrand Russell first quoted: "All traditional logic habitually assumes that precise symbols are being employed". It is therefore not applicable to this terrestrial life but only to an imagined celestial existence. "All languages are vague" and "vagueness is also a matter of degree". An important step towards dealing with vagueness was made by the philosopher Black Max (1937), who introduced the concept of vague set. He first introduced the concept of "Fuzzy Set Theory" in a complete mathematical form. It is known that the real business world is full of uncertainties in non-stochastic sense, which leads to estimation of different inventory parameters as fuzzy numbers. Zadeh [239] first introduced a new concept 'Fuzzy Set Theory' to accommodate the uncertainty in non-stochastic sense.

Introduction to fuzzy set theory and basic ideas of fuzziness was described by Zimmermann [244]. Bellman and Zadeh [8] first introduced fuzzy set theory in decision making process. After that, Zimmermann [244], Zadeh [240], Dubois and Prade [61, 64], Li et al. [133] developed fuzzy models for single-period inventory problem. Hsieh [93], Xu and Zhou [228], Liu and Liu [142] applied fuzzy programming technique to single and multiple objective decision making problem. After that extensive research work have been done in this area. The literature survey has been made separately on fuzzy imperfect production inventory, fuzzy demand, fuzzy credit period, fuzzy technique in fuzzy environment which are discussed below.

Fuzzy Production Inventory Models:
The uncertain nature of demands, production inventory costs, damageable amounts, etc. results the imprecise rate of production. For this reason, imprecise production inventory model is vary much visible in imprecise decision making problem. Lee and Yao [128] developed an economic production quantity (EPQ) model with fuzzy demand quantity and fuzzy production quantity. After that, Chang et al. [24], Lin and Yao [137], Hsieh [93] considered fuzzy economic production for production inventory. Recently, Hazari et al. [98] proposed an advertisement policy and reliability dependent imperfect production inventory control problem in bi-fuzzy environment.

Models with Fuzzy/Fuzzy Rough Demand:
In inventory control problem, the assumption or consideration of market demand i.e., decay rate of the item is the fundamental issue of the system. In modern age, this rate of demand fluctuates day by day due to many reasons. In this context, inventory model with fuzzy demand was considered by Lee and Yao [128]. After that, Kao and Hsu [107], Dutta et

al. [68] developed a single-period inventory model with fuzzy demand. In 2009, Taleizadeh et al. [201] constructed a joint-replenishment inventory control problem with fuzzy rough demand.

Models with Fuzzy Technique:
In spite of different parameters a production inventory system may be imprecise for some other phenomenons. Some times, the nature of fuzziness may be allowable for an imprecise constrains or in many events. Thus, a production inventory system may yield fuzzy differential equations or different inventory costs are expressions of fuzzy integral. Seikkla [192], Chalco-Cano and Roman-Flores [19] solved fuzzy differential equation problem. Wu [226] gives an approach of fuzzy Riemann integral and its numerical integration. As a fuzzy constraint represents a fuzzy event, it should be satisfied with some predefined necessity by Dubois and Prade [62], according to company's requirements. As like stochastic environment, for the solution of this type of problems Liu and Iwamura [140] proposed a 'here and now' approach, i.e., the chance constraint programming approach in which a minimum probability level for satisfying each of the constraint is specified. Similarly, possibilistic constraints also may be defined by Zadeh [240], Dubois and Prade [61] and others. Das et. al. [48] proposed the necessity / possibility / credibility technique for solving fuzzy decision making problem for triangular or trapezoidal fuzzy numbers only. In 2002, Liu and Liu [142] calculated the expected value of fuzzy variable for different fuzzy expected value models. Due to imprecise parameters, the objective function (i.e., profit function) becomes fuzzy in nature. Since optimization of a fuzzy objective is not well defined, one can optimize the optimistic/pessimistic returns of the objectives with some degree of possibility/necessity according to requirement as proposed by Liu and Iwamura [140].

Models with Fuzzy Credit Period:
In the present competitive business world a permissible delay in payment which is termed as trade credit period in paying for purchasing cost, is very common business practice. It influences the demand of order and reduces the holding cost. Singh et al. [195] developed a two warehouse inventory model in crisp and fuzzy environments respectively with permissible delay in payment. Recently, Das et al. [51] proposed an integrated production inventory model under interactive fuzzy credit period for deteriorating item with several markets.

1.4.4 Historical Literature Reviews on Imperfect Production Inventory Models in Fuzzy-Stochastic Environment

In this environment, the parameter(s) involved in imperfect production inventory system such as demand, production, production run time, inventory costs etc. are fuzzy-random in nature and specified by imprecise probability distributions.

Introduction of fuzzy random variables and its applications is not much old.

Kwakernaak [126] first introduced fuzzy random variables. After that, several researchers Liu [141] and others developed linear and non-linear programming methods in fuzzy stochastic environment with fuzzy stochastic or fuzzy and stochastic data. Petrovic et al. [165] considered the news boy problem with fuzzy demand and fuzzy inventory costs. They considered two fuzzy models one with (i) imprecisely described discrete demand and other with (ii) imprecisely estimated unit holding and unit shortage costs. Yao et al. [233] described a single period inventory management in fuzzy stochastic environment.

Recently, researchers have focused on the situations in which inventory parameters are random as well as imprecise. Models developed in such situations are known as fuzzy stochastic inventory models. In such mixed environment, very few models have been developed. Panda and Maiti [163] constructed a multi item fuzzy stochastic inventory model in which reliability resources are assumed to be random and available storage space are considered as imprecise in nature. Roy et al. [182] formulated an inventory model for a deteriorating item under fuzzy inflation and time discounting over a random planning horizon, which extend for a EPQ model with random planning horizon by Jana et al. [105]. Das and Maiti [50] developed production inventory model by considering one constraint in fuzzy environment and the other in random environment.

1.5 Motivation and Objective of the Thesis

In any manufacturing system, the production of defective units is a natural phenomenon occurring from the different difficulties such as raw material, labor experience, machine component, production rate etc. which are arise in a long-run production process. So, it is necessary to inspect each item after production to check whether the item is perfect or not before selling. In the literature, there is a large number of articles on the imperfect production inventory models. But very few researchers have developed the imperfect production inventory models by considering the reworking of defective units and re-manufacturing of returned items. Since in a competitive business world, advertisement, price-discount and warranty period etc. are important factors in creating a market demand in the society, it is essential to study to control inventory in the imperfect production inventory system. Moreover many business people use showrooms to attract the customer by display of stock of the units in the showroom to influence the demand. **Therefore, one objective of our research works is to analyze the effect of the reworking of defective items in the imperfect production inventory model with various demand rate**.

Recently, it is seen that, with the increased competitions in the global business, the manufacturing companies are forced to work closely in partnership with their suppliers, retailers and manufacturers. In supply chain management, the establishment of a long-term cooperative relationship between vendor/ manufacturer and buyer as an integrated Supply Chain Model (SCM) is beneficial for two parties with regard to costs/profits to get the tensionless stable sources of supply and demand as well as smooth running of business to gain optimum profit from each other. For that reason, the coordination in the supply chain is very important; otherwise, each stage of the supply chain acts independently without considering the impacts of the remaining stages in the supply chain and lack of coordination results in less profit than what could be achieved through coordination. **So, in our research works, one of the objectives is to co-ordinate the members of supply chain on imperfect production inventory models in different environment.**

Traditional inventory models generally hypothesize that a retailer pays his/her supplier at the time of purchase. But in real world, it is observed that a supplier often offers a retailer a delayed payment period known as the trade credit period to settle the account. Offering such a credit period, the supplier entices the buyer to increase the size of their order and hence reduce on-hand stock level. From the retailer's view-point, during the credit period he/she can sell the items and continue to accumulate the sales revenue and earn interest from it before payment to be made. Hence, the trade credit has an important role for the decision on the rate of supply and indirectly on the order quantity. During the past few years, many articles dealing with various type of inventory models under trade credit have appeared in various research journals. So in an integrated inventory system, the offer of a credit period plays an important role to achieve the optimum profits. **Therefore, another objective of our research works is to analyze the effect of the credit period(s) in the supply chain model.**

Many researchers have worked on imperfect production inventory model with the considerations of various types of demand rate. In the present time, the advertisement has an important role in increasing the demand of a commodity. In the same time, from the market survey it is also observed that for the coming of other new brands of the same product, there is a declination of demand rate. In such circumstances, a manufacturer wants to produce the optimum amount of item per unit time to get the maximum profit from his/her business where he/she has capability to collect sufficient raw materials, labours, machines and other related resources to produce the item. Henceforth, the production rate should be considered to be a variable. In this case, the defective rate of the produced items must be dependent on the production rate. Due to the existence of defective production, the manufacturer decides to sell the items of the perfect quality after sorting the inventory. So during the production period, the screening process has been carried out simultaneously. These concepts have motivated us to develop an imperfect production inventory model with production rate dependent defective rate and advertisement dependent demand which is discussed in **Chapter 3**. Also in this chapter, the depreciation rate in demand function has been considered in which the screening rate is less than or equal to the production rate, but greater than or equal to the demand rate.

From the literature survey, it is seen that traditional inventory models generally hypothesize that the imperfect units are disposed. But in real world, these units are reworked and transformed into a good ones. Moreover, the time horizon of commonly used EPQ model are either infinite or finite. But in the reality, there may occur an uncertainty in the time horizon. Again the demand of the goods is not dependent on exactly one parameter in the production inventory system. Basically there are different parameters on which demand may vary. This sense motivated us to develop a model "An EPQ model with promotional demand in random planning horizon: population varying genetic algorithm approach" in **Chapter 4**, where the demand of the items has three basic parts such as (i) minimum requirement of the goods, (ii) demand enhancement due to lower selling price and (iii) demand increased through the motivation of the customer by advertisement.

Most of research articles (cf. Kim [118], Goyal and Cardenas-Barron [78] and others) on the EPQ models with imperfect production process considered the inspection process for searching the defective items in which the inspection process has been as error free. But in reality, this assumption is not true in business world. Practically, the inspection process is not error free due to different types of factors related to machine and human in the system. In practice, the manufacturer usually offers a warranty for all selling items for a specific warranty period with a view to increasing the selling rate and reliability of the product. Now it is seen that if any purchased product fails to work properly within its warranty period, then the servicing center replaces it with a new item or repairs the product by replacing one or more parts. Therefore, the retailers as well as users are motivated by the displays, advertisements, selling price discount and warranty period to procure the products such as mobiles, computers etc. This phenomena has inspired and motivated us to formulate and analyze the model "A deteriorating manufacturing system considering inspection errors with discount and warranty period dependent demand" by considering warranty period dependent warranty cost which is illustrated in **Chapter 5.**

Again, in a production system, it is seen that initially the production process is in an in-control state, because all factors associated with the system are fresh. But due to continuous running of the system these factors gradually lose their perfectness and thus reliability of the production process decreases. So, after some time of production, the production process may shift from the in-control state to out-control state. If the production rate increases, the occurrence of out-control state comes very first and hence it produces more non conforming items than earlier. So in such situations a development cost is required to control the occurrence of out-control state. Already many researchers (cf. Rosenblatt and Lee [181], Hayek and Salameh [97] and others) have worked on imperfectness of a product. Keeping the above sense in mind, we are motivated to formulate a model in **Chapter 6** where reliability of the production system is continuously maintained by imposing time dependent development cost to reduce the imperfectness of the product during production. Also we assumed here that defective rate depends upon the production rate and the time length of out-control state.

It is observed that a large pile of goods in showroom in a super market will lead the customer to buy more and generate higher demand. But due to limited space facility of showroom, sometimes one or more warehouse(s) is hired on rental basis nearer the showroom. There is a good number of research papers (cf. Bhunia and Maiti [11]) published by several researchers on this ware house facility. In 2008, Dey et al. [57] considered a two ware house problem with lead time and inflation in wholesaler-retailer or manufacturer-retailer problem two storage inventory problem. This influences us to formulate a three layer supply chain model with two warehouse facility. In the recent age, Liu & Sai [139], Chen [33] and others extended the ideas of fuzzy set as fuzzy rough set. All these help us to form a "Three-layer supply chain in an imperfect production inventory model with two storage facilities under fuzzy rough environment" which is discussed in **Chapter 7.**

Normally, the rate of defectiveness is not a constant due to many factors. So it should be taken as uncertain quantity. Also in practical business world, sometimes it is seen that the demand of a retailer changes due to various factors according to his/her business policy. So in nature, it is vague and imprecise. These situation have motivated us to develop a model "A fuzzy imperfect production inventory model based on fuzzy differential and fuzzy integral method" in **Chapter 8** where we consider demand of the product, defective rate and the time at which the production system shifts from in-control to out-control state are as fuzzy.

A classical logistic system gives a forward flow, i.e., the material and related information flow until the final products are delivered to the customer. But reverse logistic manages backward process, i.e., the used and reusable parts are returned from the customers to the producer. Environmental consciousness forces companies to initiate such product recovery systems with their disposal such as metal, glass, paper etc. In this way natural resources can be saved for the future generations. These concepts have motivated us to develop, the joint decision-making problem for the two plant production and reproduction inventory model with reworking of defective units over a finite planning horizon. Also our object maximized the expected total profit and minimized expected GHG emission in the imperfect production inventory model **Chapter 9.**

In business world, there are many parameters or components which are not fixed in reality. This situation of uncertainty has motivated us to form an imperfect production inventory model in fuzzy-random environment. Also the concept of learning from our daily mistakes expertises us to introduce the concept of learning effect on the imperfect production model to reduce the rate of defective units including a extra cost for this expertise, which is shown in the model "Multi-item EPQ model with learning effect on imperfect production over fuzzy-random planning horizon" in **Chapter 10**.

In the supply chain, there exists at least one process (production, repairing etc), through which the environment becomes polluted. Therefore, in the context of global business management, an extra cost should be introduced for the emission of carbon to keep the environment fresh. In spite of this carbon emission in a production, trade credit has an intrinsic connection with the demand in supply chain management. These lacunas of supply chain motivated us to form a model entitled "Two layers supply chain imperfect production inventory model with fuzzy credit period, imperfectness depend on time and production rate" in **Chapter 11.**

1.6 Organization of the Thesis

In the proposed thesis, some real-life Imperfect Production Inventory problems in crisp, stochastic, fuzzy and fuzzy stochastic environments are considered and analyzed. The proposed thesis has been divided into seven parts and twelve chapters as follows:

Part I: General Introduction, Basic Concepts and Solution Methodologies

- **Chapter 1:** General Introduction
- **Chapter 2:** Basic Concept and Solution Methodology

Part II: Studies on Imperfect Production Inventory System in Crisp Environment

- **Chapter 3:** Imperfect production inventory model with production rate dependent defective rate and advertisement dependent demand

Part III: Studies on Imperfect Production Inventory Systems in Stochastic Environment

- **Chapter 4:** An EPQ model with promotional demand in random planning horizon: population varying genetic algorithm approach
- **Chapter 5:** A deteriorating manufacturing system considering inspection errors with discount and warranty period dependent demand
- **Chapter 6:** Two layers supply chain in an imperfect production inventory model with two storage facilities under reliability consideration

Part IV: Studies on Imperfect Production Inventory Systems in Fuzzy Environment

- **Chapter 7:** Three-layer supply chain in an imperfect production inventory model with two storage facilities under fuzzy rough environment
- **Chapter 8:** A fuzzy imperfect production inventory model based on fuzzy differential and fuzzy integral method
- **Chapter 9:** GA approach for controlling GHG emission from industrial waste in two plant production and reproduction inventory model with interval valued fuzzy pollution parameters

Part V: Studies on Imperfect Production Inventory System in Fuzzy Stochastic Environment

- **Chapter 10:** Multi-item EPQ model with learning effect on imperfect production over fuzzy-random planning horizon
- **Chapter 11:** Two layers supply chain imperfect production inventory model with fuzzy credit period, time and production rate dependent imperfectness

Part VI: Summary and Extension of the Thesis

- **Chapter 12:** Summary and Future Research Work

Part VII: Appendices, Bibliography and Index

Part I

(General Introduction, Basic Concepts and Solution Methodologies)

The Part I is divided into two chapters - Chapter 1 and Chapter 2.

Chapter 1: General Introduction

This chapter contains an introduction giving an overview of the preliminary studies along with historical reviews on integrated production inventory control system with reworked of imperfect product in crisp, fuzzy and stochastic environments.

Chapter 2: Basic Concepts and Solution Methodologies

In this chapter, Generalized Reduced Gradient (GRG) technique, Genetic Algorithm (GA), Population Varying Genetic Algorithm (PVGA), Multi-Objective Genetic Algorithm(MOGA), Fuzzy Simulation Based Genetic Algorithm (FSGA), Possibility/ Necessity/ Credibility representation, Solution of Fuzzy Differential Equation (FDE) and Fuzzy Programming Technique(FPT) have been studied which are used to solve and develop the models described in the thesis.

Part II

(Studies on Imperfect Production Inventory System in Crisp Environment)

The Part II contains Chapter 3, in which an imperfect production inventory model is derived, solved and discussed in crisp environment.

Chapter 3: Imperfect production inventory model with production rate dependent defective rate and advertisement dependent demand

In this chapter, an economic production quantity (EPQ) model with imperfect production system and advertisement dependent demand has been presented in crisp environment. The advertisement rate has been assumed to be a function of time which has been increased with respect to time at a decreasing rate i.e., it grows exponentially with respect to time but rate of growth gradually decreases. Here, the rate of producing defective has been followed to be a function of production rate. Also, the produced units have been inspected in order to screen the defective but the screening rate is less than or equal to the production rate and greater than the demand rate. For the developed EPQ model, the total profit has been maximized to obtain the optimum production rate and production run time in the system. Here, an algorithm has been developed for finding the optimal profit of the imperfect production inventory model. Finally, different numerical examples have been considered to illustrate the feasibility of the model taking different special cases in the system and then some sensitivity analysis have been carried out to get the impact of some parameters on the objective function of the model.

Part III

(Studies on Imperfect Production Inventory System in Stochastic Environment)

The Part III contains Chapter 4, 5, & 6, in which different imperfect production inventory models have been derived in stochastic environment and solved then.

Chapter 4: An EPQ model with promotional demand in random planning horizon: population varying genetic algorithm approach

One of the Economic Production Quantity (EPQ) problems that have been of interest to researchers is the production with reworking of the imperfect items including waste most disposal form and vending the units. In this chapter, an imperfect production inventory model is developed over a finite random planning horizon (which is assumed to follow the exponential distribution with known parameters) with the assumption that the decay rate of the items is satisfied from three different points of view: (i) minimum demands of the customer's requirement, (ii) demands to be enhanced for lower selling price and (iii) demands of the customers who are motivated by the advertisement. The model has been illustrated with a numerical example, whose parametric inputs are estimated from market survey. Here the model is optimized by using a population varying genetic algorithm.

Chapter 5: A deteriorating manufacturing system considering inspection errors with discount and warranty period dependent demand

This chapter deals with selling price-discount and warranty period dependent demand in an imperfect production inventory model under the consideration of inspection errors and time dependent development cost. Normally, due to long-run, a production process deteriorates with time and here we assume that the process shifts from in-control to out-of-control state at any random time. A time dependent development cost has been constructed to increase the reliability of the production system i.e., to decrease the deterioration of the system during the production run. As a result, the less amount of items are rejected. Here, two types of inspection errors such as Type-I error and Type-II error, have been considered during the period of product inspection process. In Type-I error, an inspector may choose falsely a defective item as non-defective and in Type-II error an inspector may choose falsely a non-defective item as defective. Due to these phenomena, the inspection process would consist of the following costs: cost of inspection, cost of inspection errors. The purpose of this chapter is to investigate the effects of time dependent development cost on the defective items, selling price-discount and warranty policy on the market demand and finally optimize the expected average profit under consideration of such inspection costs in infinite time horizon. Some numerical examples along with graphical illustrations and sensitivity analysis are provided to test the feasibility of the model.

Chapter 6: Two layers supply chain in an imperfect production inventory model with two storage facilities under reliability consideration

This chapter focuses on an imperfect production inventory model considering production system reliability as well as development cost to improve the system reliability and reworking of imperfect items in the environment of two layer supply chain management. Here we consider, the production system may be shifted from "in-control" state to an "out-of-control" after a time which is a random variable and distributed exponentially with mean $\frac{1}{\lambda}$, where λ is the system reliability depending on the production rate. A development cost is incurred to improve the reliability of the production system. The rate of defectiveness of the imperfect quality items (which produce in the "out-of-control" state) is also assumed as random and depends upon the production rate and time length of the "out-of-control" state. A portion of the imperfect quality items is transformed into perfect quality items after some necessary rework. Another portion of imperfect quality items, termed as 'less perfect quality items', is sold at a reduced price to the retailer and the portion which cannot be either transformed to the perfect quality items or sold at a reduce price, is being rejected. For such rejection of some items, a disposal cost per unit of rejected items is incurred to minimize the environmental pollution. Here, a retailer purchases both perfect and imperfect quality items from manufacturer to sale the items to the customers through his/her respective showrooms of finite capacities. A secondary warehouse of infinite capacity is hired by the retailer on rental basis to store the excess quantity of perfect quality items. Finally, average profit of the integrated model has been maximized by optimizing the production rate as well as defective rate of the production system and some numerical examples have been given to illustrate the feasibility of the model.

Part IV

(Studies on Imperfect Production Inventory System in Fuzzy Environment)

The Part IV contains Chapter 7, 8, & 9, in which different imperfect production inventory models have been developed in fuzzy environment and then optimized.

Chapter 7: Three-layer supply chain in an imperfect production inventory model with two storage facilities under fuzzy rough environment

This chapter focuses on an imperfect production inventory model considering product reliability and reworking of imperfect items in three layers supply chain under fuzzy rough environment. In the model, the supplier receives the raw materials, all are not of perfect quality, in a lot and delivers the items of superior quality to the manufacturer and the inferior quality items are sold at a reduced price in a single batch by the end of the cent percent screening process. Manufacturer produces a mixture of perfect and imperfect quality items. A portion of the imperfect items is transformed into perfect quality items after rework. Another portion of imperfect items, termed as 'less perfect quality items', is sold at a reduced price to the retailer and the portion which can not be either transformed to the perfect quality items or sold at a reduce price, is being rejected. Here retailer purchases both the perfect and imperfect quality items from the manufacturer to sale the items to the customers through his/her respective showrooms of finite capacities. A secondary warehouse of infinite capacity is hired by the retailer on rental basis to store the excess quantity of perfect quality items. This model considers the impact of business strategies such as optimal order size of raw materials, production rate and unit production cost in different sectors in a collaborating marketing system that can be used in the industry like textile, footwear, electronics goods etc. An analytical method has been used to optimize the production rate and raw material order size for maximization of the average profit of the integrated model. Finally, a numerical example is given to illustrate the model.

Chapter 8: A fuzzy imperfect production inventory model based on fuzzy differential and fuzzy integral method

This chapter considers a fuzzy economic production quantity (FEPQ) model with interactive fuzzy demands. At the beginning of a production process, the system is assumed to be in a controlled state i.e., under this stage only perfect items are produced. But after some time to be considered as fuzzy here, the manufacturing production process shift to an 'out-control' state i.e., during this stage the system produces both of perfect and imperfect items simultaneously. Here the defective rate of production system has been considered also as fuzzy. Here the screening process of produced items has been considered during production period. Finally some numerical examples have been illustrated to study the practical feasibility of the production inventory model along with sensitivity analysis of some parameters.

Chapter 9: GA approach for controlling GHG emission from industrial waste in two plant production and reproduction inventory model with interval valued fuzzy pollution parameters

This chapter investigates an imperfect production – reproduction inventory model for two types of quality items (item-I and item-II) produce in two different plants (plant-I and plant-II) in the same premises under single management system over a known-finite time horizon with consideration of environment pollution control through industrial waste management. Both the production plant-I and plant-II produces mixture of perfect and defective units. Some of the defective units are rework and non-reworkable defective units are continuously transferred to the raw material processing unit. Used units are collected from the customers and deposited in the raw material processing unit as raw materials for plant II. Treatment of industrial waste from both the plants and raw material processing unit is considered to protect the environment from water pollution and Green House Gas (GHG) emission as industrial waste is become a serious environmental issue. Two conflicting objectives are integrally considered of which one is maximization of the total profit out of two plants and other is minimization of Green House Gas (GHG) emission from industrial waste over the finite time horizon .

Part V

(Studies on Imperfect Production Inventory System in Fuzzy-Stochastic Environment)

The Part II contains Chapter 10 & 11, in which a imperfect production inventory model have been developed, solved and discussed in fuzzy-stochastic environment.

Chapter 10: Multi-item EPQ model with learning effect on imperfect production over fuzzy-random planning horizon

Uncertainty is certain in the world of uncertainty. This study revisits an economic production quantity (EPQ) model with shortages for stock-dependent demand of the items with reworking and disposing of the imperfect ones over a random planning horizon under the joint effect of inflation and time value of money, where the expected time length is imprecise in nature. Transmission of learning effect has been incorporated to reduce the defective production. The total expected profit over the random planning horizon is maximized subject to the imprecise space constraint. The possibility, necessity and credibility measures have been introduced to defuzzify the model. The simulation-based genetic algorithm is used to make decision for the above EPQ model in different measures of uncertainty. The model is illustrated through an example. Sensitivity analysis shows the impacts of different parameters on the objective function in the model.

Chapter 11: Two layers supply chain imperfect production inventory model with fuzzy credit period, time and production rate dependent imperfectness

This chapter focused on an integrated production inventory model with rework of the imperfect units and stock dependent demands of the customer from several retailers. There is an opportunity to build model to measure the amount of carbon emissions during the time of production and the corresponding rate of carbon emission parameters are random which follows Beta distribution. Here, the rate of imperfectness is assumed to be a function of time and production rate. One portion of produces imperfect units is transformed into perfect quality items after some necessary rework. In this chapter, a manufacturer-retailer-customer chain system is developed in which the retailer gets an upstream trade credit period ($\widetilde{M}$) from the manufacturer and retailers offers a down stream trade credit period (N_i) to customers to stimulate demand as well as sales and reduce inventory. We employ the sequential optimization structure of the extensive problem under different scenarios of trade-credit periods. The model has been developed as a profit maximization problem with respect to the manufacturer and retailers. The production time and expected profit has been optimized using develop algorithm and non-linear optimization technique Generalized Reduced Gradient method (LINGO). Finally, several numerical examples and sensitivity analysis are provided to illustrate the utilization of our model.

Part VI

(Summary and Extension of the Thesis)

Chapter 12: Summary and future research work

At the end, a summary of the thesis, its limitation and the scope of future research work have been given.

Part VII

(Appendices, Bibliography and Index)

This part of the thesis contains Appendices, Bibliography and Index.

Chapter 2

Basic Concepts and Solution Methodologies

2.1 Basic Concepts of Crisp Set Theory

Crisp Set: By crisp one means dichotomous, that is, yes or no type rather than more-or-less type. In conventional dual logic, for instance, a statement can be either true or false- and nothing in between. In set theory, an element can either belongs to a set or not and in optimization, a solution is either feasible or not. A classical set, X, is defined by crisp boundaries, i.e., there is no uncertainty in the prescription of the elements of the set. Normally, it is defined as a well defined collection of distinct elements or objects, $x \in X$, where X may be countable or uncountable.

Convex Set: A subset $S \subset \Re^n$ is said to be convex, if for any two points x_1, x_2 in S, the line segment joining the points x_1 and x_2 is also contained in S. In other words, a subset $S \subset \Re^n$ is convex, if and only if $x_1, x_2 \in S \Rightarrow \lambda x_1 + (1-\lambda)x_2 \in S$ for $\quad 0 \leq \lambda \leq 1$.

Convex Combination: Given a set of vectors $\{x_1, x_2, \cdots, x_n\}$, a linear combination $x = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$ is called a convex combination of the given vectors, if $\lambda_1, \lambda_2, \cdots, \lambda_n \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$.

Convex Function: A function $f : S \to \Re$ is said to be convex, if for any $x_1, x_2 \in S$ and $0 \leq \lambda \leq 1$, it implies that $f\{(1-\lambda)x_1 + \lambda x_2\} \leq (1-\lambda)f(x_1) + \lambda f(x_2)$.

Concave Function: A function $f : S \to \Re$ is said to be convex, if for any $x_1, x_2 \in S$ and $0 \leq \lambda \leq 1$, it implies that $f\{(1-\lambda)x_1 + \lambda x_2\} \geq (1-\lambda)f(x_1) + \lambda f(x_2)$.

2.2 Basic Concepts of Probability Distribution

Random Variable: Let S be a sample space of of some given random experiment E. It has been observed that the outcomes (i.e., sample points of S) are not always numbers. We may however assign a real numbers to each sample point according to some definite rule. Such an assignment gives us a "function defined on the sample space S". This function is called a random variable (or stochastic variable).

Probability Space: Mathematically, an order tuple (S, Ω, P) is said to be Probability Space if

- S is a non-empty set of outcomes of a random experiment E.
- Ω is a set of all events i.e., subsets of S, which is a σ-field, i.e., it satisfies the following properties:

  (i) $\emptyset \in \Omega$

  (ii) $A \in \Omega \Rightarrow A^c \in \Omega$, where A^c is the complement of A in Ω.

  (iii) $A_1, A_2, ... \in \Omega \Rightarrow A = \bigcup_{i=1}^{\infty}(A_i \in \Omega)$.
- P is a probability function for the events, i.e., $P : \Omega \to [0, 1]$.

Discrete Probability Distribution: A random variable which assumes a finite number or countably infinite number of values is called a discrete random variable.
Let X be a discrete random variable which can assume the values $x_1, x_2, x_3, ...$ (arranged in an increasing order of magnitude) with probabilities $p_1, p_2, p_3, ...$ respectively. The specification of the set of values x_i together with their probabilities p_i (i=1, 2, 3, ...) defines the discrete probability distribution of X.

- Properties: (i) $P(x_i) = p_i, 0 \leq p_i \leq 1, \forall x_i \in S(i = 1, 2, 3, ...)$,

  (ii) $\sum_{i=1}^{\infty} p_i = 1$.

Continuous Probability Distribution: A random variable which assumes an uncountable infinite number of values, it is called a continuous random variable.
If X is a continuous random variable, the number of possible values which can assume is uncountable infinite and hence the probability function cannot be defined in the same manner as for a discrete random variable. In this case, we define a function $f(x) = P(-\infty < X \leq x)$, which satisfies two conditions

$$(i) f(x) \geq 0 \qquad \text{and} \qquad (ii) \int_{-\infty}^{\infty} f(x)dx = 1$$

The function $f(x)$ is called probability density function (p.d.f.) or simply density function of the continuous random variable X.
In any physical problem, one chooses a particular type of probability distribution depending on (i) the nature of the problem, (ii) the underlying assumptions associated with the distribution of the parameters, (iii) the shape of the graph between the probability density function $f(x)$ (or distribution function $F(x)$) and x obtained after plotting the available data

and (iv) the convenience and simplicity afforded by the distribution. Some continuous probability distributions are presented here. In this thesis, Uniform distribution, Exponential distribution and Beta distribution have been used for stochastic and fuzzy stochastic models.

2.2.1 Some Probability Distributions

Uniform Distribution or Rectangular Distribution: A continuous random variable X, is said to have a uniform distribution, if its probability density function $f(x)$ (cf. Figure 2.1) is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & \text{elsewhere} \end{cases}$$

where a and b are two parameters of the distribution.

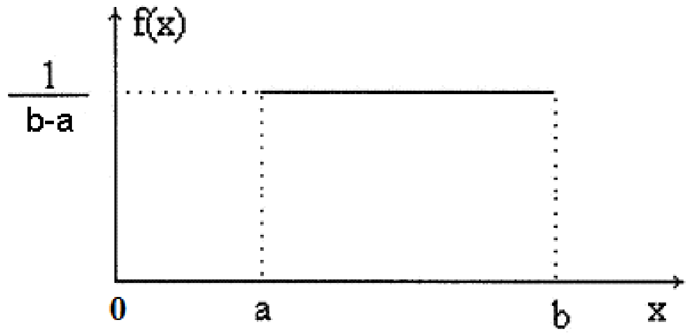

Figure 2.1: Probability density function of uniform distribution

Exponential Distribution: A continuous random variable X, is said to have an exponential distribution, if its probability density function $f(x)$ (cf. Figure 2.2) is of the form:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & 0 \leq x < \infty \\ 0, & \text{elsewhere} \end{cases}$$

Here $\lambda(> 0)$ is the parameter of the distribution.

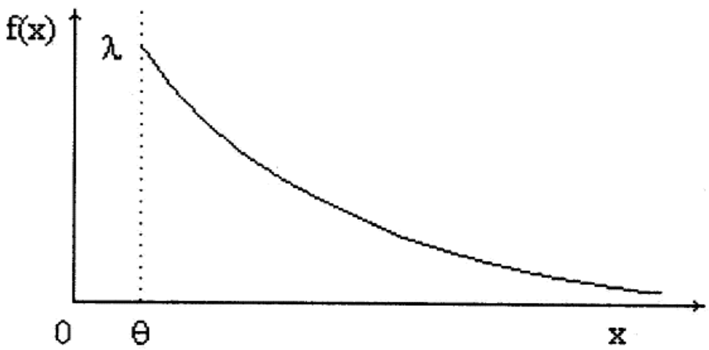

Figure 2.2: Probability density function of exponential distribution

Beta Distribution: A continuous random variable X, is said to have a beta distribution, if its probability density function $f(x)$ (cf. Figure 2.3) is given by

$$f(x) = \begin{cases} \frac{x^{l-1}(1-x)^{m-1}}{\beta(l,m)}, & 0 \leq x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

where,

$$\beta(l,m) = \frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m)} = \int_0^1 x^{l-1}(1-x)^{m-1}dx$$

l and m being two positive parameters of the distribution.

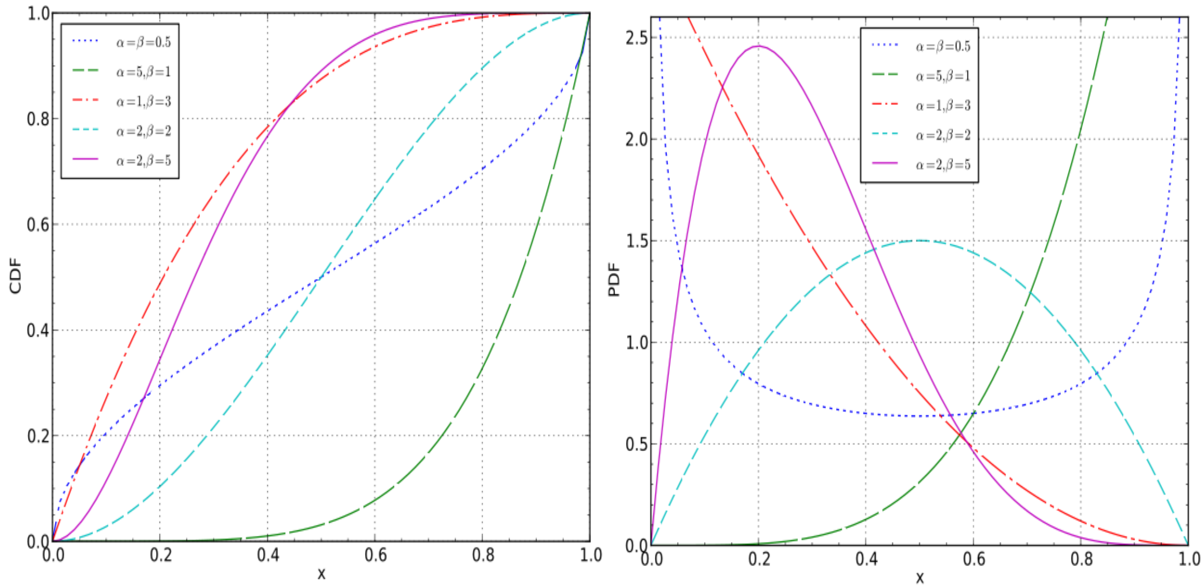

Figure 2.3: Cumulative and Probability density function of beta distribution of first kind

2.2.2 Some Statistical Terms and Tests

Statistical Hypothesis: Any statement or assertion about a statistical population or the values of its parameters is called a statistical hypothesis. There are two types of hypothesis - simple and composite.

Test of Hypothesis or Test of Significance: A test of hypothesis is a procedure which specifies a set of "rules for decision" whether to 'accept' or 'reject' the hypothesis under consideration (i.e., null hypothesis).

Null Hypothesis: A statistical hypothesis which is set up (i.e., assumed) and whose validity is tested for possible rejection on the basis of sample observations is called a null hypothesis. It is denoted by H_0 and tested against alternatives. Test of hypothesis deals with rejection or acceptance of null hypothesis only.

Alternative Hypothesis: A statistical hypothesis which differs from the null hypothesis is called an alternative hypothesis and is denoted by H_1. The alternative hypothesis is not tested, but its acceptance (rejection) depends on the rejection (acceptance) of the null hypothesis. The choice of an appropriate critical region depends on the type of alternative hypothesis, viz. whether both-sided, one-sided (right/left) or specified alternative.

Test Statistic: A function of sample observations (i.e., statistic) whose computed value determines the final decision regarding acceptance or rejection of H_0, is called a test statistic. The appropriate test statistic has to be chosen very carefully and a knowledge of its sampling distribution under H_0 (i.e., when the null hypothesis is true) is essential in framing the decision rules. If the value of the test statistic falls in the critical region, the null hypothesis is rejected.

Critical Region: The set of values of the test statistic which lead to rejection of the null hypothesis is called critical region of the test. The probability with which a true null hypothesis is rejected by the test is often referred to as "size" of the critical region. Geometrically, a sample $x_1, x_2, ..., x_n$ of size n is looked upon as just a point x, called sample point, within the region of all possible samples, called the sample space. The critical region is then defined as a subset of those sample points which lead to rejection of the null hypothesis.

Level of Significance: The maximum probability with which a true null hypothesis is rejected is known as level of significance of the test, and is denoted by α. In framing decision rules, the level of significance is arbitrarily chosen in advance depending on the consequence of statistical decision. Customarily, 5% or 1% level of significance is taken, although other levels such as 2% or $\frac{1}{2}\%$ is also used. The level of significance α is used to indicate the upper limit of the size of critical region.

Fisher's 't' Test for Comparison of Two Means (s.d.'s unknown): Consider two independent random samples of sizes n_1 and n_2 from two normal populations with means μ_1 and μ_2 respectively. It is required to test the hypothesis that the means are equal. The null hypothesis is

$$H_0 : \mu_1 = \mu_2$$

If the standard deviations of two population are assumed to be equal, then an "unbiased" estimator of the common variance is given by

$$s^2 = \frac{n_1 {S_1}^2 + n_2 {S_1}^2}{n_1 + n_2 - 2}$$

where S_1 and S_2 are the sample standard deviations.
If this is substituted for ${\sigma_1}^2$ and ${\sigma_2}^2$ in formula (2.1), the statistic is

$$t = \frac{\overline{x_1} - \overline{x_2}}{s.\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

follows **Fisher's t** distribution with degrees of freedom $(n_1 + n_2 - 2)$.
Confidence limits for $(\mu_1 - \mu_2)$ are
95% confidence limits: $(\overline{x_1} - \overline{x_2}) \pm t_{.025} \times s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$
99% confidence limits: $(\overline{x_1} - \overline{x_2}) \pm t_{.005} \times s\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$

ANOVA Test for Comparison of Means: Like so many of our inference procedures, ANOVA has some assumptions which should be in place in order to make the results of calculations completely trustworthy. They include:

(i) Subjects are chosen via a simple random sample.

(ii) Within each group/population, the response variable is normally distributed.

(iii) While the population means may be different from one group to the next, the population standard deviation is the same for all groups.

The statistical test ANOVA is used to test under the following hypothesis:

- H_0: The (population) means of all groups under consideration are equal.
- H_1: The (population) means are not all equal.

Following notations are needed to be described for one way ANOVA

k = the number of groups/populations/values of the explanatory levels of treatment.
n_i = the sample size taken from group i.
n = the (total) sample, irrespective of groups = $\sum_{i=1}^{k} n_i$.
x_{ij} = the jth response sampled from the ith group/population.
$\overline{x_i}$ = the sample mean of responses from the ith group = $\frac{1}{n_i}\sum_{j=1}^{n_i} x_{ij}$.
s_i = the sample standard deviation from the ith group = $\frac{1}{n_i - 1}\sum_{j=1}^{n_i}(x_{ij} - \overline{x})^2$.
$\overline{x}$ = the mean of all responses, irrespective of groups = $\frac{1}{n}\sum_{ij} x_{ij}$.

Splitting the Total Variability into Parts
Viewed as one sample (rather than k samples from the individual groups/populations), one might measure the total amount of variability among observations by summing the squares of the differences between each x_{ij} and $\overline{x}$:

$$\text{SST (stands for sum of squares total)} = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij} - \overline{x})^2.$$

This variability can be explicit into two sources:
(i) Variability between group means (specifically, variation around the overall mean $\bar{x}$)

$$\text{SSG} = \sum_{i=1}^{k} n_i(\overline{x_i} - \overline{x})^2$$

(ii) Variability within groups means (specifically, variation of observations about their group mean $\bar{x}_i$)

$$\text{SSE} = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(x_{ij} - \overline{x_i})^2 = \sum_{i=1}^{k}(n_i - 1){s_i}^2$$

Therefore it is clear that,

$$\text{SST} = \text{SSG} + \text{SSE}$$

Evaluation of F statistics
It is a measure of the variability between groups divided by a measure of the variability within groups.
The detail computation of F statistic is shown in the following ANOVA Table 2.1

Table 2.1: ANOVA Table

Source	SS	df	MS	F
Model/Group	SSG	k-1	MSG=$\frac{SSG}{k-1}$	$F = \frac{MSG}{MSE}$
Residual/Error	SSE	n-k	MSE=$\frac{SSE}{n-k}$	
Total	SST	n-1		

Conclusion
If the computed value of F statistic is greater than the theoretical F value with $(k-1, n-k)$ degree of freedom for an assigned level of significance α i.e., calculated F value falls in critical region for that level of significance, then the null hypothesis (H_0) is rejected with level of significance α. Otherwise it is accepted with level of significance α.

2.3 Basic Concept of Fuzzy Sets

The fuzzy set theory was developed to define and solve the complex system with sources of uncertainty or imprecision which are non-statistical in nature. Fuzzy set theory is a theory of graded concept (a matter of degree) but different from a theory of chance or probability. The term "FUZZY"was proposed by Prof. L, A. Zadeh in 1962 (cf. Zadeh [239]). A short delineation of the fuzzy set theory is given below. For developing the mathematical formulation of the model, some definitions related to fuzzy set are presented here.

Fuzzy Set: A fuzzy set is a class of objects in which there is no sharp boundary between those objects that belong to the class and those that do not. If X is a collection of objects denoted by x, then a fuzzy set $\widetilde{S}$ in X is a set of ordered pairs:

$$\widetilde{S} = \Big\{(x, \mu_{\widetilde{S}}(x))|x \in X\Big\}$$

where $\mu_{\widetilde{S}}(x)$ is called the membership function of x in $\widetilde{S}$ which maps X to the membership space M which is considered as the closed interval [0,1].

Note: When M consists of only two points 0 and 1, $\widetilde{A}$ becomes a non-fuzzy set (or Crisp set) and $\mu_{\widetilde{A}}(x)$ reduces to the characteristic function of the non-fuzzy set (or crisp set). The range of the membership function is a subset of the non-negative real numbers whose supremum is finite.

Example 2.1. $\widetilde{S}$= *"real numbers considerably larger than* 6*"*

$$\widetilde{S} = \Big\{(x, \mu_{\widetilde{S}}(x))|x \in X)\Big\}$$

where

$$\mu_{\widetilde{S}}(x) = \begin{cases} 0 & ;\textit{when}\, x \le 6 \\ (1 + (x-6)^{-2})^{-1} & ;\textit{when}\, x > 6 \end{cases}$$

Fuzzy Function: Let X and Y be the universes and $\widetilde{P}(Y)$ be the set of all fuzzy sets in Y (power set), $\widetilde{\phi} : X \to \widetilde{P}(Y)$ is a mapping. Then $\widetilde{\phi}$ is a fuzzy function iff

$$\mu_{\widetilde{\phi}(x)}(y) = \mu_{\widetilde{R}}(x, y), \forall (x, y) \in X' \times Y$$

where $\mu_{\widetilde{R}}(x, y)$ is the membership function of the fuzzy relation.

Example 2.2. *Let X be the set of all workers of a plant, $\widetilde{\phi}$ the daily output and y the number of processed work pieces. A fuzzy function could then be $\widetilde{\varphi}(x) = y$.*

Fuzzy Number: A fuzzy number is a special class of a fuzzy sets. A fuzzy number $\widetilde{M}$ is a convex normalized fuzzy set $\widetilde{M}$ of the real line $\Re$ such that

(i) It exists one $\xi_0 \in R$ with $\mu_{\widetilde{M}}(\xi_0) = 1$ (ξ_0 is called the mean value of $\widetilde{M}$).

(ii) $\mu_{\widetilde{M}}(\xi)$ is piece wise continuous.

Example 2.3. *The fuzzy set, 'approximately 9'* = $\{(6, .6), (7, .8), (8, .9), (9, 1), (11, .\ 1)\}$ *is a fuzzy number. But* $\{(7, .\ 8), (8, 1), (9, .5), (10, 1), (11, .7)\}$ *is not a fuzzy number because* $\mu(8) = 1$, $\mu(8) = .5$ *and* $\mu(10) = 1$.

2.3.1 Different Types of Fuzzy Numbers

Triangular Fuzzy Number (TFN): A TFN $\widetilde{M} = (M_0 - \Delta_1, M_0, M_0 + \Delta_2)$ (cf. Figure 2.4), where $0 < \Delta_1 < M_0$ and $0 < \Delta_2$; Δ_1, Δ_2 are determined by the decision makers. Now, the membership function of $\widetilde{M}$ is $\mu_{\widetilde{M}}(x)$ defined as follows:

$$\mu_{\widetilde{M}}(x) = \begin{cases} \frac{x - M_0 + \Delta_1}{\Delta_1} & \text{for } M_0 - \Delta_1 \leq x \leq M_0 \\ \frac{M_0 + \Delta_2 - x}{\Delta_2} & \text{for } M_0 \leq x \leq M_0 + \Delta_2 \\ 0 & \text{otherwise} \end{cases}$$

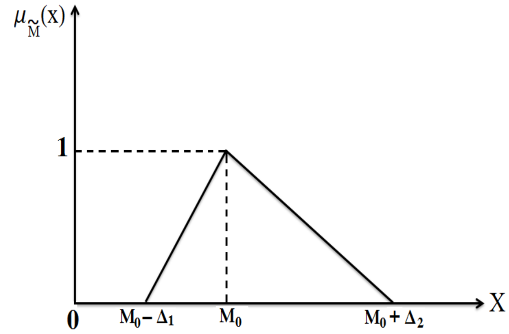

Figure 2.4: Membership function of Triangular fuzzy number (TFN)

• If M_α^L and M_α^R are left and right α-cut of $\widetilde{M} = (M_0 - \Delta_1, M_0, M_0 + \Delta_2)$ respectively then $M_\alpha^L = (M_0 - \Delta_1) + \alpha\Delta_1$ and $M_\alpha^R = (M_0 + \Delta_2) - \alpha\Delta_2$.

Trapezoidal Fuzzy Number (TrFN): A TrFN (cf. Figure 2.5) $\widetilde{M} = (M_0 - \Delta_1, M_0 - \Delta_2, M_0 + \Delta_3, M_0 + \Delta_4)$, where $0 < \Delta_2 < \Delta_1 < M_0$ and $0 < \Delta_3 < \Delta_4$; Δ_1, Δ_2,Δ_3, Δ_4 are determined by the decision makers. Now, the membership function of $\widetilde{M}$ is $\mu_{\widetilde{M}}(x)$ defined as follows:

$$\mu_{\widetilde{A}}(x) = \begin{cases} \frac{x - M_0 + \Delta_1}{\Delta_1 - \Delta_2} & \text{for } M_0 - \Delta_1 \leq x \leq M_0 - \Delta_2 \\ 1 & \text{for } M_0 - \Delta_2 \leq x \leq M_0 + \Delta_3 \\ \frac{M_0 + \Delta_4 - x}{\Delta_4 - \Delta_3} & \text{for } M_0 + \Delta_3 \leq x \leq M_0 + \Delta_4 \\ 0 & \text{otherwise} \end{cases}$$

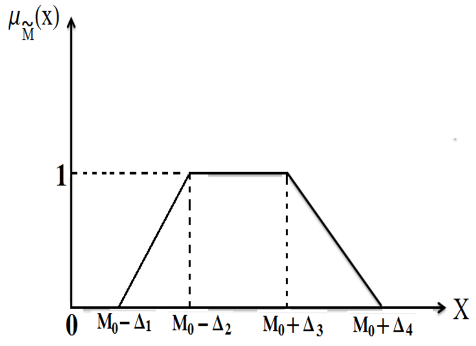

Figure 2.5: Membership function of Trapezoidal fuzzy number (TrFN)

• If M_α^L and M_α^R are left and right α-cut of $\widetilde{M} = (M_0 - \Delta_1, M_0 - \Delta_2, M_0 + \Delta_3, M_0 + \Delta_4)$ respectively then $M_\alpha^L = (M_0 - \Delta_1) + \alpha(\Delta_1 - \Delta_2)$ and $M_\alpha^R = (M_0 + \Delta_4) - \alpha(\Delta_4 - \Delta_3)$.

Parabolic Fuzzy Number (PFN): A PFN $\widetilde{M} = (M_0 - \Delta_1, M_0, M_0 + \Delta_2)$ (cf. Figure 2.6), where $0 < \Delta_1 < M_0$ and $0 < \Delta_2$; Δ_1, Δ_2 are determined by the decision makers. Now, the membership function of $\widetilde{M}$ is $\mu_{\widetilde{M}}(x)$ defined as follows:

$$\mu_{\widetilde{A}}(x) = \begin{cases} 1 - (\frac{M_0 - x}{\Delta_1})^2 & \text{for } M_0 - \Delta_1 \leq x \leq M_0 \\ 1 - (\frac{x - M_0}{\Delta_2})^2 & \text{for } M_0 \leq x \leq M_0 + \Delta_2 \\ 0 & \text{otherwise} \end{cases}$$

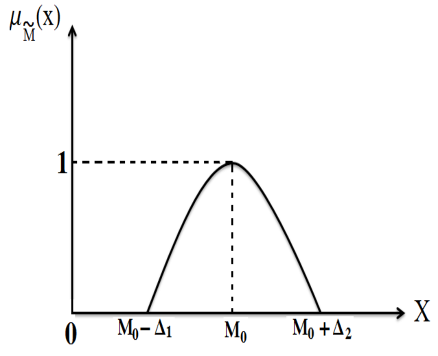

Figure 2.6: Membership function of Parabolic fuzzy number (PFN)

• If M_α^L and M_α^R are left and right α-cut of $\widetilde{M} = (M_0 - \Delta_1, M_0, M_0 + \Delta_2)$ respectively then $M_\alpha^L = M_0 - \sqrt{\alpha}\Delta_1$ and $M_\alpha^R = M_0 + \sqrt{\alpha}\Delta_2$.

General Fuzzy Number (GFN): A GFN $\widetilde{M} = (M_0 - \Delta_1, M_0 - \Delta_2, M_0 + \Delta_3, M_0 + \Delta_4)$ (cf. Figure 2.7), where $0 < \Delta_2 < \Delta_1 < M_0$ and $0 < \Delta_3 < \Delta_4$; Δ_1, Δ_2,Δ_3, Δ_4 are determined by the decision makers. Now, the membership function of $\widetilde{M}$ is $\mu_{\widetilde{M}}(x)$ defined as follows:

$$\mu_{\widetilde{A}}(x) = \begin{cases} (\frac{x-M_0+\Delta_1}{\Delta_1-\Delta_2})^2 & \text{for } M_0-\Delta_1 \le x \le M_0-\Delta_2 \\ 1 & \text{for } M_0-\Delta_2 \le x \le M_0+\Delta_3 \\ (\frac{M_0+\Delta_4-x}{\Delta_4-\Delta_3})^2 & \text{for } M_0+\Delta_3 \le x \le M_0+\Delta_4 \\ 0 & \text{otherwise} \end{cases}$$

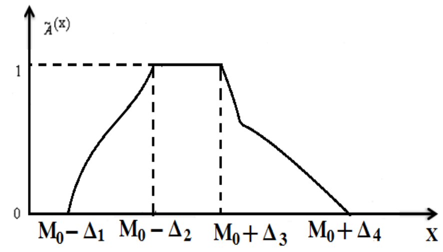

Figure 2.7: Membership function of a General fuzzy number (GFN)

• If M_α^L and M_α^R are left and right α-cut of $\widetilde{M} = (M_0-\Delta_1, M_0-\Delta_2, M_0+\Delta_3, M_0+\Delta_4)$ respectively then $M_\alpha^L = (M_0-\Delta_1)+\sqrt{\alpha}(\Delta_1-\Delta_2)$ and $M_\alpha^R = (M_0+\Delta_4)-\sqrt{\alpha}(\Delta_4-\Delta_3)$.

Proposition 2.1. *Let $\tilde{a}$ be a fuzzy number. Then*

(i) $\tilde{a}$ is called non negative fuzzy number if $\xi_{\tilde{a}}(x) = 0$, $\forall x < 0$.

(ii) $\tilde{a}$ is called non-positive fuzzy number if $\xi_{\tilde{a}}(x) = 0$, $\forall x > 0$.

(iii) $\tilde{a}$ is called positive fuzzy number if $\xi_{\tilde{a}}(x) = 0$, $\forall x \le 0$.

(iv) $\tilde{a}$ is called negative fuzzy number if $\xi_{\tilde{a}}(x) = 0$, $\forall x \ge 0$.

Proposition 2.2. *(Mizumoto and Tanaka [153]). Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. Then*

(i) $\tilde{a} \oplus \tilde{b}$, $\tilde{a} \ominus \tilde{b}$ and $\tilde{a} \otimes \tilde{b}$ are also fuzzy numbers.

(ii) $\tilde{b}$ is a positive or negative fuzzy number then $\tilde{a} \oslash \tilde{b}$ is also fuzzy numbers.

Proposition 2.3. *Let $\tilde{a}$ and $\tilde{b}$ be two closed fuzzy numbers. Then*

(i) $\tilde{a} \oplus \tilde{b}$, $\tilde{a} \ominus \tilde{b}$ and $\tilde{a} \otimes \tilde{b}$ are also closed fuzzy numbers.

(ii) $\tilde{b}$ is a positive or negative fuzzy number then $\tilde{a} \oslash \tilde{b}$ is also closed fuzzy numbers.

α**-cut set:** α-cut of a fuzzy number $\widetilde{S}$ in $\Re$ is denoted by $\widetilde{S}[\alpha]$ and is defined as,

$$\widetilde{S}[\alpha] = \left\{ x \in \Re / \mu_{\widetilde{S}}(x) \ge \alpha \right\}$$

Let $F(X)$ be the space of all compact and convex fuzzy sets on X. If $f : \Re^n \rightarrow \Re^n$ is a continuous function, then $f : F(\Re^n) \rightarrow F(\Re^n)$ is well defined function and its α-cut $\widetilde{f}(u)[\alpha]$ is given by Roman-Flores et al. [180].

$$\widetilde{f}(u)[\alpha] = \widetilde{f}([u]^\alpha), \forall \alpha \in [0,1], \forall u \in F(\Re^n) \tag{2.1}$$

where $f(A) = \{f(a)/a \in A\}$.

Proposition 2.4. *(i) If $\tilde{a}$ and $\tilde{b}$ be two closed fuzzy numbers. Then*
$(\tilde{a} \oplus \tilde{b})_\alpha = [a^L_\alpha + b^L_\alpha, a^R_\alpha + b^R_\alpha]$ *and* $(\tilde{a} \ominus \tilde{b})_\alpha = [a^L_\alpha - b^R_\alpha, a^R_\alpha - b^L_\alpha]$
(ii) If $\tilde{a}$ and $\tilde{b}$ be two closed fuzzy numbers then
$(\tilde{a} \otimes \tilde{b})_\alpha = [min\{a^L_\alpha b^L_\alpha, a^L_\alpha b^R_\alpha, a^R_\alpha b^L_\alpha, a^R_\alpha b^R_\alpha\}, max\{a^L_\alpha b^L_\alpha, a^L_\alpha b^R_\alpha, a^R_\alpha b^L_\alpha, a^R_\alpha b^R_\alpha\}]$.
(iii) If $\tilde{a}$ and $\tilde{b}$ be two non-negative closed fuzzy numbers then $(\tilde{a} \otimes \tilde{b})_\alpha = [a^L_\alpha b^L_\alpha, a^R_\alpha b^R_\alpha\}]$.

Note: $\tilde{a}_\alpha \oslash_{int} \tilde{b}_\alpha$ is well-defined when $\tilde{b}_\alpha$ does not contain zero.

2.3.2 Interval-valued Fuzzy Numbers

In order to consider the fuzzy fault tree analysis based on level (λ, ρ) interval-valued fuzzy numbers, we provide following definitions:

λ-Triangular Fuzzy Number: A fuzzy number $\widetilde{A} = (a, b, c)$ is called the level λ-triangular fuzzy number if its membership function is

$$\mu_{\widetilde{A}}(x) = \begin{cases} \lambda\frac{(x-a)}{(b-a)}, & a \le x \le b \\ \lambda\dfrac{(c-x)}{(c-b)}, & b \le x \le c \\ 0, & otherwise \end{cases}$$

where $0 < a < b < c$, $0 < \lambda \le 1$ and the level λ-triangular fuzzy number denoted by $\widetilde{A} = (a, b, c; \lambda)$.

Note: The level λ-triangular fuzzy number is known as triangular fuzzy number when $\lambda = 1$ and denoted by $\widetilde{A} = (a, b, c)$.

Interval: An interval I in R has two components $[I_L, I_R]$ and defined as $I = \{x \in R \mid I_L \le x \le I_R\}$. Mean of I are denoted by $m(I)$ and defined as $m(I) = \frac{1}{2}(I_L + I_R)$. Also half width of I are denoted by $w(I)$ and defined as $w(I) = \frac{1}{2}(I_R - I_L)$. Clearly α-cut of a fuzzy number with continuous membership function can be treated as an interval.

Interval-valued Fuzzy Set: An interval-valued fuzzy set $\widetilde{A}$ is denoted by $\widetilde{A} = [\widetilde{A}^L, \widetilde{A}^U]$ and defined as

$$\widetilde{A} = \Big\{(x, [\mu_{\widetilde{A}^L}(x), \mu_{\widetilde{A}^U}(x)\Big\}, \quad \forall x$$

where $0 \le \mu_{\widetilde{A}^L}(x) \le \mu_{\widetilde{A}^U}(x) \le 1$ and $\mu_{\widetilde{A}^L}(x), \mu_{\widetilde{A}^U}(x) \in [0,1]$.
Let $\widetilde{A}^L = (a,b,c;\lambda)$, $\widetilde{A}^U = (e,b,h;\rho)$, then the interval-valued fuzzy set is expressed as

$$\widetilde{A} = [(a,b,c;\lambda),(e,b,h;\rho)]$$

where $0 < \lambda \le \rho \le 1$ and $e < a < b < c < h$.
The interval-valued fuzzy set $\widetilde{A}$ indicates that, when the membership grade of X belongs to the interval $[\widetilde{A}^L, \widetilde{A}^U]$, where $\widetilde{A}^L(x)$ is the smallest grade and $\widetilde{A}^L(x)$ is the largest grade.
Let $\widetilde{A}^L = (a,b,c;\lambda)$ and $\widetilde{A}^U = (e,b,h;\rho)$, then

$$\mu_{\widetilde{A}^L}(x) = \begin{cases} \lambda\frac{(x-a)}{(b-a)}, & a \le x \le b \\ \lambda\dfrac{(c-x)}{(c-b)}, & b \le x \le c \\ 0, & otherwise \end{cases}$$

where $a < b < c$.

$$\mu_{\widetilde{A}^L}(x) = \begin{cases} \lambda\frac{(x-e)}{(b-e)}, & a \le x \le b \\ \lambda\dfrac{(h-x)}{(h-b)}, & b \le x \le h \\ 0, & otherwise \end{cases}$$

where $e < b < h$.

Arithmetic Operations of Interval-valued Fuzzy Numbers: Assume that there are two level λ & ρ-triangular fuzzy numbers are respectively $\widetilde{A} = (a,b,c;\lambda)$ and $\widetilde{B} = (p,q,r;\rho)$, where where $0 < \lambda \le 1$, $0 < \rho \le 1$ and $a < b < c$, $p < q < r$. Let $\widetilde{A}$=$[\widetilde{A}^L, \widetilde{A}^U]$=$[(a_1,b_1,c_1;\lambda),(a_2,b_1,c_2;\rho)]$ and $\widetilde{B}$=$[\widetilde{B}^L, \widetilde{B}^U]$ = $[(p_1,q_1,r_1;\lambda),(p_2,q_1,r_2;\rho)]$ Then arithmetic operations between the level $\widetilde{A}$ and $\widetilde{B}$ fuzzy numbers, we can get the following:
(i) Interval-valued fuzzy numbers Addition $\oplus$: $\widetilde{A} \oplus \widetilde{B} = [\widetilde{A}^L \oplus \widetilde{B}^L, \widetilde{A}^U \oplus \widetilde{B}^U]$
(ii) Interval-valued fuzzy numbers Subtraction $\ominus$: $\widetilde{A} \ominus \widetilde{B} = [\widetilde{A}^L \ominus \widetilde{B}^L, \widetilde{A}^U \ominus \widetilde{B}^U]$
(iii) Interval-valued fuzzy numbers Multiplication $\otimes$: $\widetilde{A} \otimes \widetilde{B} = [\widetilde{A}^L \otimes \widetilde{B}^L, \widetilde{A}^U \otimes \widetilde{B}^U]$

$$(iv)\; K\widetilde{A} = [K\widetilde{A}^L, K\widetilde{A}^U] = \begin{cases} [(Ka_1,Kb_1,Kc_1;\lambda),(Ka_2,Kb_1,Kc_2;\rho)], & K > 0 \\ [(Kc_1,Kb_1,Ka_1;\lambda),(Kc_2,Kb_1,Ka_2;\rho)], & K < 0 \\ [(0,0,0;\lambda),(0,0,0;\rho)], & K = 0, \end{cases}$$

2.3.3 Possibility / Necessity / Credibility Measurements

Degree of uncertainty: This interpretation was proposed by Zadeh [240] when he introduced the possibility theory and developed his theory of approximate reasoning (Zadeh [240]). $\mu_F(u)$ is then the degree of possibility that a parameter x has value u, given that all that is known about it, is that x is F. Then the values encompasses by the support of the membership functions are mutually exclusive, and the membership degrees rank these values in terms of their respective plausibility. Set functions called possibility and necessity measures can be derived so as to rank-order events in terms of unsurprising-ness and acceptance respectively.

Possibility Measure: Let $\Re$ represents the set of real numbers and $\tilde{A}$ and $\tilde{B}$ be two fuzzy numbers with membership functions $\mu_{\tilde{A}}$ and $\mu_{\tilde{B}}$ respectively. Then taking degree of uncertainty as the semantics of fuzzy number, according to Zadeh [240], Dubois and Prade [62], Liu and Iwamura [140]:

$$\text{Pos } (\tilde{A} \star \tilde{B}) = \sup\{\min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)), x, y \in \Re, x \star y\} \tag{2.2}$$

where the abbreviation Pos represent possibility and $\star$ is any one of the relations $>, <, =, \leq, \geq$. Analogously if $\tilde{B}$ is a crisp number, say b, then

$$\text{Pos } (\tilde{A} \star b) = \sup\{\mu_{\tilde{A}}(x), x \in R, x \star b\} \tag{2.3}$$

Necessity Measure: Necessity measure of an event $\tilde{A} \star \tilde{B}$ is a dual of possibility measure. The grade of necessity of an event is the grade of impossibility of the opposite event and is defined as:

$$\text{Nes } (\tilde{A} \star \tilde{B}) = 1 - \text{Pos } (\overline{\tilde{A} \star \tilde{B}}) \tag{2.4}$$

where the abbreviation Nes represents necessity measure and $\overline{\tilde{A} \star \tilde{B}}$ represents complement of the event $\tilde{A} \star \tilde{B}$.

Lemma 2.1. *For two triangular fuzzy numbers (TFN) $\widetilde{a} = (a_1, a_2, a_3)$ and $\widetilde{b} = (b_1, b_2, b_3)$ Pos $(\tilde{a} \geq \tilde{b}) > \epsilon$ iff $\frac{a_3 - b_1}{b_2 - b_1 + a_3 - a_2} > \epsilon$ $(a_2 < b_2,\ a_3 > b_1)$.*

Proof. Let us consider, Pos$(\tilde{a} \geq \tilde{b}) > \epsilon$.

If $\widetilde{a} = (a_1, a_2, a_3)$ and $\widetilde{b} = (b_1, b_2, b_3)$ be two TFNs then

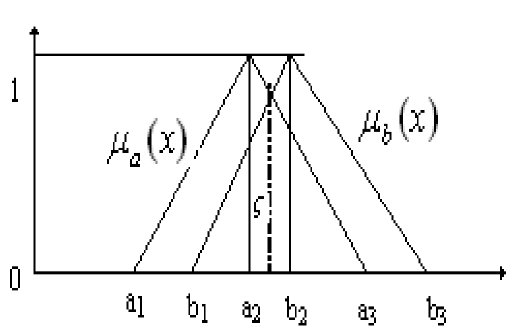

Figure 2.8: Measures of the event $(\tilde{a} \geq \tilde{b})$

$$\text{Pos }(\tilde{a} \geq \tilde{b}) = \begin{cases} 1, & \text{for} a_2 \geq b_2 \\ \zeta_2 = \frac{a_3 - b_1}{b_2 - b_1 + a_3 - a_2}, & \text{for} a_2 < b_2,\ a_3 > b_1 \\ 0, & \text{for} a_3 \leq b_1 \end{cases}$$

Hence, Pos $(\tilde{a} \geq \tilde{b}) > \epsilon$ iff $\zeta_2 = \frac{a_3 - b_1}{b_2 - b_1 + a_3 - a_2} > \epsilon, \quad (a_2 < b_2,\ a_3 > b_1)$, which is depicted in Figure 2.8. □

Note: Pos $(a \geq \tilde{b}) > \epsilon$ iff $\zeta_2 = \frac{a - b_1}{b_2 - b_1} > \epsilon, \quad (b_1 < a < b_2)$.
Therefore, it is clear that the event $-\infty < b < a_1$ is not acceptable (impossible event) with respect to the fuzzy event $\widetilde{a} \leq b$ as $\widetilde{a} < b$ implies the value of $b >$ least value of $\widetilde{a}$. On the other hand the event $b > a_2$ is certain case of the fuzzy event $\widetilde{a} \leq b$. Hence we consider the case $a_1 \leq b \leq a_2$, which gives Pos $(\tilde{a} \leq b) = \frac{b - a_1}{a_2 - a_1}$. Therefore, $Pos(\tilde{a} \leq b) > \epsilon \Rightarrow \frac{b - a_1}{a_2 - a_1} > \epsilon$.

Lemma 2.2. *For two triangular fuzzy numbers (TFN)* $\widetilde{a} = (a_1, a_2, a_3)$ *and* $\widetilde{b} = (b_1, b_2, b_3)$ *Nes* $(\tilde{a} > \tilde{b}) > \eta$ *iff* $\frac{b_3 - a_1}{a_2 - a_1 + b_3 - b_2} < 1 - \eta \ (a_2 > b_2,\ b_3 > a_1)$.

Proof. Let, we have Nes $(\tilde{a} > \tilde{b}) > \eta$.

From Lemma 2.1, it is clear that

$$\text{Pos }(\tilde{a} \leq \tilde{b}) = \begin{cases} 1, & \text{for} a_2 \leq b_2 \\ \zeta = \frac{b_3 - a_1}{a_2 - a_1 + b_3 - b_2}, & \text{for} a_2 > b_2,\ b_3 > a_1 \\ 0, & \text{for} a_1 \geq b_3 \end{cases} \tag{2.5}$$

Hence, Nes $(\tilde{a} > \tilde{b}) > \eta \Rightarrow$ (1-Pos$(\tilde{a} \leq \tilde{b})) > \eta$

Therefore, Nes $(\tilde{a} > \tilde{b}) > \eta$ iff $\zeta = \frac{b_3 - a_1}{a_2 - a_1 + b_3 - b_2} < 1 - \eta, \quad (a_2 > b_2,\ b_3 > a_1)$, which is depicted in Figure 2.9. □

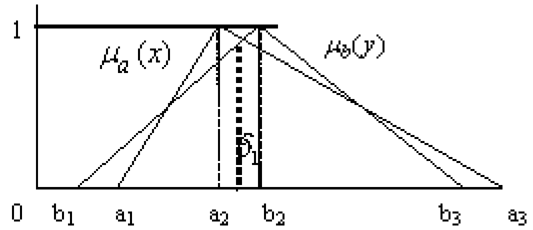

Figure 2.9: Measures of Necessity of the event $(\tilde{a} \leq \tilde{b})$

Note: Nes $(b < \tilde{a}) > \eta$ iff $\frac{b-a_1}{a_2-a_1} < 1 - \eta$ $(a_1 < b < a_2)$.

Properties of Possibility and Necessity Measures: The possibility and necessity measures have the following properties
i) Min(Nes($\tilde{a} * \tilde{b}$), Nes($\overline{\tilde{a} * \tilde{b}}$))=0.
ii) Pos($\tilde{a} * \tilde{b}$) $\geq$ Nes($\tilde{a} * \tilde{b}$).
iii) Nes($\tilde{a} * \tilde{b}$) $> 0 \Rightarrow$ Pos ($\tilde{a} * \tilde{b}$)=1.
iv) Pos($\tilde{a} * \tilde{b}$) $< 1 \Rightarrow$ Nes($\tilde{a} * \tilde{b}$)=0.

If the attitude of the DM is toward optimistic, equation (2.3) is the measure of best case and in pessimistic sense equation (2.4) gives the measure of worst case of that event.

Credibility Measure: The credibility measure of a fuzzy event $\tilde{A}$ is defined by

$$Cr(\tilde{A}) \ = \ \frac{1}{2}[\, Pos(\tilde{A}) + Nec(\tilde{A})] \ \text{ for any} \tilde{A} \in 2^R \tag{2.6}$$

Properties of credibility measure: The credibility measures have the following properties
i) $Cr(\phi) = 0$ and $Cr(R) = 1$,
ii) $Cr(A) \leq Cr(B)$ when ever $A, B \in 2^{\Re}$ and $A \subset B$
iii) $Cr(A) = 1 - Cr(A^C)$ for any $A \in 2^{\Re}$.
Thus, Cr is also a fuzzy measure defined on $(\Re, 2^{\Re})$. Here, based on the credibility measure the following form can be defined as

$$Cr(A) \ = \ [\, \rho Pos(A) + (1-\rho) Nec(A)\,] \tag{2.7}$$

for any $A \in 2^{\Re}$ and $0 < \rho < 1$. It also satisfies the above conditions.

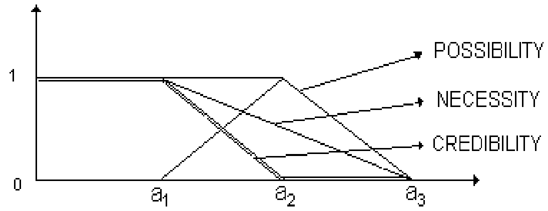

Figure 2.10: Weighted fill rate of possibility, necessity and credibility

Let $\tilde{A} \ = \ (a_1, a_2, a_3\,)\,)$ is a triangular fuzzy number and r is a crisp number. Using Liu and Liu [142], we define possibility measure and necessity measure as following.

$$Pos(\tilde{A} \geq r) = \begin{cases} 1, & r \leq a_2 \\ \dfrac{a_3 - r}{a_3 - a_2}, & a_2 \leq r \leq a_3 \\ 0, & r \geq a_3 \end{cases} \qquad Nec(\tilde{A} \geq r) = \begin{cases} 1, & r \leq a_1 \\ \dfrac{a_2 - r}{a_2 - a_1}, & a_1 \leq r \leq a_2 \\ 0, & r \geq a_2 \end{cases}$$

Using equation (2.7), the credibility measure for TFN can be defined as

$$Cr(\tilde{A} \geq r) = \begin{cases} 1, & r \leq a_1 \\ \dfrac{a_2 - \rho a_1}{a_2 - a_1} - \dfrac{(1-\rho)r}{a_2 - a_1}, & a_1 \leq r \leq a_2 \\ \dfrac{\rho(a_3 - r)}{a_3 - a_2}, & a_2 \leq r \leq a_3 \\ 0, & r \geq a_3 \end{cases} \tag{2.8}$$

$$Cr(\tilde{A} \leq r) = \begin{cases} 0, & r \leq a_1 \\ \rho\dfrac{r - a_1}{a_2 - a_1}, & a_1 \leq r \leq a_2 \\ \dfrac{a_3 - \rho a_2 - (1-\rho)r}{a_3 - a_2}, & a_2 \leq r \leq a_3 \\ 1, & r \geq a_3 \end{cases} \tag{2.9}$$

2.3.4 Expected Value of a Fuzzy Variable

Based on the credibility measure, Liu and Liu [142] presented the expected value operator of a fuzzy variable as follows.

Expected Value of a Fuzzy Variable: Let X be a normalized fuzzy variable, the expected value of the fuzzy variable X is defined by

$$E[X] = \int_0^{\infty} Cr(X \geq r)dr - \int_{-\infty}^{0} Cr(X \leq r)dr \tag{2.10}$$

Also, the expected value operation has been proved to be linear for bounded fuzzy variables, i.e., for any two bounded fuzzy variables X and Y, we have $E[\,aX + bY\,] = aE[X] + bE[Y]$ for any real numbers a and b.

Lemma 2.3. *The expected value of triangular fuzzy variable* $\tilde{A} = (a_1, a_2, a_3)$ *is defined as*

$$E[\tilde{A}] = \frac{1}{2}[(1-\rho)a_1 + a_2 + \rho a_3] \tag{2.11}$$

Proof. Using (2.8) and (2.9), from the definition of expected value of fuzzy variable $\tilde{A}$ (equation 2.10) is defined as follows,

$$E(\tilde{A}) = \int_0^{\infty} Cr(\tilde{A} \geq r)dr - \int_{-\infty}^{0} Cr(\tilde{A} \leq r)dr$$
$$= \int_0^{a_1} Cr(\tilde{A} \geq r)dr + \int_{a_1}^{a_2} Cr(\tilde{A} \geq r)dr + \int_{a_2}^{a_3} Cr(\tilde{A} \geq r)dr + 0$$
$$= \int_0^{a_1} dr + \int_{a_1}^{a_2} [\,\tfrac{a_2 - \rho a_1}{a_2 - a_1} - \tfrac{(1-\rho)r}{a_2 - a_1}\,]dr + \int_{a_2}^{a_3} [\,\tfrac{\rho a_3}{a_3 - a_2} - \tfrac{\rho r}{a_3 - a_2}\,]dr$$
$$= \tfrac{1}{2}[\,a_1(1-\rho) + a_2 + \rho a_3\,]$$

□

2.3.5 Fuzzy Extension Principle

Fuzzy Extension Principle: If $\tilde{A}, \tilde{B} \in \Re$ and $\tilde{C} = f(\tilde{A}, \tilde{B})$ where $f : \Re \times \Re \to \Re$ be a binary operation then according to fuzzy extension principle (Zadeh [240], Dubois and Prade [61]), membership function $\mu_{\tilde{C}}$ of $\tilde{C}$ is given by

$$\mu_{\tilde{C}}(z) = \sup \{\min (\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)), x, y \in \Re, \text{ and} z = f(x, y), \forall z \in \Re\} \tag{2.12}$$

for any $A \in 2^{\Re}$ and $0 < \rho < 1$, where ρ is the degree of pessimism. It also satisfies the above conditions.

Zadeh's Extension Principle: One of the basic concepts of fuzzy set theory which is used to generalize crisp mathematical concepts to fuzzy sets is the extension principle. Let X and Y be two universes and f : X $\longrightarrow$ Y be a crisp function. The extension principle tells us how to induce a mapping f : P(X) $\longrightarrow$ P(Y), where P(X) and P(Y) are the power sets of X and Y respectively. Following Zadeh [240], we define the fuzzy extension principle as follows:

We have the mapping f : X $\longrightarrow$ Y, y = f(x) which induce a function f : $\widetilde{A} \longrightarrow \widetilde{B}$ such that $\widetilde{B}$ = f($\widetilde{A}$)=$\{((y, \mu_{\tilde{B}}(y))|y = f(x), x \in X)\}$, where

$$\mu_{\tilde{B}}(y) = \begin{cases} \underset{x \in f^{-1}(y)}{sup} \quad \mu_{\tilde{A}}(x) & \text{if } f^{-1}(y) \neq \Phi, \\ 0 & \text{otherwise} \end{cases}$$

Centroid of the Fuzzy Number: The centroid value of a fuzzy function $\phi_{\widetilde{M}}$ is given by

$$C[\phi_{\widetilde{M}}] = \frac{\int_{-\infty}^{\infty} y\mu_{\phi_{\widetilde{M}}}(y)\, dy}{\int_{-\infty}^{\infty} \mu_{\phi_{\widetilde{M}}}(y)\, dy}$$

For a triangular fuzzy number (TFN) $\widetilde{A} = (a - \Delta_1, a, a + \Delta_2)$, where $0 < \Delta_1 < a$ and $0 < \Delta_2$; Δ_1, Δ_2 are determined by the decision makers. Then the centroid value, $C[\widetilde{A}] = a + \frac{1}{3}(\Delta_1 + \Delta_2)$.

2.3.6 Fuzzy Differential Equation and Integration

Seikkla Derivative: The derivative of $\widetilde{X}(t)$ followed by Seikkla, written $SD\widetilde{X}(t)$, was defined in Seikkla [192]. This definition was as follows: if $[X_1'(t, \alpha), X_2'(t, \alpha)]$ are the α-cuts of a fuzzy number for each $t \in I$, then $SD\widetilde{X}(t)$ exists and $SD\widetilde{X}(t)[X_1'(t, \alpha), X_2'(t, \alpha)]$.

Fuzzy Differential Equation: Chalco-Cano and Roman-Flores [19] Consider the fuzzy differential equation

$$\widetilde{X'}(t) = \widetilde{\phi}(t, \widetilde{X}(t)),\ \widetilde{X}(0) = \widetilde{X}_0 \tag{2.13}$$

where $f : [0,T] \times F(U) \to F(\Re^n)$ is obtained by Zadeh's extension principle (2.12) from a continuous function $\phi : [0,T] \times U \to \Re^n$ where $U \in \Re^n$. As g is continuous so ϕ is continuous [180] and by equation (2.1) we have

$$[\phi(t,X)]^\alpha = g(t,[X]^\alpha)$$

where $g(t,A) = \{g(t,a)/a \in A\}$.
Consider the deterministic differential equation (DDE), associated with FDE (2.13)

$$x'(t) = g(t,x(t)), x(0) = x_0 \tag{2.14}$$

where $x'(t)$ is the derivative (crisp) of a function $x : [0,T] \to \Re^n$.

Then according to Chalco-Cano and Roman-Flores [19], a fuzzy solution for equation (2.13) can be derived from equation (2.14) as below

• Solve DDE (2.14) and let $x(t,x_0)$ be its solution.

• Use Zadeh [240] extension principle (2.12), to $x(t,x_0)$ in relation to the parameter x_0 and obtain the extension $\widetilde{X}(t) = \widetilde{x}(t,\widetilde{X}_0)$, for each fixed t , which is a fuzzy solution of problem (2.13) provided the conditions of following theorem holds.

Theorem 2.1. *[19] Let U be an open set in $\Re^n$ and $X_0[\alpha] \subset U$. Suppose that g is continuous, and that for each $c \subset U$ there exists a unique solution $x(.,c)$ of the deterministic problem (2.14) and that $x(t,.)$ is continuous on U for each $t \in [0,T]$ fixed. Then, there exists a unique fuzzy solution $\widetilde{X}(t) = \widetilde{x}(t,X_0)$ of the FDE (2.13).*

Proposition 2.5. *[227] Let $\Re$ be a set of all fuzzy numbers, $\Re_{cl}$ be a set of all closed fuzzy numbers and $\Re_b$ be a set of all bounded fuzzy numbers. We say that*

(i) $\tilde{f}(x)$ fuzzy-valued function if $\tilde{f} : X \to \Re$;

(ii) $\tilde{f}(x)$ closed-fuzzy-valued function if $\tilde{f} : X \to \Re_{cl}$;

(iii) $\tilde{f}(x)$ bounded-fuzzy-valued function if $\tilde{f} : X \to \Re_b$.

The Fuzzy Riemann Integral: We discuss two kinds of fuzzy Riemann integrals [227]. One is based on the crisp interval and the other one is considered on the fuzzy interval. We call them as fuzzy Riemann integral of type-I and type-II, respectively.

The Fuzzy Riemann Integral of Type-I: Let $\tilde{f}(x)$ be a closed- and bounded-fuzzy-valued function on [a, b]. Suppose that $\int_a^b f_\alpha^L(x)\,dx$ and $\int_a^b f_\alpha^R(x)\,dx$ are Riemann-integrable on [a, b] $\forall\alpha$. Let

$$A_\alpha = \left[\int_a^b f_\alpha^L(x)\,dx, \int_a^b f_\alpha^R(x)\,dx\right]$$

Then we say that $\tilde{f}(x)$ is a fuzzy Riemann-integrable on [a, b] with type-I, denoted as $\tilde{f}(x) \in \Re\Re_1$ on [a, b], and the membership function of $\int_a^b \tilde{f}(x)\,dx$ is defined by, for $r \in A_0$,

$$\xi_{\int_a^b \tilde{f}(x)\,dx}(r) = \sup_{0\leq\alpha\leq 1} \alpha l_{A_\alpha}(r).$$

Proposition 2.6. *Let $\tilde{f}(x)$ be a closed- and bounded-fuzzy-valued function on [a, b]. If $\tilde{f}(x) \in \Re\Re_1$ on [a, b] then fuzzy Riemann-integral $\int_a^b \tilde{f}(x)\,dx$ is closed fuzzy number. Furthermore, the $\alpha-$cut set of $\int_a^b \tilde{f}(x)\,dx$ is $\left[\int_a^b \tilde{f}(x)\,dx\right]_\alpha = [\int_a^b f_\alpha^L(x)\,dx, \int_a^b f_\alpha^R(x)\,dx]$.*

Proposition 2.7. *If $\tilde{f}(x)$ be a closed- and bounded-fuzzy-valued function on [a, b] and $f_\alpha^L(x)$ and $f_\alpha^R(x)$ are continuous on [a, b] $\forall\alpha$ then $\left[\int_a^b \tilde{f}(x)\,dx\right]_\alpha = [\int_a^b f_\alpha^L(x)\,dx, \int_a^b f_\alpha^R(x)\,dx]$.*

Proposition 2.8. *[227] If $\tilde{f}(x)$ and $\tilde{g}(x)$ be a closed- and bounded-fuzzy-valued function on [a, b] and $\tilde{f}(x)$, $\tilde{g}(x) \in \Re\Re_1$ on [a, b] then $\tilde{f}(x) \oplus \tilde{g}(x) \in \Re\Re_1$ on [a, b]. Moreover, we have*

$$\int_a^b(\tilde{f}(x) \oplus \tilde{g}(x))\,dx = \int_a^b \tilde{f}(x)\,dx \oplus \int_a^b \tilde{g}(x)\,dx \quad .$$

The Fuzzy Riemann Integral of Type-II: In order to define the fuzzy Riemann integral of type-II, we need to consider the "length" between $\tilde{a}$ and $\tilde{b}$ for $\tilde{b} \succ \tilde{a}$ ($\tilde{b} \succ \tilde{a}$ means $\tilde{b} \succeq \tilde{a}$ and $b_\alpha^L \succeq a_\alpha^R$ for all α). Now $(\tilde{b} \ominus \tilde{a})_\alpha^L = b_\alpha^L - a_\alpha^R$ and $(\tilde{b} \ominus \tilde{a})_\alpha^R = b_\alpha^R - a_\alpha^L$ (by Proposition 8.3). We shall consider the interval $[a_\alpha^R, b_\alpha^L]$ for the lower bound case and the interval $[a_\alpha^L, b_\alpha^R]$ for the upper bound case. Then we have the following definition.

Proposition 2.9. *[227] Let $\tilde{f}(\tilde{x})$ be a bounded- and closed-fuzzy-valued function defined on the closed fuzzy real number system $(\Re_{\boldsymbol{R}}/\sim)_{\boldsymbol{R}}$ and $\tilde{f}(x)$ be induced by $\tilde{f}(\tilde{x})$.*

Suppose that $\tilde{b} \succeq \tilde{a}$.
(i) If $\tilde{f}(x)$ is non-negative and $f_\alpha^L(x)$ and $f_\alpha^R(x)$ are Riemann integrable on $[a_\alpha^R, b_\alpha^L]$ and $[a_\alpha^L, b_\alpha^R]$, respectively, for all α then we let

$$A_\alpha = \begin{cases} \left[\int_{a_\alpha^R}^{b_\alpha^L} f_\alpha^L(x)\,dx, \int_{a_\alpha^L}^{b_\alpha^R} f_\alpha^R(x)\,dx\right] & \text{if } b_\alpha^L > a_\alpha^R \\ \left[0, \int_{a_\alpha^L}^{b_\alpha^R} f_\alpha^R(x)\,dx\right] & \text{if } b_\alpha^L \leq a_\alpha^R \end{cases}$$

(ii) If $\tilde{f}(x)$ is non-positive and $f_\alpha^L(x)$ and $f_\alpha^R(x)$ are Riemann integrable on $[a_\alpha^L, b_\alpha^R]$, and $[a_\alpha^R, b_\alpha^L]$ respectively, for all α then we let

$$A_\alpha = \begin{cases} \left[\int_{a_\alpha^L}^{b_\alpha^R} f_\alpha^L(x)\,dx, \int_{a_\alpha^R}^{b_\alpha^L} f_\alpha^R(x)\,dx\right] & \text{if } b_\alpha^L > a_\alpha^R \\ \left[\int_{a_\alpha^L}^{b_\alpha^R} f_\alpha^R(x)\,dx\right] & \text{if } b_\alpha^L \leq a_\alpha^R \end{cases}$$

Under the above conditions, we say that $\tilde{f}(\tilde{x})$ is fuzzy Riemann-integrable on the fuzzy interval $[\tilde{a}, \tilde{b}]$ with type-II, denoted as $\tilde{f}(\tilde{x}) \in \Re R_{II}$ on $[\tilde{a}, \tilde{b}]$, and the membership function of $\int_{\tilde{a}}^{\tilde{b}} \tilde{f}(\tilde{x})\, d\tilde{x}$ is defined by,

$$\xi_{\int_{\tilde{a}}^{\tilde{b}} \tilde{f}(\tilde{x})\, dx}(r) = \sup_{0 \le \alpha \le 1} \alpha l_{A_\alpha}(r), \quad \text{for } r \in A_0$$

Proposition 2.10. *[227] (i) Let [a, b] and [c, d] be closed intervals. We say that $[a, b] \succeq_{int} [c, d]$ if and only if $a \geq c$ and $b \geq d$.*

(ii) Let $\widetilde{a}$ and $\widetilde{b}$ be closed fuzzy numbers. We say that $\widetilde{a} \geq \widetilde{b}$ if and only if $\widetilde{a}_\alpha \geq_{int} \widetilde{b}_\alpha$ for all α.

(iii) Let $\widetilde{a}$ and $\widetilde{b}$ be two fuzzy numbers. We say that $\widetilde{a}$ is equal to $\widetilde{b}$, denoted as $\widetilde{a} = \widetilde{b}$ if and only if $\widetilde{a}_\alpha = \widetilde{b}_\alpha$ for all α.

Proposition 2.11. *[227] Let $\widetilde{a}$ and $\widehat{b}$ be two closed fuzzy numbers. Suppose that $\widetilde{b} \succeq \widetilde{a}$. Then we only have two cases.*

(a) $a_\alpha^R \geq b_\alpha^L, \forall \alpha \in$ [0, 1].

(b) $\exists \alpha_0 \in [0, 1]$ such that $a_\alpha^R \geq b_\alpha^L$ for $0 \leq \alpha \leq \alpha_0$ and $a_\alpha^R \leq b_\alpha^L$ for $1 \geq \alpha > \alpha_0$.

2.3.7 Fuzzy-Rough Set

In this section, we discuss some basic concepts, theorems and lemmas on fuzzy rough theory by Xu and Zhou [228].

Lower and Upper Approximation: In Xu and Zhou [228] proposed some definitions and discussed some important properties of fuzzy rough variable. Let U be a universe, and X be a set representing a concept. Then its lower and upper approximation is defined by

$\underline{X} = \{x \in U \mid R(x) \subset X\}$ and $\overline{X} = \bigcup_{x \in X} R(x)$ respectively.

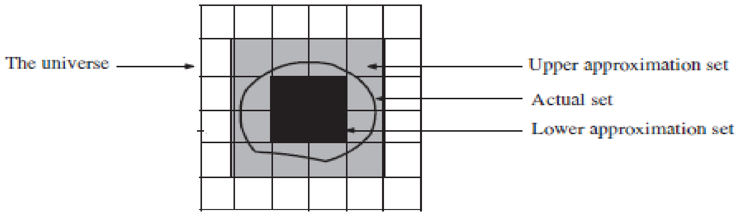

Figure 2.11: Rough Set

Rough Set: The collection of all sets having the same lower and upper approximations is called a rough set, denoted by $(\underline{X}, \overline{X})$.The figure of the rough set is depicted in Figure 2.11.

Example 2.4. *Let ξ focus on the continuous set in the one dimension real space R. There are still some vague sets which cannot be directly fixed and need to be described by the rough approximation. Let, set R be the universe, a similarity relation is defined as $a \simeq b$ if and only if $|a-b| \leq 10$. Let us defined for the set $[20,50]$, its lower approximation $\underline{[20,50]}$ = [30,40] and its upper approximation $\overline{[20,50]} = [10,60]$. Then the upper and lower approximation of the set [20,50] make up a rough set $([30,40],[10,60])$ which is the collection of all sets having the same lower approximation $[30,40]$ and upper approximation [10,60].*

Fuzzy Rough Variable: A fuzzy rough variable ξ is a fuzzy variable with uncertain parameter $\rho \in X$, where X is approximated by $(\underline{X},\overline{X})$ according to the similarity relation R, namely, $\underline{X} \subseteq X \subseteq \overline{X}$.
For convenience, we usually denote $\rho \vdash (\underline{X},\overline{X})_R$ expressing that ρ is in some set A which is approximated by $(\underline{X},\overline{X})$ according to the similarity relation R, namely, $\underline{X} \subseteq A \subseteq \overline{X}$.

Example 2.5. *Let's consider the LR fuzzy variable ξ with the following membership function,*

$$\mu_\xi(x) = \begin{cases} L\left(\dfrac{\rho - x}{\alpha}\right) & \text{if } \rho-\alpha < x < \rho \\ 1 & \text{if } x = \rho \\ L\left(\dfrac{x-\rho}{\beta}\right) & \text{if } \rho < x < \rho+\beta \end{cases}$$

Where $L(x) = 1-x$ and $\rho \vdash ([1,2],[0,3])$ then ξ is a fuzzy rough variable.

Theorem 2.2. *If fuzzy rough variables $\tilde{\bar{c}}_{ij}$ are defined as $\tilde{\bar{c}}_{ij}(\lambda) = (\bar{c}_{ij1},\bar{c}_{ij2},\bar{c}_{ij3},\bar{c}_{ij4})$ with $\bar{c}_{ijt} \vdash ([c_{ijt2},c_{ijt3}],[c_{ijt1},c_{ijt4}])$, for $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$, $t = 1,2,3,4$, $x = (x_1,x_2,\ldots,x_m)$, $0 \leq c_{ijt1} \leq c_{ijt2} < c_{ijt3} \leq c_{ijt4}$.*
then $E[\tilde{\bar{c}}_1^T x]$,$E[\tilde{\bar{c}}_2^T x]$,…,$E[\tilde{\bar{c}}_n^T x]$ is respectively equivalent to

$$\frac{1}{16}\sum_{j=1}^{n}\sum_{t=1}^{4}\sum_{k=1}^{4} c_{1jtk}x_j,\ \frac{1}{16}\sum_{j=1}^{n}\sum_{t=1}^{4}\sum_{k=1}^{4} c_{2jtk}x_j, \ldots,\ \frac{1}{16}\sum_{j=1}^{n}\sum_{t=1}^{4}\sum_{k=1}^{4} c_{njtk}x_j.$$

Proof. The proof of the theorem is in reference Xu and Zhou [228]. □

Theorem 2.3. *If fuzzy rough variables $\tilde{\bar{a}}_{rj},\tilde{\bar{b}}_r$ defined as follows, $\tilde{\bar{a}}_{rj}(\lambda) = (\bar{a}_{rj1},\bar{a}_{rj2},\bar{a}_{rj3},\bar{a}_{rj4})$ with $\bar{a}_{rjt} \vdash ([a_{rjt2},a_{rjt3}],[a_{rjt1},a_{rjt4}])$, $\tilde{\bar{b}}_r(\lambda) = (\bar{b}_{r1},\bar{b}_{r2},\bar{b}_{r3},\bar{b}_{r4})$ with $\bar{b}_{rt} \vdash ([b_{rt2},b_{rjt3}],[b_{rt1},b_{rt4}])$, for $r = 1,2,...,p$, $j = 1,2,...,n, t = 1,2,3,4$, $0 \leq a_{rt1} \leq a_{rt2} < a_{rt3} \leq a_{rt4}$, $0 \leq b_{rt1} \leq b_{rt2} < b_{rt3} \leq b_{rt4}$.*

Then $E[\tilde{\tilde{a}}_{rj}^T x] \leq E[\tilde{\tilde{b}}_{rj}], r = 1, 2, ..., p$ *is equivalent to*

$$\frac{1}{16}\sum_{j=1}^{n}\sum_{t=1}^{4}\sum_{k=1}^{4} a_{rjtk}x_j \leq \frac{1}{16}\sum_{t=1}^{4}\sum_{k=1}^{4} b_{rtk},\ r = 1, 2, ..., p$$

Proof. The proof of the theorem is in reference Xu and Zhou [228]. □

Lemma 2.4. *Assume that* ξ *and* η *are the introduction of variables with finite expected values. Then for any real numbers a and b, we have*

$$E[a\xi + b\eta]\ =\ aE[\xi] + bE[\eta].$$

Proof. The proof of the Lemma is in reference Xu and Zhou [228]. □

Single-objective Fu Ro Model: Let us consider the following single-objective decision making model with fuzzy rough coefficients

$$\begin{cases} Max & \{f(x,\xi)\} \\ s.t & \begin{cases} g_r(x,\xi) \leq 0,\ r = 1, 2, ..., p \\ x \in X \end{cases} \end{cases} \tag{2.15}$$

where x is a n-dimensional decision vector, $\xi = (\xi_1, \xi_2, \xi_3, ..., \xi_n)$ is a Fu-Ro vector, $f(x, \xi)$ is objective function. Because of the existence of Fu-Ro vector ξ, problem (3) is not well-defined. That is, the meaning of maximizing $f(x, \xi)$ is not clear and constraints $g_r(x, \xi) \leq 0, r = 1, 2, ..., p$ do not define a deterministic feasible set.

Equivalent Crisp model for Single Objective Problem with Fu Ro Parameters: For the single-objective model with Fu-Ro parameters, we cannot deal with it directly, we should use some tools to make it have mathematical meaning, we then can solve it. In this subsection, we employ the expected value operator to transform the fuzzy rough model into Fu-Ro EVM i.e., crisp model. Based on the definition of the expected value of fuzzy rough events $f(x, \xi)$, $g_r(x, \xi)$ and Theorems 1, 2 the Fu-Ro EVM is proposed as follows,

$$\begin{cases} Max & \mathrm{E}[f(x,\xi)] \\ s.t & \begin{cases} E[g_r(x,\xi)] \leq 0,\ r = 1, 2, ..., p \\ x \in X \end{cases} \end{cases} \tag{2.16}$$

where x is n-dimensional decision vector and ξ is n-dimensional fuzzy rough variable.

2.4 Optimization Techniques

2.4.1 Fuzzy Programming Technique (FPT)

To solve a multi-objective programming problem by a Fuzzy Programming Technique(FPT), the first step is to assign, for each objective $f_1(x)$, $f_2(x)$, $f_3(x)$, ..., $f_k(x)$, $(k \geq 2)$, two values

U_j and L_j which are upper and lower bounds of the j-th objective for j=1, 2, ..., k. Here, $L_j =$ aspired level of achievement, $U_j =$ higher acceptable level of achievement for minimization and $L_j =$ higher acceptable level of achievement, $U_j =$ aspired level of achievement for maximization. The steps of the fuzzy programming technique are as follows:

Step-1: Solve the multi-objective programming problem as a single objective problem using, only one objective at a time and ignoring the other.

Step-2: From the results of step-1, determine the corresponding values for every objective at each solution derived. According to each solution and value for every objective, pay off matrix can be formulated as follows:

$$\begin{pmatrix} f_1^*(x^1) & f_2(x^1) & .... & f_k(x^1) \\ f_1(x^2) & f_2^*(x^2) & .... & f_k(x^2) \\ .... & .... & .... & .... \\ f_1(x^k) & f_2(x^k) & .... & f_k^*(x^k) \end{pmatrix}$$

where x^1, x^2, ..., x^k are the ideal solution of the objectives $f_1(x)$, $f_2(x)$, $f_3(x)$, ..., $f_k(x)$ respectively.

Step-3: From the step-2, find the desired goal L_j and worst tolerable value U_j of $f_j(x)$, j=1, 2, ..., k as follows:

$$U_j = \max(f_j(x^1), f_j(x^2), ..., f_j(x^{(j-1)}), f_j^*(x^j), f_j(x^{(j+1)}), ..., f_j(x^k))$$
$$L_j = \min(f_j(x^1), f_j(x^2), ..., f_j(x^{(j-1)}), f_j^*(x^j), f_j(x^{(j+1)}), ..., f_j(x^k))$$

Let, $T_{f_j} = U_j - L_j$ be the tolerances for the fuzzy constraints.

Step-4: The membership functions (MFs) of these objectives may be linear and/ or non-linear. Suppose, $\mu_j(f_j(x))$ be the membership function corresponding to the j-th (j=1, 2, ..., k) objective function of $f_1(x)$, $f_2(x)$, $f_3(x)$, ..., $f_k(x)$. Using Zimmermann [245] method, the multi-objective programming problem reduces to the following nonlinear single objective problem

$$\begin{aligned} & \text{Max } \lambda \\ & \text{such that} \\ & \mu_j(f_j(x)) \geq \lambda, \quad (j = 1, 2, ..., k) \\ & \phi_r(x) \leq b_r, \quad (r = 1, 2, ..., m) \\ & x_i \geq 0, \quad (i = 1, 2, ..., n) \\ & \text{and } \ x = (x_1, x_2, ..., x_n)^T \\ & \text{where } \ \lambda \in [0, 1]. \end{aligned}$$

2.4.2 Generalized Reduced Gradient (GRG) Technique

The GRG technique is a method for solving $SONLP$ problems for handling equality as well as inequality constraints. Consider the $SONLP$ problem as

$$\left.\begin{array}{ll} \text{Find} & x = (x_1, x_2, ...x_n)^T \\ \text{which maximizes} & f(x) \\ \text{subject to} & x \in X \\ \text{where } X = & \left\{ x : \begin{array}{ll} g_j(x) \leq 0, & j = 1, 2, ..., m \\ h_r(x) = 0, & r = 1, 2, ..., p \\ x_i \geq 0, & i = 1, 2, ..., n \end{array} \right\} \end{array} \right\} \tag{2.17}$$

By adding a non-negative slack variable $s_j\,(\geq\ 0),\ j = 1, 2, ..., m$ to each of the above inequality constraints, the problem (2.17) can be stated as

$$\left.\begin{array}{ll} \text{Maximize} & f(x) \\ \text{subject to} & x \in X \\ \text{where } X = & \left\{ x : \begin{array}{ll} x = (x_1, x_2, ...x_n)^T & \\ g_j(x) + s_j = 0, & j = 1, 2, ..., m \\ h_r(x) = 0, & r = 1, 2, ..., p \\ x_i \geq 0 & i = 1, 2, ...n \\ s_j \geq 0, & j = 1, 2, ...m \end{array} \right\} \end{array} \right\} \tag{2.18}$$

where the lower and upper bounds on the slack variables, $s_j,\ j = 1, 2, ..., m$ are taken as a zero and a large number (infinity) respectively.
Denoting s_j by x_{j+n}, $g_j(x) + s_j$ by ξ_j, $h_r(x)$ by ξ_{m+r}, the above problem can be rewritten as,

$$\left.\begin{array}{ll} \text{Maximize} & f(x) \\ \text{subject to} & x \in X \\ \text{where } X = & \left\{ x : \begin{array}{ll} x = (x_1, x_2, ...x_{n+m})^T & \\ \xi_j(x) = 0, & j = 1, 2, ...m + p \\ x_i \geq 0 & i = 1, 2, ...n + m \end{array} \right\} \end{array} \right\} \tag{2.19}$$

This GRG technique is based on the idea of elimination of variables using the equality constraints. Theoretically, $(m + p)$ variables (dependent variables) can be expressed in terms of remaining $(n - p)$ variables (independent variables). Thus one can divide the $(n + m)$ decision variables arbitrarily into two sets as

$$x = (y, z)^T$$

where, y is $(n - p)$ design or independent variables and z is $(m + p)$ state or dependent variables and

$$\begin{array}{rcl} y & = & (y_1, y_2, ..., y_{n-p})^T \\ z & = & (z_1, z_2, ..., z_{m+p})^T \end{array}$$

Here, the decision variables are completely independent and the state variables are dependent on the design variables used to satisfy the constraints $\xi_j(x) = 0, \ (j = 1, 2, ..., m+p)$. Consider the first variations of the objective and constraint functions as follows:

$$df(x) = \sum_{i=1}^{n-p} \frac{\partial f}{\partial y_i} dy_i + \sum_{i=1}^{m+p} \frac{\partial f}{\partial z_i} dz_i = \nabla_y^T f \, dy + \nabla_z^T f \, dz \tag{2.20}$$

$$\begin{aligned} d\xi_j(x) &= \sum_{i=1}^{n-p} \frac{\partial \xi_j}{\partial y_i} dy_i + \sum_{i=1}^{m+p} \frac{\partial \xi_j}{\partial z_i} dz_i \\ \text{or} \quad d\xi &= C \, dy + D \, dz \end{aligned} \tag{2.21}$$

$$\begin{aligned} \text{where} \quad \nabla_y^T f &= \left(\frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2}, ..., \frac{\partial f}{\partial y_{n-p}} \right) \\ \text{and} \quad \nabla_z^T f &= \left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, ..., \frac{\partial f}{\partial z_{m+p}} \right) \end{aligned}$$

$$C = \begin{bmatrix} \frac{\partial \xi_1}{\partial y_1} & \cdots \cdots \cdots & \frac{\partial \xi_1}{\partial y_{n-p}} \\ \frac{\partial \xi_2}{\partial y_1} & \cdots \cdots \cdots & \frac{\partial \xi_2}{\partial y_{n-p}} \\ \cdots & \cdots \cdots \cdots \cdots \cdots & \cdots \\ \cdots & \cdots \cdots \cdots \cdots \cdots & \cdots \\ \frac{\partial \xi_{m+p}}{\partial y_1} & \cdots \cdots \cdots & \frac{\partial \xi_{l+m}}{\partial y_{n-p}} \end{bmatrix}, \qquad D = \begin{bmatrix} \frac{\partial \xi_1}{\partial z_1} & \cdots \cdots \cdots & \frac{\partial \xi_1}{\partial z_{m+p}} \\ \frac{\partial \xi_2}{\partial z_1} & \cdots \cdots \cdots & \frac{\partial \xi_2}{\partial z_{m+p}} \\ \cdots & \cdots \cdots \cdots \cdots \cdots & \cdots \\ \cdots & \cdots \cdots \cdots \cdots \cdots & \cdots \\ \frac{\partial \xi_{m+p}}{\partial z_1} & \cdots \cdots \cdots & \frac{\partial \xi_{m+p}}{\partial z_{m+p}} \end{bmatrix},$$

$$\begin{aligned} dy &= (dy_1, \, dy_2, ..., dy_{n-p})^T \\ \text{and} \quad dz &= (dz_1, dz_2, ..., dz_{m+p})^T \end{aligned}$$

Assuming that the constraints are originally satisfied at the vector x $(\xi(x) = 0)$, any change in the vector dx must correspond to $d\xi = 0$ to maintain feasibility at $x + dx$. Thus, equation (2.21) can be solved as

$$\begin{aligned} & Cdy + Ddz = 0 \\ \text{or} \quad & dz = -D^{-1}Cdy \end{aligned} \tag{2.22}$$

The change in the objective function due to the change in x is given by the equation (2.21), which can be expressed, using equation (2.22) as

$$df(x) = (\nabla_y^T f - \nabla_z^T f D^{-1} C) dy$$

$$\text{or} \quad \frac{df(x)}{dy} = G_R \tag{2.23}$$

$$\text{where} \quad G_R = \nabla_y^T f - \nabla_z^T f D^{-1} C \tag{2.24}$$

is called the generalized reduced gradient. Geometrically, the reduced gradient can be described as a projection of the original $n-$dimensional gradient into the $(n - m)$ dimensional feasible region described by the design variables.

A necessary condition for the existence of minimum of an unconstrained function is that the components of the gradient vanish. Similarly, a constrained function assumes its minimum value when the appropriate components of the reduced gradient are zero. In fact, the reduced gradient G_R can be used to generate a search direction S to reduce the value of the constrained objective function. Similarly, to the gradient ∇f that can be used to generate a search direction S for an unconstrained function. A suitable step length λ is to be chosen to minimize the value of $f(x)$ along the search direction. For any specific value of λ, the dependent variable vector z is updated using equation (2.22). Noting that equation (2.21) is based on using a linear approximation to the original non-linear problem, so the constraints may not be exactly equal to zero at λ, i.e., $d\xi \neq 0$. Hence, when y is held fixed, in order to have

$$\xi_j(x) + d\xi_j(x) = 0, \;\; j = 1, 2, ..., m + p \tag{2.25}$$

following must be satisfied.

$$\xi(x) + d\xi(x) = 0 \tag{2.26}$$

Using equation (2.21) for $d\xi$ in equation (2.26), following is obtained

$$dz = D^{-1}(-\xi(x) - Cdy) \tag{2.27}$$

The value dz given by equation (2.27) is used to update the value of z as

$$z_{update} = z_{current} + dz \tag{2.28}$$

The constraints evaluated at the updated vector x, and the procedure of finding dz using equation (2.28) is repeated until dz is sufficiently small.

2.4.3 Single Objective Genetic Algorithm (GA)

There are several non-analytic methods for solving combination decision making problem like, neural network, simulation algorithm (cf. Kumar et. al. [125], Chan et al. [21]). Use of GA in complex decision making problem is already well established (cf. Mickelwicz [150], Roy et al. [182], Chan et al. [20]). GAs are inspired by evolutionary biology. **Mainly, because of the global searching ability and converging criteria, GAs are widely used in many areas.** It consists of a population of artificial agents mimicking the animals' behavior in the real world. Each agent follows some simple rules and interacts with the others to share information to lead the behavior to convergence.

It has unique characteristics compared to other meta-heuristic methods. The following

advantages have been added in the revised paper (Goldberge [75]).
• GA works with the coding of the parameters, not the parameters themselves.
• It is a population-based solution, not based on a single point.
• Good for noisy environments.
• It uses probabilistic transition rules, not deterministic rules.
• It trades-off between exploration and exploitation.
• Inherently parallel, easily distributed for all variables.
• It is capable of working with any kinds of the objective functions and constraints in linear and/or non-linear forms within any solution space (discrete or continuous).

General structure of the GA for the optimization problem with out and with constrains are presented below:

Algorithm 2.1. *1. Set iteration counter* $T = 0$.

2. *Initialize probability of crossover* p_c *and probability of mutation* p_m.
3. *Initialize* $P(T)$.
4. *Evaluate* $P(T)$.
5. *Repeat*
    a. *Select* N *solutions from* $P(T)$, *for mating pool using Roulette-wheel selection process. Let this set be* $P(T)^1$.
    b. *Select solutions from* $P(T)^1$, *for crossover depending on* p_c .
    c. *Made crossover on selected solutions for crossover to get population* $P(T)^2$.
    d. *Select solutions from* $P(T)^2$, *for mutation depending on* p_m.
    e. *Made mutation on selected solutions for mutation to get population* $P(T+1)$.
    f. $T \leftarrow T + 1$.
    g. *Evaluate* $P(T)$.
6. *Until(Termination condition does not hold).*
7. *Output: Fittest solution (chromosome) of* $P(T)$.

Constraints Handling in GA

The main idea of handling constraints is to design chromosomes carefully by genetic operators to keep all these within the feasible solution set. To ensure that the chromosomes (solutions) are feasible, we have to check all new chromosomes (x) generated by genetic operators. We suggest that a function is designed for each target optimization problem, the output value 1 means that the chromosome is feasible, 0 for infeasible. The algorithm for finding the feasibility of an individual (solution) (x) for the optimization problem is as follows:

```
for j = 1 to l do
  if(g_j(x) ≤ 0)
    continue;
   else
    return 0;
  endif
endfor
for k = 1 to m do
  if(h_k(x) = 0)
    continue;
   else
    return 0;
  endif
endfor
 return 1
```

The above genetic algorithm can be depicted pictorially by the following flowchart. The same

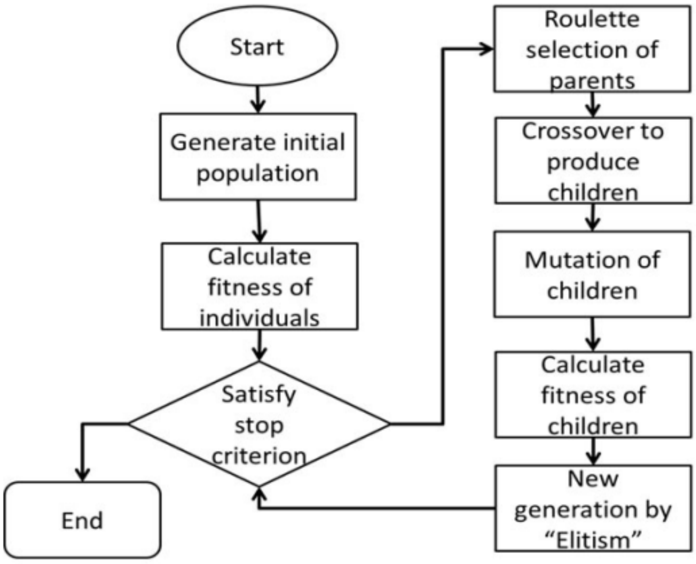

Figure 2.12: Flowchart of GA

phenomenon is followed to create a genetic algorithm for an optimization problem. Here, the potential solutions of a problem are analogous with the chromosomes and chromosome of better offspring with the better solution of the problem. Crossover and mutation happen among a set of potential solutions to get a new set of solutions and it continues until terminating conditions are encountered. A GA for a particular problem must have the following six components.

(a) A genetic representation for potential solutions (**chromosomes**) to the problem.

(b) A way to create an **initial population** of potential solutions(chromosomes).

(c) A way to **evaluate fitness** of each solution.

(d) An evolution function that plays the role of environment, rating solutions in term of their fitness, i.e., **selection process** for mating pool.

(e) Genetic operators such as **crossover** and **mutation** that alter the composition of children.

(f) Values of different parameters such as **population size**, **probability of crossover**, **probability of mutation** etc that the genetic algorithm uses.

The above GA components are described elaborately in the followings.

Procedures for Different GA Components:
(a) Chromosome Representation: The concept of chromosome is normally used in the GA to stand for a feasible solution to the problem. A chromosome has the form of a string of genes that can take on some value from a specified search space. The representation of chromosome depends on properties and requirements of a problem. Normally, there are two types of chromosome representation such as (i) the binary vector representation based on bits and (ii) the real number representation. In this research work, the real number representation scheme has been used. Here, a 'K dimensional real vector', X=(x_1, x_2, ... x_K) is used to represent a solution, where x_1, x_2, ... x_K represent different decision variables of the problem.

(b) Initialization: A set of solutions (chromosomes) is called a population. N such solutions X_1, X_2, X_3, ... X_N are randomly generated from search space by random number generator such that each X_i satisfies the constraints of the problem. This solution set and it is taken as initial population and is the starting point for a GA to evolve to desired solutions. At this step, probability of crossover p_c and probability of mutation p_m are also initialized. These two parameters are used to select chromosomes from the mating pool for genetic operations such as crossover and mutation respectively.

(c) Constraint Checking: For constrained optimization problems, at the time of generation of each individuals X_i of $P(1)$, constraints are checked using a separate subfunction check_constraint(X_i), which returns 1 if X_i satisfies the constraints, otherwise it returns 0. If check_constraint(X_i)=1, X_i is included in $P(1)$ otherwise X_i is again generated and it continues until constraints are satisfied.

(d) Fitness Value: All chromosomes in the population are evaluated using a fitness function. This fitness value is a measure of whether the chromosome is suited for the environment under consideration. Chromosomes with higher fitness will receive larger probabilities of inheritance in subsequent generations, while chromosomes with low fitness will more likely be eliminated. The selection of a good and accurate fitness function is thus a key to the success of solving any problem quickly. In this thesis, value of a objective function due to the solution X_i, is taken as fitness of X_i. Let it be $f(X_i)$.

(e) Selection Process to Create Mating Pool: Selection in the GA is a scheme used to select some solutions from the population for mating pool. From this mating pool, the pairs of individuals in the current generation are selected as parents to reproduce offspring. There are several selection schemes, such as roulette wheel selection, ranking selection, stochastic universal sampling selection, local selection, truncation selection, tournament selection, etc. Here, the Roulette wheel selection process has been used in different cases. This process consists of the following steps:

(i) Find total fitness of the population $F = \sum_{i=1}^{N} f(X_i)$

(ii) Calculate the probability of selection pr_i of each solution X_i by the formula pr_i=f(X$_i$)/F.

(iii) Calculate the cumulative probability qr_i for each solution X_i by the formula $qr_i = \sum_{j=0}^{i} pr_j$

(iv) Generate a random number 'r' from the range [0,1].

(v) If $r < qr_1$, then select X_1 otherwise select X_i $(2 \leq i \leq N)$ where $qr_{i-1} \leq r < qr_i$.

(vi) Repeat step (iv) and (v) N times to select N solutions from current population. Clearly one solution may be selected more than once.

(vii) Let us denote this selected solution set by $P^1(T)$.

(f) Crossover: The crossover is a key operator in the GA and it is used to exchange the main characteristics of parent individuals and pass them on the children. It consists of the following two steps:

(i) Selection for crossover: For each solution of $P^1(T)$, generate a random number r from the range $[0, 1]$. If $r < p_c$ then the solution is taken for crossover, where p_c is the probability of crossover.

(ii) Crossover process: Crossover is performed on the selected solutions. For each pair of coupled solutions Y_1 and Y_2, a random number c is generated from the range $[0, 1]$. Then Y_1, Y_2 are replaced by their offsprings Y_{11} and Y_{21} respectively where,

$$Y_{11} = cY_1 + (1 - c)Y_2$$
$$Y_{21} = cY_2 + (1 - c)Y_1$$

provided that Y_{11}, Y_{21} satisfy the constraints of the problem.

(g) Mutation: The mutation operation is needed after the crossover operation to maintain the population diversity and recover the possible loss of some good characteristics. It also consists of the following two steps:

(i) Selection for mutation: For each solution of $P^1(T)$, generate a random number r from the range $[0, 1]$. If $r < p_m$, then the solution is taken for mutation, where p_m is the probability of mutation.

(ii) Mutation process: To mutate a solution $X = (x_1, x_2, ..., x_K)$ select a random integer ξ in the range $[1, K]$. Then replace x_ξ by randomly generated value within the boundary of ξ^{th} component of X.

(h) Selection of Offsprings: Maximum population growth in a generation is assumed as forty percent. So, not all offsprings belong to the parent set for next generation. At first, offspring set is arranged in descending order in fitness. Then better solutions are selected and entered into parent set such that the population size does not exceeds $Maxsize$.

(i) Termination Condition: Algorithm terminates when difference between maximum fitness ($Maxfit$) of chromosome, (i.e., fitness of the best solution of the population) and average fitness ($Avgfit$) of the population becomes negligible. In other words when fitness of all chromosomes in $P(t)$ are almost equal. If the algorithm does not terminates (converge) under above condition then to exit from infinite loop the algorithm will terminate after $Maxgen$ iterations.

(j) Implementation: With the above function and values the algorithm is implemented using C-programming language in a personal computer consist of Intel 3.07 GHZ processor, 448 MB RAM and Microsoft Windows XP operating system.

(k) Convergency of the GA: The convergence of GA is very important. Following, Goldberg [75] the convergence of the method is predict the proportion of optimal population $p(t)$ in the total population as a function of the number of generations t and the population mean fitness at generation t is given by $\overline{f}(t) = np(t)$ and the variance $\sigma^2(t) = np(t)(1 - p(t))$, here $n = pop\_size$.

The increment in population mean fitness can be computed as:

$$\overline{f}(t^s) - \overline{f}(t) = \frac{\sigma^2(t)}{\overline{f}(t)}$$

At the time of optimizing bit counting function $\overline{f}(t+1) = \overline{f}(t^s)$, so the increment of the population average fitness becomes $\overline{f}(t+1) - \overline{f}(t) = \frac{\sigma^2(t)}{\overline{f}(t)}$

Therefore increase of the proportion of the population becomes

$$n(p(t+1) - p(t)) = \frac{np(t)(1 - p(t))}{np(t)}$$

Approximating the difference equation with the corresponding differential equation we obtain a simple convergence model expressing the proportion $p(t)$ in function of the number of generations t

$$\frac{dp(t)}{dt} = \frac{1 - p(t)}{n}$$

The above differential equation leads to a convergent solution

$$p(t) = 1 - (1 - p(0))e^{-t/n}$$

To calculate the convergence speed we compute the number of population n it takes to let the proportion $p(t)$ come arbitrarily close to 1 or $1 - \epsilon$.
we may also compute the generation at which the global optimum is expected to be found with a given probability. The probability that at least one of the strings in the population consists of all ones is given by $Prob(opt) = 1 - [1 - p^l(t)]^n$, where l =no of generations.
The convergency rate of the proportion $p(t)$ for different no generation t yields the following graph for three different popsize:

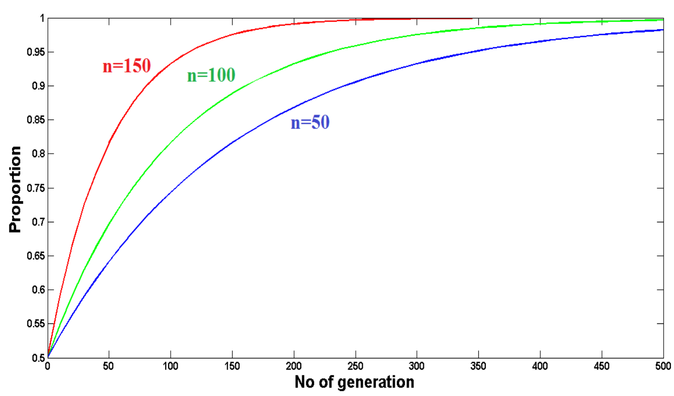

Figure 2.13: Proportion vs no of generation

2.4.4 Genetic Algorithm with Varying Population (GAVP)

Behaviour and performance of a GA is directly affected by the interaction between the parameters, i.e., selection process of chromosomes for mating pool, p_c, p_m, etc. In this Genetic Algorithm with Varying Population (GAVP) (cf. Last and Eyal [127]), a subset of better children is included with the parent population for next generation and maximum size of this subset is a percentage of the size of its parent set. To control memory overflow at the run time of the GAVP, an upper limit of population size is imposed (Maxsize). In most of the GA, movement from old population to new population takes place only when average fitness of new population is better than the old one and proved the asymptotic convergence of the algorithm by Banach fixed point theorem. In this GA with varying population size, chromosomes are classified into young, middle-age, and old according to their age and lifetime. Following comparison of fuzzy numbers using possibility theory (cf. Dubois & Prade, [61]), crossover probability is measured as a function of parents' age interval (a fuzzy rule base on parents' age limit is also used for this purpose). In this GAVP, a subset of better children is included with the parent population for next generation and maximum size of this subset is a percentage of the size of its parent set following entropy measure.

GAVP Procedures:

The GAVP procedure content the following additional and modified components:

- **Diversity Preservation:** At the time of generation of P(1) diversity is maintained using entropy originating from information theory. Following steps are used for this purpose.

  (i) Probability, pr_{jk}, that the value of the i-th gene (variable) of the j-th chromosome is different from the i-th gene of the k-th chromosome is calculated by using the formula
  $$pr_{jk} = 1 - \frac{|x_{ji} - x_{ki}|}{B_{ir} - B_{il}},$$
  where $[B_{il}, B_{ir}]$ is the variation domain of the i-th gene.

  (ii) Entropy of the i-th gene, $E_i(M)$, $i = 1, 2, \cdots, n$ is calculated using the formula
  $$E_i(M) = \sum_{j=1}^{M-1} \sum_{k=j+1}^{M} -pr_{jk} \ln pr_{jk},$$
  where M is the size of the current population.

  (iii) Average entropy of the current population is calculated by the formula
  $$E(M) = \frac{1}{n} \times \sum_{i=1}^{n} E_i(M).$$

  (iv) Incorporating the above three steps a separate sub-function check_diversity(X_i) is developed.

  Every time a new chromosome X_i is generated, the entropy between this one and previously generated individuals is calculated. If this information quantity is higher than a threshold, E_T, fixed at the beginning, X_i is included in the population otherwise X_i is again generated until diversity exceeds the threshold, E_T. This method induces a good distribution of initial population.

- **Determination of fitness and lifetime:** Value of the objective function due to the solution X_i, is taken as fitness of X_i. Let it be $Z(X_i)$. At the time of initialization age of each solution is set to zero. Following Michalewicz [150], at the time of birth life-time of X_i is computed by using the following formula:

$$\text{lifetime}(X_i) = \begin{cases} Minlt + \dfrac{K(Z(X_i) - Minfit)}{Avgfit - Minfit}, & \text{if } Avgfit \geq Z(X_i) \\ \dfrac{Minlt + Maxlt}{2} + \dfrac{K(Z(X_i) - Avgfit)}{Maxfit - Avgfit}, & \text{otherwise} \end{cases}$$

where $Maxlt$ and $Minlt$ refers to maximum and minimum allowed lifetime of a chromosome, $K = (Maxlt - Minlt)/2$. $Maxfit$, $Avgfit$, $Minfit$ represent the best, average and worst fitness of the current population. According to the age, a chromosome can belong to any one of age intervals - young, middle-aged or old.

- **Crossover:** Crossover process consists of two major sub-processes which are discussed below. For GAVP procedure the probability of crossover is based on the fuzzy rule as presented in the following Table 2.2.

Table 2.2: Fuzzy rule base for crossover probability

Parent 2	Parent 1		
	Young	Middle-aged	Old
Young	Low	Medium	Low
Middle-aged	Medium	High	Medium
Old	Low	Medium	Low

- **Reduction process of p_m:** Let $p_m(0)$ is the initial value of p_m. The probability of mutation in T-th generation $p_m(T)$ is calculated by the formula $p_m(T) = p_m(0)exp(-T/\alpha_1)$, where α_1 is calculated so that the final value of p_m is small enough (10^{-2} in our case). So α_1=Maxgen/log[$\frac{p_m(0)}{10^{-2}}$], where Maxgen is the expected number of generations that the GAVP can run for convergence.

The other components remain unchanged for the GAVP procedure.

2.4.5 Fuzzy Simulation Based Genetic Algorithm (FSGA)

To get the optimal solution of a single objective imprecise model, a fuzzy simulation based genetic algorithm (FSGA) has been used in this thesis. It contains two different simulation algorithms, one of them based on the possibility/necessity/credibility measures to satisfy the fuzzy constraints. First Algorithm is similar as Algorithm 2.1 and second Algorithm is described below.

Algorithm 2.2. *Algorithm to determine a feasible set of solutions for the maximized Z obtained from the maximization of $\tilde{f}(X)$.*
We know that Nes($\tilde{f}(x) \leq Z) \geq \mu \Rightarrow$ Pos($\tilde{f}(x) > Z) \leq 1 - \mu$. Now roughly find a point R_0 from fuzzy number $\tilde{R}$, which approximately minimizes $\tilde{f}(x)$. Let this value be z_0 (for simplicity one can take $z_0 = 0$ also) and ϵ be a positive number. Set $Z = z_0 - \epsilon$ and if Pos($\tilde{f}(x) > Z) \leq 1 - \mu$ then increase Z with ϵ. Again check Pos($\tilde{f}(x) \leq Z) \geq \mu$ and credibility condition Cr($\tilde{f}(x) \leq Z) \geq \mu \Rightarrow \frac{1}{2}\Big\{Pos(\tilde{f}(x) \leq Z) + Nes(\tilde{f}(x) \leq Z)\Big\} \geq \mu$. At this stage decrease value of ϵ and again try to improve Z. When ϵ becomes sufficiently small then we stop and final value of Z is taken as the value of Z. Using this criterion, required algorithm is developed as below. In the algorithm the variable F_0 is used to store initial assumed value of Z and F is used to store value of Z in each iteration. As like the objective function $\tilde{f}(x)$, the imprecise constraint is also converted to a deterministic constant by same manner and then use the following steps:

1. Set $Z = z_0 - \epsilon$, $F = z_0 - \epsilon$, $tol = 0.0001$.

2. Generate R_0 uniformly from the $1 - \alpha_2$ cut set of fuzzy number $\tilde{R}$.

3. Set $z_0 =$ value of $f(x)$ for $R = R_0$.

4. If $z_0 < Z$.

5. then go to step 11.

6. End If

7. Repeat step-2 to step-6 for N_2 times.

8. Set $F = Z$.

9. Set $Z = Z + \epsilon$.

10. Go to step-2.

11. If $(Z = F_0)$// In this case optimum value of $Z < z_0 - \epsilon$.

12. Set $Z = F_0 - \epsilon$, $F = F - \epsilon$, $F_0 = F_0 - \epsilon$.

13. Go to step-2.

14. End If

15. If $(\epsilon < to1)$

16. go to step-21

17. End If

18. $\epsilon = \epsilon/10$

19. $Z = F + \epsilon$

20. Go to step-2.

21. Output F.

22. End algorithm.

So for a feasible value of decision variables, we determine Z using the above algorithms and to optimize Z we use GA. When such a fuzzy simulation algorithm is used to determine Z in the algorithm, this GA is named as fuzzy simulation based genetic algorithm (FSGA). This is used to determine fuzzy objective function values.

2.4.6 Multi-Objective Genetic Algorithm (MOGA)

Multi-objective problem is solved with the Multi-objective Genetic Algorithm (MOGA) developed for this purpose. The MOGA is illustrated as follows.

We assume that there are M objective functions. In order to cover both minimization and maximization of objective functions, we use the operator between two solutions and as to denote that solution is better than solution on a particular objective. Similarly, for a particular objective implies that solution is worse than solution on this objective. For example, if an objective function is to be minimized, the operator would mean the $<$ operator, whereas if the objective function is to be maximized, the operator would mean the $>$ operator. The following definition covers mixed problems with minimization of some objective functions and maximization of the rest of them.

Dominating Criteria: A solution X_1 is said to dominate the other solution X_2, if the following both conditions (i) and (ii) are true: (i) The solution X_1 is no worse than X_2 in all objectives, or for all $j = 1, 2, ..., M$. (ii) The solution X_1 is strictly better than X_2 in at least one objective, or for at least one $j = 1, 2, .., M$. If any of the above condition is violated, the solutions X_1 does not dominate the solution X_2. If X_1 dominates the solution X_2, it is also customary to write any of the following:
i) X_2 is dominated by X_1, ii) X_1 non-dominated by X_2, iii) X_1 is non-inferior to X_2.
It is intuitive that if a solution X_1 dominates another solution X_2, the solution X_1 is better than X_2 in the parlance of multi-objective optimization. Since the concept of domination

allows a way to compare solutions with multiple objectives, most multi-objective optimization methods use this domination concept to search for non-dominated solution.

Crowding Distance: Crowding distance of a solution is measured using the following rule.

Step-1: Sort the population set according to every objective function values in ascending order of magnitude.

Step-2: For each objective function, the boundary solutions are assigned an infinite distance value. All other intermediate solutions are assigned a distance value equal to the absolute normalized difference in the function values of two adjacent solutions. This calculation is continued with other objective functions.

Step-3: The overall crowding distance value is calculated as the sum of the individual distance values corresponding to each objective.

Each objective function is normalized before calculating the crowding distance. Following algorithm is used for this purpose.

set k = number of solutions in F
for each k
{
  set $F[k]_{distance} = 0$
}
for each m
{
  sort F, in ascending order of magnitude of m-th objective
  set $F[1]_{distance} = F[m]_{distance} = M$ where M is a large number
  for $i = 2$ to $k - 1$
  {
$F[i+1]_{distance} = F[i]_{distance} + (F[i+1]_m - F[i-1]_m)/(f_m^{max} - f_m^{min})$
  }
}

Here, $F[i]_m$ refers to the m-th objective function value of $F[i]$. f_m^{max} and f_m^{min} are the maximum and minimum values of the m-th objective function.

Non-Dominated Sorting of a Population: In this case, first, for each solution we calculate two entities: i) domination count n_p, the number of solutions which dominate the solution p, and ii) S_p, a set of solutions that the solution p dominates. All solutions in the first non-dominated front will have their domination count as zero. Now, for each solution p with $n_p = 0$, we visit each member(q) of its set S_p and reduce its domination count by one. In doing so, if for any member q the domination count becomes zero, we put it in a separate list Q. These members belong to the second non-dominated front. Now, the above procedure is continued with each member of Q and the third front is identified. This process continues until all fronts are identified.

The other components are same as like single objective genetic algorithm.

Procedure of MOGA: The stepwise discussion of MOGA is as follows:

Step-1: Generate initial population P_1 of size N.

Step-2: $i \leftarrow 1$ [i represent the number of current generation.]

Step-3: Select solution from P_i for crossover.

Step-4: Made crossover on selected solution to get child set C_1.

Step-5: Select solution from P_i for mutation.

Step-6: Made mutation on selected solution to get solution set C_2.

Step-7: Set $P_i' = P_i \bigcup C_1 \bigcup C_2$.

Step-8: Partition P_i' into subsets $F_1, F_2, \cdots, F_k$, such that each subset contains non-dominated solutions of P_i' and every solutions of F_i dominates every solu.s of F_{i+1} for $i = 1, 2, \cdots, k-1$.

Step-9: Select largest possible integer l, so that no of solu.s in the set $F_1 \bigcup F_2 \bigcup \cdots \bigcup F_l \leq N$.

Step-10: Set $P_{i+1} = F_1 \bigcup F_2 \bigcup \cdots \bigcup F_l$.

Step-11: Sort F_{l+1} in decreasing order by crowding distance.

Step-12: Set M = number of solutions in P_{i+1}.

Step-13: Select first $N - M$ solutions from set F_{l+1}.

Step-14: Insert these solution in solution set P_{i+1}.

Step-15: Set $i \leftarrow i + 1$.

Step-16: If termination condition does not hold, goto step-3.

Step-17: Output P_i.

Step-18: End.

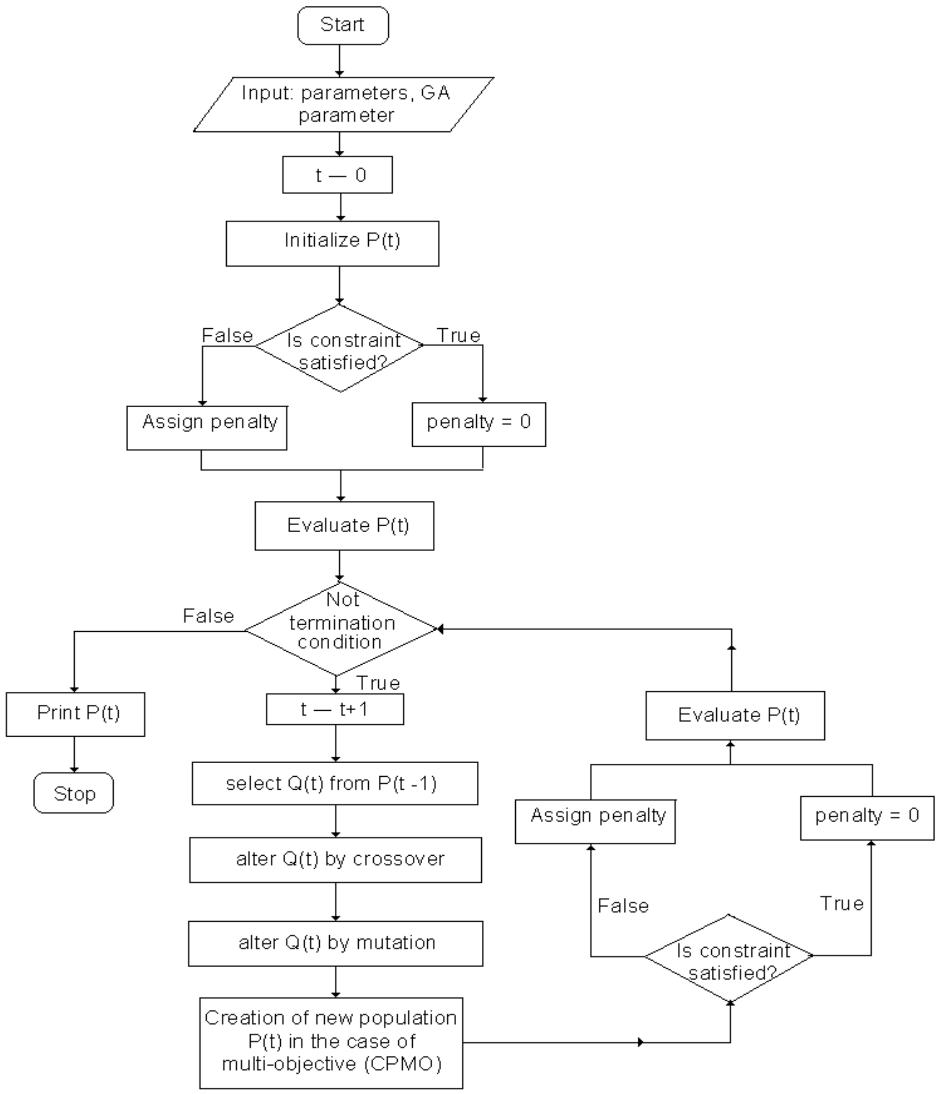

Figure 2.14: Flow-chart of MOGA

Part II

Studies on Imperfect Production Inventory System in Crisp Environment

Chapter 3

Imperfect production inventory model with production rate dependent defective rate and advertisement dependent demand [1]

3.1 Introduction

The origin of the Economic Production Quantity (EPQ) model can be traced back to 1918, when E.W. Taft made an extension to the Economic Order Quantity (EOQ) model developed by Harris [89]. The EPQ model is commonly used in the manufacturing sector of the business world to determine the optimal production quantity which optimizes the objective function in the system.

There exists a considerable amount of research works such as Cho [41], Goyal and Gunasekaran [79] in which the items to be produced in a manufacturing system had been considered as perfect. But, this is not a realistic assumption because, in any manufacturing system, the production of defective units is a natural phenomenon to occur from the different difficulties arisen in a long-run production process. Now, the defective items can be treated as a result of imperfect production. At first in 1986, Rosenblatt and Lee [181] considered such type of items in a imperfect production system. After that, Salameh and Jaber [184] presented a modified EPQ model that accounts imperfect quality items. Then Hu [96], Khan et

[1] This model published in **Computers** & **Industrial Engineering, 104 (2017) 9-22**, ELSEVIER.

al. [113], Krishnamoorthi and Panayappan [124], Tripathy and Pattnaik [208], Sivashankari and Panayappan [196], Karimi-Nasab et al. (cf. [110], [108]) and others also worked on EPQ models considering the imperfect production.

Table 3.1: Summary of related literature for EPQ/EOQ models with screening

Author(s)	EOQ /EPQ	Inspection rate	Defective rate /Deterioration rate	Screening rate & production rate	Demand rate	Production rate
Chakraborty & Giri [17]	EPQ	Constant	Constant	Equal	Constant	Constant
Cheng [32]	EPQ	Constant	Random	Equal	Constant	Constant
Chiu et al. [36]	EPQ	Constant	Random	-	Constant	Constant
Cho [41]	EPQ	-	-	-	Advertisement	Variable
Goyal & Gunasekaram [77]	EPQ	-	Constant	-	Advertisement	Variable
Goyal & Giri [79]	EPQ	-	Time varying	-	Time varying	Time varying
Hazari et al. [98]	EPQ	Constant	Reliability dependent	Equal	Advertisement	Variable
Karimi-Nasab et al. [110]	EPQ	-	Random	Equal	Constant	Variable
Lin et al. [138]	EPQ	Constant	Set up cost Dependent	Not Equal	Constant	-
Lo et al. [144]	EOQ	-	-	-	Linear trend	-
Manna et al. [147]	EPQ	Constant	Constant	Equal	Stock dependent	Variable
Manna et al. [148]	EPQ	Constant	Constant	Equal	Advertisement and price	Demand dependent
Rosenblatt & Lee [181]	EPQ	Constant	Probabilistic	Equal	Constant	
Roy et al. [182]	EOQ	Constant	Uniform	Equal	Stock dependent	
Zhang & Gerchak [241]	EOQ	Constant	Random	Equal	Constant	Constant
Present model	EPQ	Not constant	Production rate dependent	Not equal	Advertisement and time dependent	Decision variable

Now from the literature survey on imperfect production inventory models, it is seen that there exist two classes of the models on the basis of inspection methods to sort out the defective units from the perfect one. In one class of research, it has been studied that over time the produced items deteriorate in manufacturing system. In this field, the different researchers (Lee and Rosenblatt [130], Lee and Park [131], Kim et al. [119], Jaber et al. [102], Lin et al. [138], Tai [200], Rahim [171], Rahim and Ben-Daya [170], Manna et al. [148]) examined an inspection method on the produced items based on deteriorating production process. Then, Guu and Zhang [86] used an entire lot of inspection at the end of a process. After that, an inspection of the last k units had been used by Yeh and Chen [236]. On the other hand, in another class of research the imperfect production process has been investigated in which the defective units are produced during the production time due to machinery fault, labour, raw materials etc. In this field, there are also many research articles in which different inspection methods had been used by Hsu et al. [95], Cheng [32], Salameh and Jaber [184], Chiu [35], Tripathy et al. [208], Hayek and Salameh [97], Chiu et al. [36]. Zhang and Gerchak [241] used an inspection of a fraction of a lot in EOQ model. Recently, Manna et al. [147] have developed a ‘Three-layer supply chain in an imperfect production inventory model with two storage facilities under fuzzy rough environment’ with continuous screening process.

In the classical inventory model, Harris [89] considered an EOQ model with constant demand rate. Later, this model was discussed by Wilson [225]. Silver and Meal [193] extended the EOQ model for varying demand rate. Then Donaldson [59], Lo et al. [144] derived an inventory model for linear trend in demand. Other researchers such as Silver and Peterson [194], Roy et al. [182], Goyal and Giri [79] studied an inventory model with time varying demand. Karimi-Nasab et al. [109] developed a multi objective distribution-pricing model for multiperiod price-sensitive demands. In the literature there are many papers on demand. Recently, in the competitive market on a long run business system, it is seen that the demand faces a competition and sale is destroyed by depreciation. In such circumstances, the advertisement policy has a positive effect to the demand rate. The advertisement through IT by electronic media, print media, etc. improves the communication between customers and companies and it helps us to give the accurate information flow to the customer about the product of the companies. So in the present era of information technology (IT), the advertisement plays an important role to control the demand of the product in the market. For this reason, Cho [41] developed an optimal production and advertising policies in crisp environment. Recently, Hazari et al. [98] developed an imperfect production inventory model in bi-fuzzy environment with the same idea.

This chapter extends the traditional EPQ model by accounting an imperfect production inventory model with advertisement dependent demand rate in which the advertisement rate is increasing with time at a decreasing rate to recover the sale which is destroyed by depreciation and it increases the acceptability of the product in the competition market. Here, the production rate is considered as a decision variable. This chapter also considers the issue that the imperfect items are sold as a single batch at the end of 100% screening process at a reduced price. The screening rate has been considered to be different with the production rate and it has been varied proportionally to the production rate. In this chapter, a new type of defective rate has been considered depending upon the production rate. Under these considerations, a mathematical model has been developed to get the maximum profit from the system. Finally, some examples have been provided to illustrate the feasibility of the model numerically. The detailed comparative statement of the proposed model with the existing literature has been given in Table 3.1.

3.2 Notations and Assumptions

The following notations and assumptions have been used to develop the proposed model:

3.2.1 Notations

$q(t)$	:	on hand inventory of produced item (perfect and imperfect quality) in the production center.

$q_1(t)$	:	on hand inventory of perfect quality item after screening the produced item.
$q_2(t)$	:	on hand inventory of imperfect quality item after screening the produced item.
Q_0	:	minimum stock level maintained for perfect quality item.
$D(t)$	:	demand rate of the market.
x	:	percentage of screening item in the screening center per unit time.
η	:	depreciation rate.
$v(t)$	:	advertisement rate.
c_p	:	production cost per unit item.
c_{sr}	:	screening cost per unit item.
h_c	:	inventory holding cost per unit time per unit item in production center.
h_m	:	inventory holding cost per unit time per unit item of perfect quality.
h'_m	:	inventory holding cost per unit time per unit item of imperfect quality.
s	:	selling price per unit for perfect quality item.
s'	:	selling price per unit for imperfect quality item.
P	:	production rate of the manufacturer ($P \geq D(t)$).
T	:	total time-length of the business period.
t_1	:	duration of production run time.
t_2	:	duration of screening run time.
θ	:	defective production rate, i.e., the rate of producing defective units.

3.2.2 Assumptions

(i) It is a single item production inventory model in infinite time horizon.

(ii) Here, it is assumed that the manufacturer has capability to collect sufficient raw materials, labours, machines and other related resources to produce the item. So, he/she wants to produce the optimum amount of item per unit time to get the maximum profit from his/her business. In this sense, here the production rate (P) has been assumed to be a variable.

(iii) It is well known that the quality of a product depends on raw material, labour experience, machine component, production rate etc. Since here the production rate has been considered as a variable, hence the defective rate of the produced items must be dependent on the production rate. For this reason, the defective production rate (θ) has been considered as follows:

$$\theta = \theta_0 - \frac{\theta_1}{P} \tag{3.1}$$

where θ_0 and θ_1 be the positive constants.

(iv) Due to the existence of defective production, there exist some imperfect quality items in the production center. Hence, the manufacturer decides to sale the perfect quality item after sorting the items in the inventory. So in this chapter, it is assumed that during the production period, the screening process has been occurred simultaneously and the screening rate is less than or equal to production rate but greater than or equal to demand rate.

(v) Unit production cost depends on produced-quantity, raw material, wear-tear and development costs and is of the form, $c_p(P) = c_r + \frac{L}{P} + MP$; where c_r is raw material cost, L is labor charge, and M is tool wear and tear cost respectively. This assumption is based on the fact that c_r, the raw material cost is independent of production rate. Also, $\frac{L}{P}$ is a per unit cost component that decreases with the increased in production rate. MP is a per unit cost component that increases with production rate and includes tool cost and rework cost that might result from increased tool at higher production rates.

(vi) In this chapter, since the production rate (P) has been considered as a variable according to assumption (ii), hence in the manufacturing system there may not be shortages to avoid the shortage cost in a profit maximization problem in which the demand is time dependent. In this sense, here shortages are not allowed.

(vii) The initial and ending inventory levels of perfect quality item are not restricted to zero. i.e., the minimum stock level (Q_0) is maintained initially and at ending business period which is treated as a safety stock.

(viii) Normally, the demand of an item is considered as a constant or variable. When it is taken as a variable, then it may be a function of time and/or initial stock or on-hand inventory. In the present time, it is seen that the sale of a product depends upon the promotion of the product in public life. So the advertisement has an important role in increasing of the demand of a commodity. In that sense, here we have considered the advertisement dependent demand function of the perfect quality items. In the same time, from the market survey it is also observed that for the coming of other brands of the same product, there is a declination of demand rate. Incorporating this idea, there is a consideration of depreciation rate η in demand function at time t in $0 < t < T$. Therefore the rate of change of demand of an item satisfies the following differential equation:

$$\dot{D}(t) = v(t) - \eta D(t), \text{ with } D(0) = D_0. \tag{3.2}$$

where $v(t) = v_0 - v_1 e^{-v_2 t},\ 0 \leq t \leq T$, where v_0, v_1 and v_2 are known parameters and $0 < \eta < 1$.
Therefore the demand function $D(t)$ is obtained as

$$D(t) = D_0 e^{-\eta t} + \frac{v_0}{\eta}(1 - e^{-\eta t}) + \frac{v_1}{v_2 - \eta}(e^{-v_2 t} - e^{-\eta t}),\ \ 0 \leq t \leq T \tag{3.3}$$

which gives the demand rate of the item.

Lemma 3.1. *The demand rate D(t) increases with respect to time provided that the parameters* v_0, v_1, v_2, η, *and* D_0 *satisfy the relations* $\eta > v_2$ *and* $\frac{v_0}{\eta} - \frac{v_1}{\eta - v_2} > D_0$.

Proof. Now, the demand rate increases with respect to time, if $\dot{D}(t) > 0$, so from equation (3.2), we have $v(t) - \eta D(t) > 0$ with $D(0) = D_0$. After simplifying, it is obtained that
$(v_0 - v_1 e^{-v_2 t}) - \eta\Big[D_0 e^{-\eta t} + \frac{v_0}{\eta}(1 - e^{-\eta t}) + \frac{v_1}{v_2-\eta}(e^{-v_2 t} - e^{-\eta t})\Big] > 0$, [by equation (3.3)]
i.e., $\frac{v_1 v_2}{\eta - v_2} + \eta\Big[\frac{v_0}{\eta} + \frac{v_1}{v_2-\eta} - D_0\Big]e^{(v_2-\eta)t} > 0$
which is possible when $\eta > v_2$ and $\frac{v_0}{\eta} - \frac{v_1}{\eta - v_2} > D_0$ for $0 < t < T$.
Now, the proof is complete. □

Lemma 3.2. *The advertisement rate* $v(t) = v_0 - v_1 e^{-v_2 t}$*;* v_0*,* v_1*, and* $v_2 > 0$ *is an increasing from* $(v_0 - v_1)$ *at a decreasing rate.*

Proof. Here $v(t) = v_0 - v_1 e^{-v_2 t}$ and $v(0) = (v_0 - v_1)$ is the constant advertisement rate before the production start. Since $v(0) > 0$, so $v_0 > v_1$.
Therefore, $\frac{d}{dt}v(t) = v_1 v_2 e^{-v_2 t} > 0$, since v_1 and $v_2 > 0$.
i.e., $\dot{v}(t) = \frac{d}{dt}v(t) > 0$ which shows that $v(t)$ is an increasing with time.
Also, $\frac{d}{dt}\dot{v}(t) = \frac{d^2}{dt^2}v(t) = -v_1 v_2^2 e^{-v_2 t} < 0$ which shows that $\dot{v}(t)$ is decreasing with time.
i.e., rate of increasing of $v(t)$ is decreasing with time.
Hence $v(t)$ is an increasing from $(v_0 - v_1)$ at a decreasing rate.
Now, the proof is complete. □

Lemma 3.3. *The demand rate* $D(t)$ *is an increasing with time at a decreasing rate.*

Proof. Here $D(t) = D_0 e^{-\eta t} + \frac{v_0}{\eta}(1 - e^{-\eta t}) + \frac{v_1}{v_2-\eta}(e^{-v_2 t} - e^{-\eta t})$, [by equation (3.3)]
$\dot{D}(t) = (\frac{v_0}{\eta} - D_0 - \frac{v_1}{\eta - v_2})\eta e^{-\eta t} + \frac{v_1 v_2}{\eta - v_2} e^{-v_2 t} > 0$
$\frac{d}{dt}\dot{D}(t) = -(\frac{v_0}{\eta} - D_0 - \frac{v_1}{\eta - v_2})\eta^2 e^{-\eta t} - \frac{v_1 v_2^2}{\eta - v_2} e^{-v_2 t} < 0$, [By Lemma 3.1]
which shows that $\dot{D}(t)$ is decreasing with time.
Again since $\dot{D}(t) > 0$ so $D(t)$ is increasing. Hence $D(t)$ is an increasing and the rate of increasing is decreasing with time. Now, the proof is complete. □

Lemma 3.4. *The defective production rate* (θ) *is an increasing with respect to production rate* P*.*

Proof. From the assumption (iii) and equation (3.1), the defective production rate (θ) is the function of production rate (P) such as $\theta = \theta_0 - \frac{\theta_1}{P}$, where θ_0, θ_1 are positive constants.

Differentiating θ with respect to P, we get

$\frac{d\theta}{dP} = \frac{\theta_1}{P^2} > 0$, for all P and $\theta_1 > 0$.

Hence the defective rate θ increases with the increase of the production rate P.

Now, the proof is complete. □

The change of θ with respect to P is depicted in Figure 3.1.

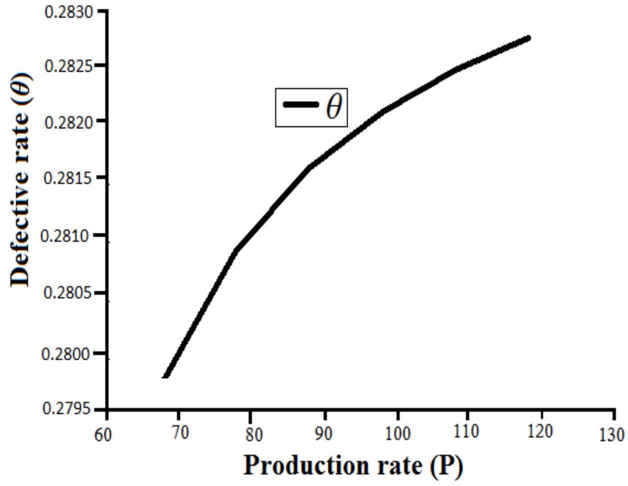

Figure 3.1: Graphical representation of the production rate vs defective rate

Lemma 3.5. *The rate of change of defective production rate of items decreases with the increase of the production rate P.*

Proof. From the assumption (iii) and equation (3.1), the defective production rate (θ) as the function of production rate (P) such as $\theta = \theta_0 - \frac{\theta_1}{P}$, where θ_0, θ_1 are positive constants.

Differentiating θ with respect to P, we gate

$\frac{d\theta}{dP} = \frac{\theta_1}{P^2} > 0$, for all P and $\frac{d^2\theta}{dP^2} = -\frac{2\theta_1}{P^3} < 0$, for all $P > 0$ and $\theta_1 > 0$.

Therefore $\frac{d\theta}{dP}$ is monotonic decreasing function of P.

Hence the rate of change of defective production rate of the items decreases with the increase of the production rate P. Now, the proof is complete. □

3.3 Mathematical Formulation of the Proposed Model

It is an imperfect production inventory model involving a manufacturer which produces and sells the item during the business time period T. The manufacturer produces the items at the rate of P up to a period t_1. Amongst the items there must exist perfect quality items including some imperfect quality items. Here the screening process has been considered to sortout up to time t_2 at a screening rate (xP) such that demand of perfect quality item is fulfilled during the business period at a time T. Since the production rate (P) is greater than the screening rate (xP) then the extra quantity $(P - xP)$ per unit time will be stocked in screening center up to a period t_2. Also, the rate of stocking the perfect quality item after screening is $(1-\theta)xP$ which must be greater than the customer demand rate $D(t)$. After fulfilling the customer demand at each instant, the extra quantity $(1 - \theta)xP - D(t)$ per unit time will be stocked in perfect quality item center up to a period T. A stock of perfect quality items, Q_0 is maintained initially and finally during the business period. Here the defective items will be stocked at the rate θxP up to the screening period t_2. At the end of the screening process, defective items are sold in a single lot at a reduced price s'. The following Figure 3.2 shows the flow of items in the production system.

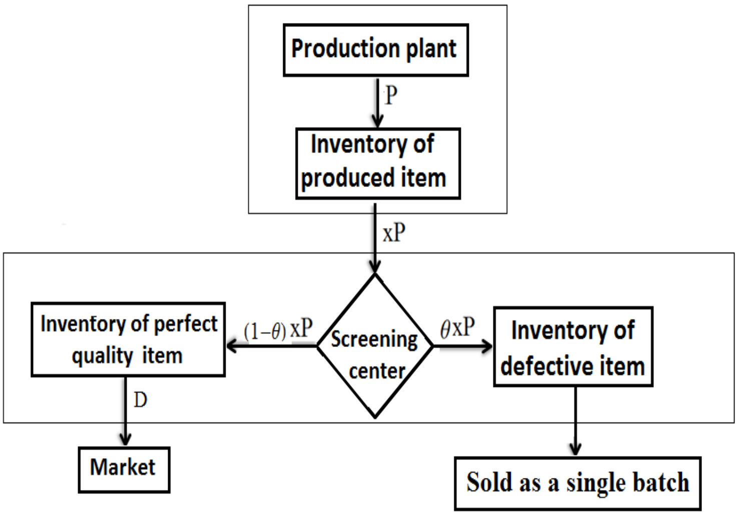

Figure 3.2: Schematic representation of the production inventory model

3.3.1 Formulation of Inventory Level of Produced Item

In this production system, production starts at t=0 with production rate(P) and continues up to $t = t_1$. The produced products continuously are transferred to the screening cell at the rate xP.

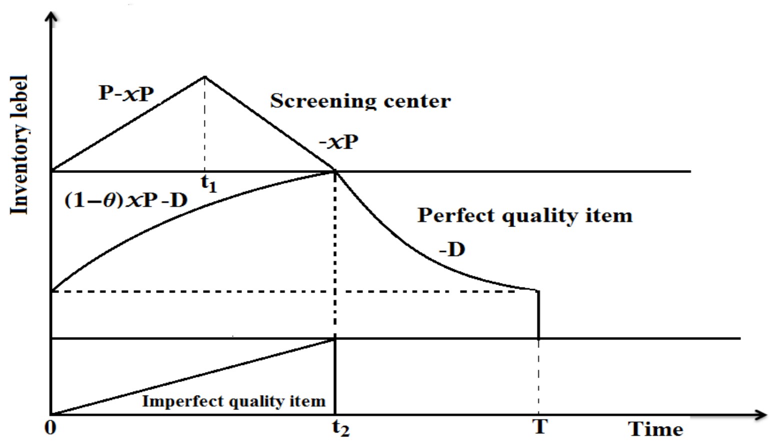

Figure 3.3: Pictorial representation of production inventory model

So, the inventory level $q(t)$ in production center is described as follows

$$\frac{dq(t)}{dt} = \begin{cases} P - xP, & 0 \le t \le t_1 \\ -xP, & t_1 \le t \le t_2 \end{cases} \tag{3.4}$$

subject to the conditions that $q(0) = 0$, $q(t_1) = Q$ and $q(t_2) = 0$.
Consequently, the solutions of the above differential equations (3.4) are given by

$$q(t) = \begin{cases} (1 - x)Pt, & 0 \le t \le t_1 \\ xP(t_2 - t), & t_1 \le t \le t_2 \end{cases} \tag{3.5}$$

Using the condition $q(t_1) = Q$ and equation (3.5), it is obtained that

$$Q = (1 - x)Pt_1 = xP(t_2 - t_1)$$

Thus, we have $(1 - x)Pt_1 = xP(t_2 - t_1)$, which implies that t_1 is a function of x and t_2 as

$$t_1 = xt_2 \tag{3.6}$$

3.3.2 Formulation of Inventory Level of Perfect Quality Item

In this screening system, the screening starts with the commencement of production at the rate (xP) and continues up to time t_2. Here 100% screening is performed on the produced products but screening rate (xP) must always be greater than or equal to the demand rate $(D(t))$. The screening cell separates the items in two categories: one is perfect and other is imperfect. After the screening, the perfect quality items meet the demand with a rate $D(t)$ during business period. Here, we consider the inventory level $q_1(t)$ and the safety stock Q_0 of perfect quality items.

So, the inventory level $q_1(t)$ of perfect quality items in the screening center is described as follows:

$$\frac{dq_1(t)}{dt} = \begin{cases} (1-\theta)xP - D(t), & 0 \le t \le t_2 \\ -D(t), & t_2 \le t \le T \end{cases} \tag{3.7}$$

subject to the conditions that $q_1(0) = Q_0$, $q_1(T) = Q_0$.
Consequently, the solutions of the above differential equations (3.7) are given by

$$q_1(t) = \begin{cases} Q_0 + (1-\theta)xPt - \Big[\frac{D_0}{\eta}(1-e^{-\eta t}) + \frac{v_0}{\eta^2}(\eta t + e^{-\eta t} - 1) \\ \quad + \frac{v_1}{v_2-\eta}\{\frac{1}{v_2}(1-e^{-v_2 t}) - \frac{1}{\eta}(1-e^{-\eta t})\}\Big], & 0 \le t \le t_2 \\ Q_0 - \Big[\frac{D_0}{\eta}(e^{-\eta T} - e^{-\eta t}) + \frac{v_0}{\eta^2}(\eta(t-T) + e^{-\eta t} - e^{-\eta T}) \\ \quad + \frac{v_1}{v_2-\eta}\{\frac{1}{v_2}(e^{-v_2 T} - e^{-v_2 t}) - \frac{1}{\eta}(e^{-\eta T} - e^{-\eta t})\}\Big], & t_2 \le t \le T \end{cases}$$

Lemma 3.6. *In a manufacturing system, at any time during the screening period* $(0, t_2)$ *the inventory of the perfect quality items must be greater than the customer demand at that time, i.e., there does not occur any shortages provided that* $(1-\theta)xP\eta > v_0$ *and* $\eta Q_0 + \frac{v_1}{v_2} > D_0$.

Proof. $q_1(t)$ is the inventory of perfect quality items and $D(t)$ is the customer demand at time t. Now

$$q_1(t) - D(t) = (\frac{1}{\eta} - 1)D(t) + \{(1-\theta)xP - \frac{v_0}{\eta}\}t + \frac{1}{\eta}(\eta Q_0 - D_0 + \frac{v_1}{v_2})$$

According to assumption $(viii)$, we have $0 < \eta < 1$. i.e., $\frac{1}{\eta} > 1$. i.e., $\frac{1}{\eta} - 1 > 0$.
Therefore, using the above relation we have $q_1(t) - D(t) > 0$ for all t if $(1-\theta)xP - \frac{v_0}{\eta} > 0$ and $\eta Q_0 - D_0 + \frac{v_1}{v_2} > 0$. i.e., if $(1-\theta)xP\eta > v_0$ and $\eta Q_0 + \frac{v_1}{v_2} > D_0$.
Now, the proof is complete. □

Lemma 3.7. *In a manufacturing system, the screening period* (t_2) *must satisfy the following relation in terms of production rate* (P)*, production period* (t_1) *and business period* (T)*.*

$$t_2 = \frac{1}{(1-\theta)xP}\Big[\frac{D_0 - v_0\eta}{\eta^2}(1-e^{-\eta T}) + \frac{v_0 T}{\eta} + \frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2}(1-e^{-v_2 T}) - \frac{1}{\eta}(1-e^{-\eta T})\Big\}\Big]$$

Proof. Satisfying the continuity condition of $q_1(t)$ at $t = t_2$ a relation is obtained as follows

$$Q_0 + (1-\theta)xPt_2 - \frac{D_0}{\eta}(1-e^{-\eta t_2}) - \frac{v_0}{\eta^2}(\eta t_2 + e^{-\eta t_2} - 1) - \frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2}(1-e^{-v_2 t_2}) - \frac{1}{\eta}(1-e^{-\eta t_2})\Big\} = Q_0 - \Big[\frac{D_0}{\eta}(e^{-\eta T} - e^{-\eta t_2}) + \frac{v_0}{\eta^2}(\eta(t_2 - T) + e^{-\eta t_2} - e^{-\eta T}) + \frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2}(e^{-v_2 T} - e^{-v_2 t_2}) - \frac{1}{\eta}(e^{-\eta T} - e^{-\eta t_2})\Big\}\Big]$$

i.e., $$t_2 = \frac{1}{(1-\theta)xP}\Big[\frac{D_0}{\eta}(1-e^{-\eta T}) + \frac{v_0}{\eta^2}(\eta T + e^{-\eta T} - 1) + \frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2}(1-e^{-v_2 T}) - \frac{1}{\eta}(1-e^{-\eta T})\Big\}\Big]. \tag{3.8}$$

Now, the proof is complete. □

3.3.3 Formulation of Inventory Level of Imperfect Quality Item

At the end of the screening process, all imperfect quality items are sold as a single lot. So the inventory level $q_2(t)$ of imperfect quality items in the screening center is described as follows:

$$\frac{dq_2(t)}{dt} = \theta xP, \quad 0 \le t \le t_2 \tag{3.9}$$

subject to the conditions that $q_2(0) = 0$ and $q_2(t_2) = 0$.
Consequently, the solution of the above differential equation (3.9) is given by

$$q_2(t) = \theta xPt, \quad 0 \le t \le t_2$$

3.3.4 The Profit Function of the Proposed Model

Total production cost (PC) in the production system during the cycle $(0, T)$ is given by

$$PC = c_p \int_0^{t_1} P\,dt = c_p P t_1$$

Total screening cost (SC) in the production system during the cycle $(0, T)$ is given by

$$SC = c_{sr} \int_0^{t_2} xP\,dt = c_{sr} xP t_2$$

Total advertizement cost (AC) in the production system during the cycle $(0, T)$ is given by

$$AC = c_a \int_0^{T} v(t)\,dt = c_a\Big[v_0 T + \frac{v_1}{v_2}(e^{-v_2 T} - 1)\Big]$$

Total holding cost (HC) in the production system during the cycle $(0, T)$ is given by

$$
\begin{aligned}
HC &= h_c \int_0^{t_1} q(t)\,dt + h_m \int_0^{T} q_1(t)\,dt + h'_m \int_0^{t_2} q_2(t)\,dt \\
&= \frac{h_c}{2}\Big\{(1-x)Pt_1^2 + xP(t_2-t_1)^2\Big\} + \frac{h_m}{2}\Big[Q_0T + (1-\theta)xP\frac{t_2^2}{2} - \frac{D_0}{\eta^2}(\eta t_2 + e^{-\eta t_2} - 1) \\
&-\frac{v_0}{\eta^3}(\eta^2\frac{t_2^2}{2} - e^{-\eta t_2} + 1 - \eta t_2) - \frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2^2}(t_2v_2 - 1) - \frac{1}{\eta^2}(\eta t_2 - 1)\Big\} \\
&-\frac{D_0}{\eta^2}\Big((1+T-t_2)e^{-\eta T} - e^{-\eta t_2}\Big) - \frac{v_0}{\eta^3}\Big(-\frac{\eta^2}{2}(t_2-T)^2 + e^{-\eta t_2} - (1+T-t_2)e^{-\eta T}\Big) \\
&-\frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2^2}(1+v_2T-v_2t_2)e^{-v_2T} - \frac{1}{\eta^2}(1+\eta T-\eta t_2)e^{-\eta T}\Big\}\Big] + \frac{h'_m}{2}\theta xPt_2^2
\end{aligned}
$$

Total set up cost in the production system during the cycle $(0, T)$ $=A_m$
Total revenue (TR) in the production system during the cycle $(0, T)$ is given by

$$
\begin{aligned}
TR &= s\int_0^T D(t)\,dt + s'\theta xPt_2 \\
&= s\Big[\frac{D_0\eta - v_0}{\eta^2}(1-e^{-\eta T}) + \frac{v_0T}{\eta} + \frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2}(1-e^{-v_2T}) - \frac{1}{\eta}(1-e^{-\eta T})\Big\}\Big] + s'\theta xPt_2
\end{aligned}
$$

Total Profit $(TP(P, T))$ in the production system during the cycle $(0, T)$ is given by

$$
\begin{aligned}
TP(P,T) &= TR - PC - SC - HC - AC - A_m \\
&= s\Big[\frac{D_0}{\eta}(1-e^{-\eta T}) + \frac{v_0}{\eta^2}(\eta T + e^{-\eta T} - 1) + \frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2}(1-e^{-v_2T}) - \frac{1}{\eta}(1-e^{-\eta T})\Big\}\Big] \\
&+s'\theta xPt_2 - c_pPt_1 - c_{sr}xPt_2 - \frac{h_c}{2}\Big\{(1-x)Pt_1^2 + xP(t_2-t_1)^2\Big\} \\
&-\frac{h_m}{2}\Big[Q_0t_2 + (1-\theta)xP\frac{t_2^2}{2} - \frac{D_0}{\eta^2}(\eta t_2 + e^{-\eta t_2} - 1) - \frac{v_0}{\eta^3}(\frac{\eta^2t_2^2}{2} - e^{-\eta t_2} + 1 - \eta t_2) \\
&-\frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2^2}(t_2v_2 + e^{-v_2t_2} - 1) - \frac{1}{\eta^2}(\eta t_2 + e^{-\eta t_2} - 1)\Big\} + Q_0(T-t_2) \\
&-\frac{D_0}{\eta^2}\Big((1+T-t_2)e^{-\eta T} - e^{-\eta t_2}\Big) - \frac{v_0}{\eta^3}\Big(-\frac{\eta^2}{2}(t_2-T)^2 + e^{-\eta t_2} - (1+T-t_2)e^{-\eta T}\Big) \\
&-\frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2^2}\Big((1+v_2T-v_2t_2)e^{-v_2T} - e^{-v_2t_2}\Big) - \frac{1}{\eta^2}\Big((1+\eta T-\eta t_2)e^{-\eta T} - e^{-\eta t_2}\Big)\Big\}\Big] \\
&-\frac{h'_m}{2}\theta xPt_2^2 - c_a\Big[v_0T + \frac{v_1}{v_2}(e^{-v_2T} - 1)\Big] - \frac{A_m}{T}
\end{aligned}
$$

Average Profit $AP(P,T)$ in the production system during the cycle $(0,T)$ is given by

$$\begin{aligned}
AP(P,T) &= \frac{1}{T}\Big[TP(P,T)\Big] \\
&= \frac{s}{T}\Big[\frac{D_0}{\eta}(1-e^{-\eta T}) + \frac{v_0}{\eta^2}(\eta T + e^{-\eta T} - 1) + \frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2}(1-e^{-v_2 T}) - \frac{1}{\eta}(1-e^{-\eta T})\Big\}\Big] \\
&+ \frac{x}{T}\Big\{s'\theta - c_p - c_{sr}\Big\}Pt_2 - \frac{x}{2T}\Big\{(1-x)h_c + \theta h'_m\Big\}Pt_2^2 - \frac{h_m}{2T}\Big[Q_0 T + (1-\theta)xP\frac{t_2^2}{2} \\
&- \frac{1}{\eta^3}(D_0\eta - v_0)\Big\{\eta t_2 - 1 + (1+T-t_2)e^{-\eta T}\Big\} - \frac{v_0}{2\eta}(2t_2 T - T^2) \\
&- \frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2^2}\{v_2 t_2 - 1 + (1+v_2 T - v_2 t_2)e^{-v_2 T}\} - \frac{1}{\eta^2}\{\eta t_2 - 1 \\
&+ (1+\eta T - \eta t_2)e^{-\eta T}\}\Big\}\Big] - \frac{c_a}{T}\Big[v_0 T + \frac{v_1}{v_2}(e^{-v_2 T} - 1)\Big] - \frac{A_m}{T} \qquad (3.10)
\end{aligned}$$

Proposition 3.1. *Average profit $AP(P,T)$ is a function of two independent variables P and T. Now, it is supposed that for some parametric values involved in the system of equations $\frac{\partial}{\partial P}\{AP(P,T)\} = 0$ and $\frac{\partial}{\partial T}\{AP(P,T)\} = 0$, there exists at least one positive point (P^*,T^*) at which (i) $\frac{\partial^2}{\partial P^2}\{AP(P,T)\}$ and $\frac{\partial^2}{\partial T^2}\{AP(P,T)\}$ both are negative and (ii) $\frac{\partial^2}{\partial P^2}\{AP(P,T)\}$ $\frac{\partial^2}{\partial T^2}\{AP(P,T)\}$-$[\frac{\partial^2}{\partial T\partial P}\{AP(P,T)\}]^2 > 0$ then we obtain the maximum average profit at the point (P^*,T^*).*

Here, it is considered that $\frac{\partial}{\partial P}\{AP(P,T)\} = F(P,T)$ and $\frac{\partial}{\partial T}\{AP(P,T)\} = G(P,T)$, (see appendix A). Due to complexity of the equations, $F(P,T) = 0$ and $G(P,T) = 0$, it is not possible to show the existence of the solution analytically. Now it is supposed that there exists at least one positive point (P^*,T^*) for which $F(P^*,T^*) = 0$ and $G(P^*,T^*) = 0$ for some parametric values involved in the system. Let at (P^*,T^*), $\frac{\partial}{\partial P}\{F(P,T)\} = \Delta_1$, $\frac{\partial}{\partial T}\{G(P,T)\} = \Delta_2$ and $\frac{\partial}{\partial T}\{F(P,T)\} = \Delta_3$.

Lemma 3.8. *The maximum average profit $AP(P^*,T^*)$ exist if $\Delta_1 < 0$, $\Delta_2 < 0$ and $\Delta_1\Delta_2 > \Delta_3^2$.*

Proof. Now, from the optimization calculus, it is known that a function of two variables, $\phi(u,v)$ is maximum at the stationary point (a,b) if $\frac{\partial^2}{\partial u^2}\{\phi(a,b)\}$ $\frac{\partial^2}{\partial v^2}\{\phi(a,b)\}$-$[\frac{\partial^2}{\partial u\partial v}\{\phi(a,b)\}]^2 > 0$, $\frac{\partial^2}{\partial u^2}\{\phi(a,b)\} < 0$ and $\frac{\partial^2}{\partial v^2}\{\phi(a,b)\} < 0$.
Here $P = P^*$ and $T = T^*$ are the solution of $\frac{\partial}{\partial P}\{AP(P,T)\} = 0$ and $\frac{\partial}{\partial T}\{AP(P,T)\} = 0$.
Now if $\Delta_1 < 0$ then $\frac{\partial}{\partial P}\{F(P^*,T^*)\} < 0$, since $\frac{\partial}{\partial P}\{F(P,T)\} = \Delta_1$ at (P^*,T^*).
i.e., $\frac{\partial^2}{\partial P^2}\{AP(P,T)\} < 0$ at (P^*,T^*), since $F(P,T) = \frac{\partial}{\partial P}\{AP(P,T)\}$.

Again if $\Delta_2 < 0$ then $\frac{\partial}{\partial T}\{G(P^*, T^*)\} < 0$, since $\frac{\partial}{\partial T}\{G(P, T)\} = \Delta_2$ at (P^*, T^*).
i.e., $\frac{\partial^2}{\partial T^2}\{AP(P, T)\} < 0$ at (P^*, T^*), since $G(P, T) = \frac{\partial}{\partial T}\{AP(P, T)\}$.
Again if $\Delta_1\Delta_2 > \Delta_3^2$ then $\{\frac{\partial}{\partial P}F(P, T)\}\{\frac{\partial}{\partial T}G(P, T)\} - \frac{\partial}{\partial T}\{F(P, T)\} > 0$, since at (P^*, T^*), $\Delta_1 = \frac{\partial}{\partial P}\{F(P, T)\}$, $\Delta_2 = \frac{\partial}{\partial T}\{G(P, T)\}$ and $\Delta_3 = \frac{\partial}{\partial T}\{F(P, T)\}$.
i.e., $\frac{\partial^2}{\partial P^2}\{AP(P, T)\}\frac{\partial^2}{\partial T^2}\{AP(P, T)\}$-$[\frac{\partial^2}{\partial P\partial T}\{AP(P, T)\}]^2 > 0$ at (P^*, T^*), since $F(P, T) = \frac{\partial}{\partial P}\{AP(P, T)\}$ and $G(P, T) = \frac{\partial}{\partial T}\{AP(P, T)\}$. Therefore allc onditions (Proposition 3.1) for the existence of maximum value of $AP(P, T)$ at the point (P^*, T^*) have been satisfied. So the maximum average profit $AP(P^*, T^*)$ exist if $\Delta_1 < 0$, $\Delta_2 < 0$ and $\Delta_1\Delta_2 > \Delta_3^2$. Now, the proof is complete. □

Lemma 3.9. *If $\Delta_1 > 0$ and $\Delta_2 < 0$ then the maximum average profit $AP(P^*, T^*)$ does not exist.*

Proof. Here $P = P^*$ and $T = T^*$ are the solution of $\frac{\partial}{\partial P}\{AP(P, T)\} = 0$ and $\frac{\partial}{\partial T}\{AP(P, T)\} = 0$.
Now if $\Delta_1 > 0$ and $\Delta_2 < 0$ then $\frac{\partial}{\partial P}\{F(P^*, T^*)\} > 0$ and $\frac{\partial}{\partial T}\{G(P^*, T^*)\} < 0$,
since $\Delta_1 = \frac{\partial}{\partial P}\{F(P, T)\}$ and $\Delta_2 = \frac{\partial}{\partial T}\{G(P, T)\}$ at (P^*, T^*).
i.e., $\frac{\partial^2}{\partial P^2}\{AP(P, T)\} > 0$ and $\frac{\partial^2}{\partial T^2}\{AP(P, T)\} < 0$ at (P^*, T^*), since $F(P, T) = \frac{\partial}{\partial P}\{AP(P, T)\}$ and $G(P, T) = \frac{\partial}{\partial T}\{AP(P, T)\}$. Therefore one condition (Proposition 3.1) for the existence of maximum value of $AP(P, T)$ at the point (P^*, T^*) is not satisfy. So if $\Delta_1 > 0$ and $\Delta_2 < 0$ then there does not exist the maximum average profit $AP(P^*, T^*)$. Now, the proof is complete. □

Lemma 3.10. *If $\Delta_1 < 0$ and $\Delta_2 > 0$ then the maximum average profit $AP(P^*, T^*)$ does not exist.*

Proof. The proof is similar to that in Lemma 3.9. Hence we omit it. □

Lemma 3.11. *If $\Delta_1\Delta_2 - \Delta_3^2 < 0$ then the maximum average profit $AP(P^*, T^*)$ does not exist.*

Proof. Here $P = P^*$ and $T = T^*$ are the solution of $\frac{\partial}{\partial P}\{AP(P, T)\} = 0$ and $\frac{\partial}{\partial T}\{AP(P, T)\} = 0$.
Now, if $\Delta_1\Delta_2 - \Delta_3^2 < 0$ then $\{\frac{\partial}{\partial P}F(P, T)\}\{\frac{\partial}{\partial T}G(P, T)\} - \frac{\partial}{\partial T}\{F(P, T)\} < 0$, since at

(P^*, T^*), $\Delta_1 = \frac{\partial}{\partial P}\{F(P,T)\}$, $\Delta_2 = \frac{\partial}{\partial T}\{G(P,T)\}$ and $\Delta_3 = \frac{\partial}{\partial T}\{F(P,T)\}$.
i.e., $\frac{\partial^2}{\partial P^2}\{AP(P,T)\}\frac{\partial^2}{\partial T^2}\{AP(P,T)\}$-$[\frac{\partial^2}{\partial P \partial T}\{AP(P,T)\}]^2 < 0$ at (P^*, T^*), since $F(P,T) = \frac{\partial}{\partial P}\{AP(P,T)\}$ and $G(P,T) = \frac{\partial}{\partial T}\{AP\ (P,\ T\ )\}$. Therefore one condition (Proposition 3:1) for the existence of maximum value of $AP(P,T)$ at the point (P^*, T^*) has been not satisfied. So, if $\Delta_1\Delta_2 - \Delta_3^2 < 0$ then there does not exist the maximum average profit $AP(P^*, T^*)$. Now, the proof is complete. □

3.4 Solution Procedure

From equation (3.10) it is seen that in the proposed model, the objective function AP is highly nonlinear. Here P and T are two decision variables. θ is a function of P according to equation (3.1). Also t_2 is a function of P and T obtained according to equation (3.8) in Lemma 3.7. t_1 is a function of t_2 obtained according to equation (3.6). Since the objective function is highly nonlinear, hence to get the optimal solution of the proposed model the following algorithms have been developed.

Algorithm 3.1. *For a fixed x, suppose $x = x_0$, the value of y can be obtained from $\psi(x, y) = 0$ as follows:*

step 1*: For $x = x_0$, compute $\psi(x_0, y) = 0$.*

step 2*: Select (y_1, y_2) such that $\psi(x_0, y_1)\psi(x_0, y_2) < 0$. Then by Rolle's theorem there exist a root of $\psi(x_0, y) = 0$, between y_1 and y_2.*

step 3*: Calculate $m = \frac{(y_1+y_2)}{2}$, be the midpoint of the interval (y_1, y_2).*

step 4*: Compute the signs of $\psi(x_0, y_1)$, $\psi(x_0, m)$, and $\psi(x_0, y_2)$.*

step 5*: If $\psi(x_0, y_1)\psi(x_0, m) < 0$, then a root of $\psi(x_0, y) = 0$ lies between y_1 and m. In this case, replace m by y_2. Otherwise, a root of $\psi(x_0, y) = 0$ lies between m and y_2, then replace m by y_1.*

step 6*: Repeat steps 3 through 5 until $|y_1 - y_2| < 10^{-\varepsilon}$ where ε is a tolerance limit.*

step 7*: Then the root of $\psi(x_0, y) = 0$ is m such that $m = \frac{(y_1+y_2)}{2}$.*

Algorithm 3.2. *Since there is no possibilities to get the general explicit solution due to absence of linearness, hence to get the maximum profit of the proposed model following procedure has been devised according to Lemma* 3.8*. Here, the optimal values of P, T, θ, t_1, t_2 and $AP(P,T)$ are denoted by P^*, T^*, θ^*, t_1^*, t_2^* and $AP^*(P^*,T^*)$ respectively.*

step 1*: Set an interval (P_{10}, P_{11}) where $P_{10} \in (0, P_0)$ and $P_{11} \in (0, P_0)$. Here $P \leq P_0$ where P_0 also is initialized.*

step 2*: Compute T_{0F}, T_{1F}, T_{0G} and T_{1G} for T from $F(P_{10},T) = 0$, $F(P_{11},T) = 0$, $G(P_{10},T) = 0$ and $G(P_{11},T) = 0$ respectively by Algorithm* 3.1.

step 3*: Compute $\triangle_{P_{10}} = T_{0F} - T_{0G}$ and $\triangle_{P_{11}} = T_{1F} - T_{1G}$.*

step 4*: If $\triangle_{P_{10}}\triangle_{P_{11}} < 0$, i.e., the signs of $\triangle_{P_{10}}$ and $\triangle_{P_{11}}$ are opposite, then compute $P_{1m} = \frac{(P_{10}+P_{11})}{2}$.*

step 5*: Compute T_{1mF} and T_{1mG} for T from $F(P_{1m},T) = 0$ and $G(P_{1m},T) = 0$ respectively by Algorithm* 3.1.

step 6*: Calculate $\triangle_{P_{1m}} = \triangle_{T_{1mF}} - \triangle_{T_{1mG}}$.*

step 7*: Compare $\triangle_{P_{1m}}$ with $\triangle_{P_{10}}$. If $\triangle_{P_{10}}\triangle_{P_{1m}} < 0$, i.e., the signs of $\triangle_{P_{10}}$ and $\triangle_{P_{1m}}$ are opposite, then replace P_{11} by P_{1m}. Otherwise replace P_{10} by P_{1m}.*

step 8*: Repeat steps* 4 *through* 7 *until the absolute values of $(P_{10} - P_{1m})$ or $(\triangle_{P_{10}} - \triangle_{P_{1m}})$ or $(\triangle_{P_{10}} - \triangle_{P_{1m}})$ are within the tolerance limits.*

step 9*: The root of $F(P,T) = 0$ and $G(P,T) = 0$ is (P^r, T^r) where $P^r = P_{1m}$ and $T^r = \frac{T_{0F}+T_{1F}}{2}$ or $\frac{T_{0G}+T_{1G}}{2}$.*

step 10*: Compute $\triangle_1$, $\triangle_2$ and $\triangle_3$ at the point (P^r, T^r) where $\Delta_1 = \frac{\partial}{\partial P}\{F(P,T)\}$, $\Delta_2 = \frac{\partial}{\partial T}\{G(P,T)\}$ and $\Delta_3 = \frac{\partial}{\partial T}\{F(P,T)\}$.*

step 11*: If $\Delta_1 < 0$, $\Delta_2 < 0$ and $\Delta_1\Delta_2 > \Delta_3^2$, then according Lemma* 3.8 *then (P^r, T^r) be the optimal solution. So $P^* = P^r$, $T^* = T^r$ and calculate t_2^* by equation* (3.8)*, t_1^* by equation* (3.6)*, θ^* by equation* (3.1)*. Also calculate $AP^*(P^*,T^*)$ by equation* (3.10).

step 12: *If* $\Delta_1 > 0$, $\Delta_2 < 0$ *by Lemma* 3.9, $\Delta_1 < 0$, *or* $\Delta_2 > 0$ *by Lemma* 3.10, *or* $\Delta_1\Delta_2 - \Delta_3^2 < 0$ *by Lemma* 3.11, *then* (P^r, T^r) *is not optimal solution. In this case, goto step* 2 *and change the value of* (P_{10}, P_{11}).

step 13: *Print the optimal values* P^*, T^*, θ^* t_1^*, t_2^* *and* $AP^*(P^*, T^*)$.

3.5 Numerical Illustrations

In this section, we illustrate some numerical examples to study the feasibility of the proposed imperfect production inventory model. The values of the parameters of the model, considered in these numerical examples, are not elected from any real life case study, but these values have been seemed to be realistic. All these examples have been solved to find the optimal values of production period (t_1), screening time (t_2), defective rate (θ), production rate (P) and business period (T) along with the optimal average profit $(AP(P, T))$ of the system.

Example 3.1. *In this model, the manufacturer's perspective is to maximize the average profit* $AP(P, T)$. *To get the maximum average profit, a manufacturer has taken different parametric values to be shown in Table 3.2 and 3.3.*

Table 3.2: Values of screening, defective, demand, advertisement and other parameters

Parameter	x	θ_0	θ_1	$D_0\,(unit)$	η	v_0	v_1	v_2	$Q_0\,(unit)$
Value	0.80	0.33	0.59	21	0.27	12	11	0.25	32

Table 3.3: Values of cost parameters and selling prices of items per unit

Parameter	c_r	c_{sr}	L	M	c_a	h_c	h_m	h'_m	A_m	s	s'
Value ($)	25	0.28	400	0.06	6	0.4	0.60	0.20	400	75	49

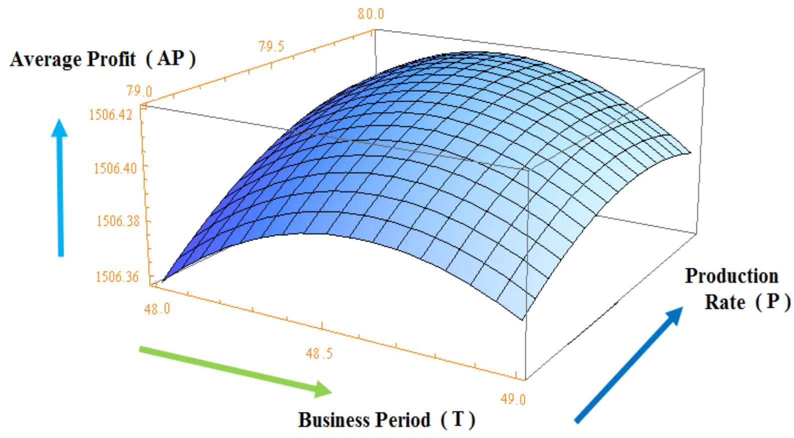

Figure 3.4: Concavity of the average profit (AP) for numerical example 1.1

Numerically for Example 3.1, Figure 3.4 shows the graphical representation of the average profit function of P and T. From this figure, it is guarantied that the average profit function is concave. So there exists a unique solution of (P, T) that maximizes the average profit $AP(P, T)$. Now, to get the optimum computational result (Table 3.4) for the non-linear profit function $AP(P, T)$ using Table 3.2 and 3.3, our developed Algorithms 3.1 and 3.2 have been used.

Table 3.4: Optimal result of the illustrated model

Production rate (P^*)	Defective rate (θ^*)	Production time (t_1^*)	Screening time (t_2^*)	Business period (T^*)	Average Profit $(AP^*(P^*, T^*))$
79.39	0.28	35.66	44.57	48.78	1506.42

From Table 3.4, it is observed that the manufacturer gets the optimum average profit of amount \$ 1506.42 when the production rate and production time of the system are 79.39 unit and 35.66 unit respectively. Here defective rate is 0.28. Screening is performed up to time 44.57 unit. For this example the business ends after 48.78 unit. The Figure 3.5 shows the graphical representation of the inventory of perfect quality items after the screening and the demand rate from which it is implied that there does not occur shortages during the screening period for this example.

Example 3.2. *In this example, using the same data as in Example* 3.1 *except the safety stock* (Q_0) *of perfect quality item to be considered here, as zero and following result in Table 3.5 has obtained:*

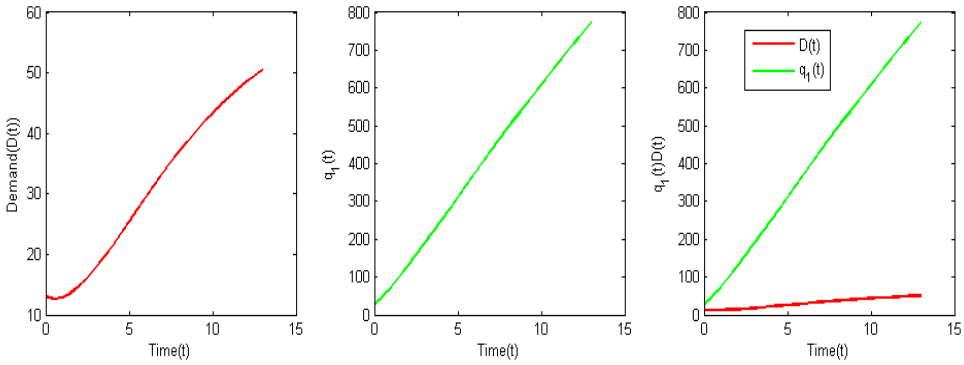

Figure 3.5: Comparison between the inventory of perfect quality items and the demand rate

Table 3.5: Optimal results of the illustrated model when $Q_0 = 0$

Production rate (P^*)	Defective rate (θ^*)	Production time (t_1^*)	Screening time (t_2^*)	Business period (T^*)	Average Profit $(AP^*(P^*, T^*))$
79.39	0.28	35.66	44.57	48.78	1516.03

From Table 3.5, it is explored that production rate (P^*), defective rate (θ^*), production time (t_1^*), screening time (t_2^*) and total time (T^*) are not changed from the result in Example 3.1 due to safety stock $Q_0 = 0$ but the average profit $(AP^*(P^*, T^*))$ has increased because total holding cost has reduced by the absence of safety stock (Q_0).

Example 3.3. *In this example, using the same data as in Example* 3.1 *except screening rate* (x)*, here the production rate and screening rate have been considered as equal, i.e.,* $P = xP$ *which implies that* $x = 1$ *and* $t_1 = t_2$ *by equation* (3.8).

The corresponding computational results are shown in Table 3.6.

Table 3.6: Optimal results of the illustrated model when $x = 1$

Production rate (P^*)	Defective rate (θ^*)	Production time (t_1^*)	Screening time (t_2^*)	Business period (T^*)	Average Profit $(AP^*(P^*, T^*))$
63.72	0.27	54.46	54.46	58.66	1608.62

From Table 3.6, it is seen that production rate (P^*), defective rate (θ^*), production time (t_1^*), screening time (t_2^*), business period (T^*) and average profit $(AP^*(P^*, T^*))$ have changed from

Example 3.1, due to the change of screening rate (x). Here production rate is less than that of Example 3.1 due to increase of screening rate, since the stock place is fixed and demand is independent of screening rate. Since defective rate (θ^*) is dependent on production rate and production rate decreases then defective rate also decreases. Again production time (t_1^*) and total time (T^*) decrease due to the decrease of production rate (P^*). Simultaneously screening time (t_2^*) is decreased due to the increase of screening rate. Ultimately, the average profit ($AP^*(P^*, T^*)$) has been increased due to the equality of production rate and screening rate because of decrease of total holding cost.

Example 3.4. *In this example, using the same data as in Example* 3.1 *except the advertisement parameters* (v_0, v_1, v_2, c_a)*, here all parameters associated with the advertisement have been considered as zero, i.e., we do not consider advertisement dependent demand.*

In this case, we get the following optimum results in Table 3.7.

Table 3.7: Optimal results of the illustrated model when $v_0 = 0, v_1 = 0, v_2 = 0, c_a = 0$

Production rate (P^*)	Defective rate (θ^*)	Production time (t_1^*)	Screening time (t_2^*)	Business period (T^*)	Average Profit ($AP^*(P^*, T^*)$)
110.20	0.31	0.78	0.98	5.18	438.95

Since here the advertisement is not considered, hence the rate of change of demand decreases with respect to time. Henceforth, the business period (T^*) and average profit ($AP^*(P^*, T^*)$) are less than those of Example 3.1 which are shown in Table 3.7.

3.5.1 Sensitivity Analysis

In this section, we examine the effect of changes in the system parameters to study the sensitivity analysis of the proposed model with respect to some parameters based on the preceding numerical Example 3.1 as follows.

Sensitivity analysis 3.1. *Using the numerical Example* 3.1 *mentioned earlier, the effect of under or over estimation of various parameters on replenishment policy to maximize the net profit has been studied.*

Here, we employ, $\triangle x = (x' - x)/x' \times 100\%$, $\triangle \eta = (\eta' - \eta)/\eta' \times 100\%$, $\triangle P^* = (P^{*'} - P^*)/P'^* \times 100\%$, $\triangle T^* = (T^{*'} - T^*)/T'^* \times 100\%$, $\triangle AP^*(P^*, T^*) = \{AP^{*'}(P^*, T^*) - AP^*(P^*, T^*)\}/AP'^*(P^*, T^*) \times 100\%$ as measure of sensitivity, where x, η, P^*, T^*, $AP^*(P^*,T^*)$ be the true values and x', η', $P^{*'}$, $T^{*'}$, $AP^{*'}(P^*,T^*)$ be also the corresponding estimated values. The sensitivity analysis has been shown by increasing or decreasing the parameters x and η by different percentage(%), taking one or more at a time with keeping the others at their true values. The results are presented in following Table 3.8 and Table 3.9.

Table 3.8: Sensitivity analysis of the illustrated model w.r.t. screening rate

Percentage of change of screening rate ($\triangle x$)	Percentage of change of production rate ($\triangle P^*$)	Percentage of change of defective rate ($\triangle \theta^*$)	Percentage of change of total time ($\triangle T^*$)	Percentage of change of average profit ($\triangle AP^*(P^*,T^*)$)
-25	+31.88	+0.39	-14.13	-7.20
-10	+10.65	+0.16	-6.02	-2.63
+10	-8.71	-0.15	+6.84	+2.45
+20	-15.97	-0.30	+14.87	+4.83
+25	-19.16	-0.38	+19.50	+6.01

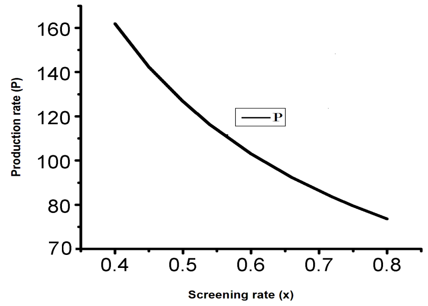

Figure 3.6: Screening rate vs production rate

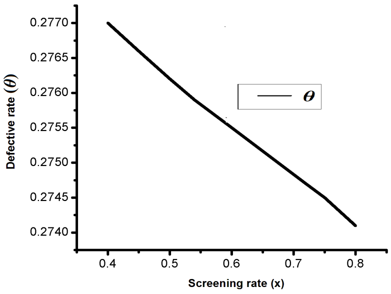

Figure 3.7: Screening rate vs defective rate

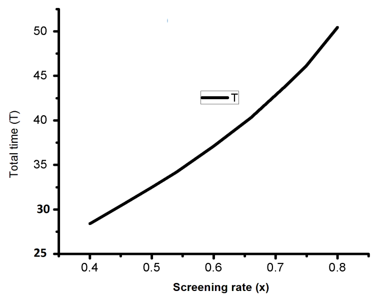

Figure 3.8: Screening rate vs total time

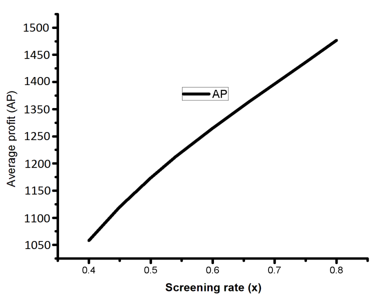

Figure 3.9: Screening rate vs average profit

• Here from Table 3.8, Figure 3.6 and 3.7 it is observed that production rate (P^*) and defective rate (θ^*) are both decreases respectively due to the increasing of screening rate (x). Also from Table 3.8, Figure 3.8 and 3.9 it is also seen that total time (T^*) and average profit $(AP^*(P^*, T^*))$ both are increases respectively due to the increasing of screening rate (x).
If screening rate increases, then business period and average profit also increase but production rate and defective rate decreases due to limited holding space of perfect items and production rate dependent defective rate respectively.

Table 3.9: Sensitivity analysis of the illustrated model w.r.t. depreciation rate of demand

Percentage of change of depreciation rate $(\triangle\eta)$	Percentage of change of production rate $(\triangle P^*)$	Percentage of change of defective rate $(\triangle\theta^*)$	Percentage of change of total time $(\triangle T^*)$	Percentage of change of average profit $(\triangle AP^*(P^*, T^*))$
-20	+22.06	+0.29	+4.89	+19.18
-15	+15.68	+0.22	+3.61	+13.83
-5	+4.73	+0.07	+1.15	+4.28
+5	-4.32	-0.07	-1.11	-3.98
+15	-8.29	-0.15	-2.18	-7.70
+20	-15.31	-0.29	-4.20	-14.43

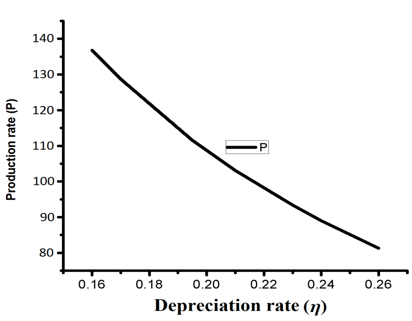

Figure 3.10: Depreciation rate vs production

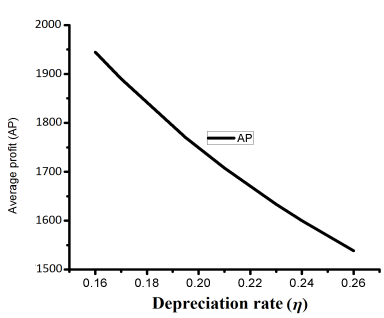

Figure 3.11: Depreciation rate vs average profit

• Here from Table 3.9, Figure 3.10, 3.11, 3.12 and 3.13 it is observed that production rate (P^*), defective rate (θ^*), total time (T^*) and average profit $(AP^*(P^*, T^*))$ are decreasing respectively due to increasing of depreciation rate (η).

Sensitivity analysis 3.2. *In this case, using the same data as those in Example* 3.1 *except production cost parameters* L *and* M*. Here we experiment for the different values of* L *and* M*. The results are given in following Table 3.10.*

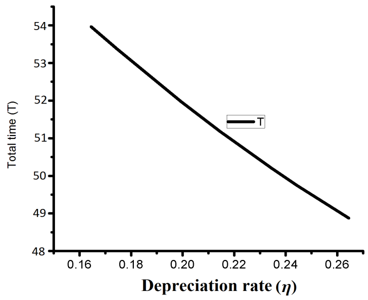

Figure 3.12: Depreciation rate vs total time

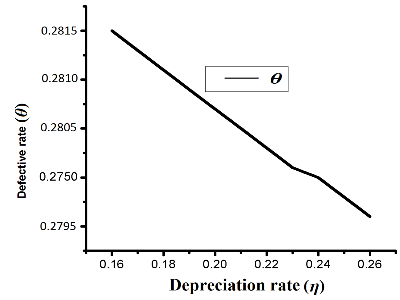

Figure 3.13: Depreciation rate vs defective rate

Table 3.10: Sensitivity analysis of the illustrated model when L and M changes

Parameters $(L,\ M)$	Production rate (P^{*})	Defective rate (θ^{*})	Production time (t_1^{*})	Screening time (t_2^{*})	Business period (T^{*})	Average Profit $(AP^{*}(P^{*},T^{*}))$
L=300, M=0.08	79.34	0.278	34.93	43.67	47.87	1487.44
L=350, M=0.07	79.36	0.279	35.30	44.12	48.32	1496.91
L=400, M=0.06	79.39	0.280	35.66	44.57	48.78	1506.42
L=450, M=0.05	84.28	0.285	33.46	41.82	48.58	1516.80
L=500, M=0.04	101.22	0.290	27.20	34.01	47.49	1538.92

• Table 3.10 shows that when L **increases and** M **decreases simultaneously**, the production rate (P^{*}), defective rate (θ^{*}) and average profit $(AP^{*}(P^{*},T^{*}))$ are increasing respectively but production period t_1^{*}, screening period t_2^{*} and business period T^{*} initially increase, after that decrease.

Sensitivity analysis 3.3. *In this case, using the same data as those in Example* 3.1 *except production cost parameters* c_a *and* v_0. *Here we experiment for the different values of* c_a *and* v_0. *The results are given in following Table 3.11.*

Table 3.11: Sensitivity analysis of the illustrated model when c_a and v_0 changes

Parameters $(c_a,\ v_0)$	Production rate (P^{*})	Defective rate (θ^{*})	Production time (t_1^{*})	Screening time (t_2^{*})	Business period (T^{*})	Average Profit $(AP^{*}(P^{*},T^{*}))$
$c_a = 5,\ v_0 = 11$	72.47	0.274	36.51	45.63	49.83	1379.53
$c_a = 6,\ v_0 = 12$	79.39	0.280	35.66	44.57	48.78	1506.42
$c_a = 7,\ v_0 = 13$	86.33	0.285	34.91	43.63	47.83	1627.24
$c_a = 8,\ v_0 = 14$	93.27	0.290	34.22	42.78	46.98	1741.95
$c_a = 9,\ v_0 = 14$	93.27	0.290	34.16	42.70	46.90	1728.89

• Table 3.11 shows that when c_a **and** v_0 **increase simultaneously**, the production rate (P^*), defective rate (θ^*) and average profit ($AP^*(P^*, T^*)$) are also increasing respectively but production period t_1^*, screening period t_2^* and business period T^* are decrease.

3.6 Conclusion

In this chapter, an imperfect production inventory model has been considered in which the screening of produced items has been incorporated with the advertisement dependent demand. Normally, such type of model is very useful for the product of saree (cloths), the commodities made by brass etc. specially for developing countries. Here advertisement and demand rate both are increasing with time at a decreasing rate. A realistic approach of screening has been adopted in which the screening rate is less than or equal to the production rate as well as greater than the demand rate. Further, the rate of producing defective items has been considered as a function of production rate. Then, to get the optimal production period, production rate, rate of producing defective items and business period, the profit function has been optimized by developing algorithms. Finally, the numerical illustrations with sensitivity analysis have been given to study the feasibility of the proposed model simulating the effect of changes in the various parameters involved in the objective function. This generalization is vital to accommodate many more real world complications. So, from this study the following conclusions can be drawn:

(i) The advertisement has a good impact to increase the customer demand.

(ii) Increasing the screening rate, the average profit can be increased by decreasing the defective rate through less production. So, it can be concluded that screening rate and production rate can not be increased simultaneously to get the optimal profit.

(iii) When depreciation rate increases, the selling rate (customer demand) decreases. Henceforth, production rate, total business period and average profit decrease.

(iv) When the advertisement cost and demand rate of customer increases simultaneously, initially the average profit increases, after that the average profit decreases. Because at first advertisement attracts more customers. As a result, the demand rate increases. But later, advertisement cost increases, the rate of demand does not increase as much as previous due to market saturation. So from this study, any manufacturer company can find the optimal advertisement rate and corresponding cost.

(v) If the salary of the labour (i.e., labour cost) is increased then the tool material cost is decreased due to the sincere care of the labour. So, it is concluded that when the labour cost is increased, then the average profit is also increased due to the more decrease to the tool material cost.

Part III

Studies on Imperfect Production Inventory Systems in Stochastic Environment

Chapter 4

An EPQ model with promotional demand in random planning horizon: population varying genetic algorithm approach[1]

4.1 Introduction

One of the key features of an Economic Production Quantity (EPQ) model is that the quality level is not subject to control, i.e., the defective items sometimes may also be produced during the production run time. These defective items may be discarded or sold at a reduced price (cf. Das and Maiti [48], Gumasta et al. [83]) or may be repaired / reworked (cf. Manna et al. [147]) and then that items are reached to the customers. In 2000, Salameh and Jaber [184] developed an EPQ model with imperfect quality items. Then Goyal and Cardenas-Barron [78] extended the EPQ model of Salameh and Jaber [184] considering the rework of imperfect items. Khouja and Mehrez [114] and Khouja [115] extended the imperfect production process with flexible production rates to reduce the imperfect rate. Husseini et al. [92] and Sana et al. [185, 186] also discussed the volume of flexibility policy in production. In practice, it is observed that the screening process is essentially required to sort out the imperfect items which are to be reworked. So, in most of the existing literature, the imperfect quantities are either rejected or fully screened in a single cycle for a single item. **But, till now, no one has considered a multi-item multi-cycle EPQ model for imperfect quality items with rework**.

Moreover, the inflation and time value of money play an important role in long time

[1]This model published in **Journal of Intelligent Manufacturing**, SPRINGER, 27(1), 2016, 1-17.

business concerns, especially in the developing countries. Therefore, the effect of inflation and time value of money can not be ignored in real situations. To relax the assumption of inflation, Buzacott [12] and Dey et al. ([55], [57]) simultaneously developed an EOQ model with a constant inflation rate to all associated costs. **From these literature survey, it is known that no one has considered inflation and time value money together in a imperfect production inventory model.**

In the present competitive market, to increase the sale of items, the inventory/stock is decoratively displayed to attract the customers. In this context, many researchers considered the stock dependent demand (cf. Chan et al. [20, 22], Das et al. [49]) instead of constant demand (cf. Dey et al. [55], Jana et al. [105]). Wadhwa et al. [209] also established the impact of product availability for impulse demand. All these factors are virtually depending on selling price and mode or regularity of the advertisement. Basically, the demand rate may change due to different market conditions. But, there may be some dedicated customers who are accustomed to a particular brand and they wait for that product until it is available. Therefore, the demand of any item must have a minimum value. So, the demand of an existing product can be considered as $D = A + B$, where A represents the minimum requirement from the loyal customers who buy this product regularly and B represents the varying demand from the disloyal customers. The volume of these disloyal customers depends on the attractive contract terms, e.g. long-term relationships, quantity discounts, low selling price for high quality items, terms of payment and delivery. Another relevant and important scenarios are often accompanied by an intensive promotion campaign or advertisement. In all the existing literature, the different demand parametric values are taken by intuitions (Barron and Sana [13]). **But, till now, no one has considered the estimation of different demand parameters from market survey in imperfect production inventory model**.

Normally, it is seen that the production inventory models are developed in infinite planning horizon (cf. Wang and Chan [218]) because it is assumed that in production inventory process, the related various parameters remain constant over the future infinite time. But, in reality, it is not correct due to several reasons such as variation of inventory costs, changes of product specifications and designs, technological changes due to environmental conditions, availability of product, etc. Moreover, for seasonal products like fruits, vegetables, warm garments, etc. business period may not be finite. Das et al. [49], Ali et al. [5] and others supported this idea. Rather for seasonal products (like, different type of juice, medicines,etc) the planning horizon is not fixed, it varies over the years and may be considered as a random variable with some probability. Moon [155] considered such type of horizon with exponential distribution for an EOQ model. Then, a lot of research works has been done in this field (cf. Roy et al. [182]). Very recently, Jana et al. [105] considered the random planning horizon for an EOQ model with shortages. **But, none have considered the random planning horizon for a imperfect production rework model**. The detailed comparative statement of the proposed model with the existing literature has been given in Table 4.1.

Table 4.1: Summary of related literature for multi-item EOQ/EPQ models

Author(s)	EOQ /EPQ	Production rate	Demand rate	Reworked of imperfect items	Inflation and Time value of money	Multi -item
Das et al. [49]	EPQ	constant	stock dependent	No	No	No
Jana et al. [105]	EOQ	NA	stock dependent	NA	Yes	Yes
Roy et al. [182]	EOQ	NA	stock dependent	NA	Yes	No
Yang et al. [232]	EOQ	NA	stock dependent	NA	No	No
Taleizadeh & Nematollahi [203]	EOQ	NA	uniform	NA	No	No
Present model	EPQ	Demand dependent	advertise and selling price dependent	Yes	Yes	Yes

In this chapter, a multi-item production inventory model with promotional effected demand under space constraints has been formulated. In the underlying EPQ model, the production rate has been considered on the basis of demand of the market and the production cost is taken as Khouja function (Khouja [115]) in the random planning horizon. The present study provides the rework facility of the unsatisfactory products. A constant rate of inflation is also introduced in the uncertain time scale. **To our knowledge, these issues have not been considered together by any researcher earlier. Moreover, here we perform a numerical study from a market survey with a statistical test which enhances the feasibility of the model.**

4.2 Notations and Assumptions

The following notations and assumptions have been used to developing the proposed model:

4.2.1 Notations

The following notations are used throughout the entire chapter (for the j-th item).

$q_i^j(t)$: On hand inventory of i-th cycle at time t for perfect quality, $(i = 1, 2, ..., N^j)$.
$q_L^j(t)$: On hand inventory of last cycle at time t for perfect quality.
Q^j : Maximum inventory level for perfect quality.
P^j : The production rate of each cycle.
ν^j : Advertisement effort.
D^j : Demand rate of each cycle for perfect quality.
N^j : Number of fully accommodated cycles to be made during the prescribed time horizon.

T^j	:	Duration of a complete cycle.
t_1^j	:	Duration of production in each cycle.
β^j	:	Percentage of perfect quality.
δ^j	:	Percentage of reworking to be perfect quality items from imperfect quality items.
c_p^j	:	Production cost per unit.
c_{sr}^j	:	Screening cost per unit item.
r_c^j	:	Reworking cost per unit item.
h_c^j	:	The inventory holding cost per unit time for perfect quality item.
s^j	:	Selling price per unit for perfect quality item.
r	:	The discount rate.
k	:	The inflation rate which is varied by the social economical situations.
R	:	(=r-k) The discount rate minus the inflation rate.
H	:	The length of time horizon.
M	:	Total number of items.
a^j	:	Space required for storing j-th item per unit.
B	:	Total space available in the system.

4.2.2 Assumptions

The following assumptions have been used to develop this model.

(i) In this production system, the multiple items are produced in the form of perfect and imperfect quality. The perfect quality items are directly ready for sale and some of the imperfect quality items are reworked to make as good as perfect. The rest imperfect quality items which may be too much expensive to make as perfect quality items, are disposed.

(ii) In this model the production rate (P^j) has been considered as $P^j = P_0^j + P_1^j D^j$ where (a) P_0^j = Minimum production (which is constant) per unit time and (b) $P_1^j D^j$ = The portion of the production which varies directly from the market demand per unit time.

(iii) Here, demand of the items are inversely and directly proportional to the selling price and advertisement respectively with constant coefficients which are estimated from the market survey. The mathematical form of the demand rate of each item is: $D^j(\nu^j, s^j) = D_0^j + D_1^j(s^j) + D_2^j(\nu^j)$, where (a) D_0^j = minimum demand(which is independent of advertisement effort and selling price) for each cycle. (b) $D_1^j(s^j) = A^j \frac{s_{max}^j - s^j}{s^j - s_{min}^j}$ where A^j is positive constant and $s^j \in [s_{min}^j, s_{max}^j]$ for each j. Here $\frac{d}{ds^j} D_1^j(s^j) = A^j \frac{s_{min}^j - s_{max}^j}{(s^j - s_{min}^j)^2} < 0$. This shows that the demand $D_1^j(s^j)$ is decreasing function of the selling price s^j. When $s^j = s_{max}^j$ then $D_1^j(s^j) = 0$ and when $s^j = s_{min}^j$ then the market has unlimited demand. i.e., $D_1^j(s^j) \in [0, \infty)$, for $s^j \in [s_{min}^j, s_{max}^j]$ for j=1,2,...,M.

(c) $D_2^j(\nu^j) = \kappa^j(1 - \frac{1}{\nu^j+1})$ where κ^j is positive constant for each j and it is estimated from previous data by a curve fitting method. Here $\frac{d}{d\nu^j}D_2^j(\nu^j) = \frac{\kappa^j}{(\nu^j+1)^2} > 0$ and $\frac{d^2}{d\nu^{j2}}D_2^j(\nu^j) = \frac{-2\kappa^j}{(\nu^j+1)^3} < 0$ which shows that $D_2^j(\nu^j)$ is increasing with ν^j and the rate of increasing is decreasing for each j. When $\nu^j = 0$, $D_2^j(\nu^j) = 0$ and $\nu^j \to \infty$, $D_2^j(\nu^j) \to \kappa^j$. i.e., the demand is κ^j (which is maximum) for unlimited advertisement effort. i.e., $D_2^j(\nu^j) \in [0, \kappa^j]$, $\nu^j \in [0, \infty)$ for j=1,2,...,M.

(iv) The ratio of perfect and imperfect quantity items and the rework rate are constant through out the time horizon.

(v) The length of time horizon, H is a random variable and the corresponding real value is h which follows an exponential distribution with probability density function (p.d.f)

$$f(h) = \begin{cases} \lambda e^{-\lambda h}\,, & h \geq 0 \\ 0\,, & otherwise \end{cases} \tag{4.1}$$

Here, $\lambda(> 0)$ is the parameter of the distribution.

(vi) The time horizon completely accommodates first N^j cycles and end during $(N^j + 1)$th cycle.

(vii) Lead time is negligible and shortages are not allowed.

(viii) The effects of inflation and time value of money have been considered.

(ix) The initial and terminal inventory levels in each cycle are zero.

(x) The promotional effort is an important management strategy to introduce a new product to the customer when it is lunched to the market. The promotional efforts are free gift, discount offer, delivery facilities, better services and advertising etc. Now-a-days, this strategy is applied for any management system.

(xi) Per unit production cost follows Khouja [115] function and it is of the form: $c_p^j(P^j) = c_r^j + \frac{L^j}{(P^j)^{\eta_j}} + (P^j)^{\rho_j}M^j$; where c_r^j is the raw material cost which is independent of production rate. L^j is the labor charge and M^j is the tool wear and tear cost respectively. Here, η^j and ρ^j are so chosen to best fit the production cost function. Also, $\frac{L^j}{(P^j)^{\eta_j}}$ is a per unit cost component that decreases with increase of the production rate. $(P^j)^{\rho_j}M^j$ is a per unit cost component that increases with the production rate and that may result from the increased tool at a higher production rate.

4.3 Mathematical Formulation of the Proposed Model

In this model, simultaneously more than one item are produced in both perfect and imperfect forms. The produced items are screened 100% and the repairable items are reworked and returned to the inventory of the good items. The perfect quality items are disposed to the customers as per their demand rate (as discussed in assumption (iii)). Such EPQ model is formulated in random time horizon scale which is taken as exponential distribution. The model also assumes the existence of a pair of mutually exclusive events for the last cycle. For the development of this model, we assume the time horizon H (random variable) to accommodate completely N^j cycles of equal time period T^j, for j-th item.

4.3.1 Formulation for i-th $(1 \leq i \leq N^j)$ Cycle of j-th Item

In this case, the initial stock of the i-th cycle is zero and it starts the production with a rate P^j. As the production and rework continues, the inventory begins to pile up continuously after meeting the demand of rate (D^j) of the customers. The production of the i-th cycle continues upto the time $(i-1)T^j + t_1^j$ and again it starts at time iT^j for next cycle. Each cycle ends with zero inventory. During the cycle period $[(i-1)T^j, iT^j]$, the inventory rate increases with the production rate of perfect item and the rework rate of the imperfect item as well as it decreases along with the demand rate.
Hence, the differential equation to describe the inventory level $q_i^j(t) \in (i-1)T^j \leq t \leq iT^j$ is given by

$$\frac{dq_i^j(t)}{dt} = \begin{cases} \beta^j P^j + \delta^j(1-\beta^j)P^j - D^j, & (i-1)T^j \leq t \leq (i-1)T^j + t_1^j \\ -D^j, & (i-1)T^j + t_1^j \leq t \leq iT^j \end{cases}$$

subject to the boundary conditions: $q_i^j[(i-1)T^j] = 0$ and $q_i^j[iT^j] = 0$
The above differential equation is linear with unit integrating factor. Therefore, the inventory level is given by the solution:

$$q_i^j(t) = \begin{cases} \Big[\{\beta^j + \delta^j(1-\beta^j)\}(P_0^j + P_1^j D^j) \\ -D^j\Big][t-(i-1)T^j], & (i-1)T^j \leq t \leq (i-1)T^j + t_1^j \\ D^j(iT^j - t), & (i-1)T^j + t_1^j \leq t \leq iT^j \end{cases} \tag{4.2}$$

The above $q_i^j(t)$ is a continuous function on the defined range of interval $[(i-1)T^j, iT^j]$, therefore

$$q_i^j((i-1)T^j + t_1^j - 0) = q_i^j((i-1)T^j + t_1^j + 0)$$
$$\text{i.e.,}\{\beta^j + \delta^j(1-\beta^j)\}(P_0^j + P_1^j D^j)t_1^j = D^j T^j \tag{4.3}$$

Therefore, the present value of production cost of the jth item during the ith $(1 \leq i \leq N^j)$ cycle is given by

$$PC_i^j = c_p^j \int_{(i-1)T^j}^{(i-1)T^j + t_1^j} P^j e^{-Rt}\, dt = \frac{c_p^j}{R}(P_0^j + P_1^j D^j)(1 - e^{-Rt_1^j})e^{-R(i-1)T^j}$$

Present value of screening cost of the ith $(1 \leq i \leq N^j)$ cycle is given by

$$SC_i^j = c_{sr}^j \int_{(i-1)T^j}^{(i-1)T^j+t_1^j} P^j e^{-Rt}\, dt = \frac{c_{sr}^j}{R}(P_0^j + P_1^j D^j)(1 - e^{-Rt_1^j})e^{-R(i-1)T^j}$$

Present value of reworked cost of the ith $(1 \leq i \leq N^j)$ cycle is given by

$$\begin{aligned} RC_i^j &= r_c^j \int_{(i-1)T^j}^{(i-1)T^j+t_1^j} \delta^j(1-\beta)P^j e^{-Rt}\, dt \\ &= \frac{r_c^j}{R}\delta^j(1-\beta^j)(P_0^j + P_1^j D^j)(1 - e^{-Rt_1^j})e^{-R(i-1)T^j} \end{aligned}$$

Present value of holding cost of the inventory for the ith $(1 \leq i \leq N^j)$ cycle is given by

$$\begin{aligned} HC_i^j &= h_c^j[\int_{(i-1)T^j}^{(i-1)T^j+t_1^j} q_i^j(t)e^{-Rt}\, dt + \int_{(i-1)T^j+t_1^j}^{iT^j} q_i^j(t)e^{-Rt}\, dt] \\ &= \frac{h_c^j}{R^2}\Big[\{\beta^j + \delta^j(1-\beta^j)\}(P_0^j + P_1^j D^j) - D^j\Big]\Big\{1 - (1 + Rt_1^j)e^{-Rt_1^j}\Big\}e^{-R(i-1)T^j} \\ &\quad + \frac{h_c^j D^j}{R^2}\Big[e^{-RT^j} - \{R(T^j - t_1^j) + 1\}e^{-Rt_1^j}\Big]e^{-R(i-1)T^j} \end{aligned}$$

Present value of sales revenue for the ith $(1 \leq i \leq N^j)$ cycle is given by

$$SR_i^j = s^j \int_{(i-1)T^j}^{iT^j} D^j e^{-Rt}\, dt = \frac{D^j s^j}{R}(1 - e^{-RT^j})e^{-R(i-1)T^j}$$

Therefore, total profit after completing N^j full cycles is given by

$$\begin{aligned} TP(T^j) &= \sum_{i=1}^{N^j}\Big[SR_i^j - PC_i^j - SC_i^j - RC_i^j - HC_i^j\Big] \\ &= \Big[\frac{s^j D^j}{R}(1 - e^{-RT^j}) - \frac{1}{R}\{c_p^j + c_{sr}^j + r_c^j\delta^j(1-\beta^j)\}(P_0^j + P_1^j D^j)(1 - e^{-Rt_1^j}) \\ &\quad - \frac{h_c^j}{R^2}\Big\{\{\beta^j + \delta^j(1-\beta^j)\}(P_0^j + P_1^j D^j) - D^j\Big\}\{1 - (1 + Rt_1^j)e^{-Rt_1^j}\} \\ &\quad - \frac{h_c^j D^j}{R^2}\Big\{e^{-RT^j} - \{R(T^j - t_1^j) + 1\}e^{-Rt_1^j}\Big\}\Big]\frac{1 - e^{-N^j RT^j}}{1 - e^{-RT^j}}, \text{ (See Appendix } \mathbf{B}\text{).} \end{aligned}$$

Since $f(h)$ is the p.d.f of the planning horizon H, therefore the expected total profit in N^j complete cycles is given by

$$\begin{aligned} E[TP(T^j)] &= \sum_{N^j=0}^{\infty} \int_{N^jT^j}^{(N^j+1)T^j} TP(N^j, T^j) f(h)\, dh \\ &= \Big[\frac{s^j D^j}{R}(1-e^{-RT^j}) - \frac{1}{R}\{c_p^j + c_{sr}^j + r_c^j \delta^j (1-\beta^j)\}(P_0^j + P_1^j D^j)(1-e^{-Rt_1^j}) \\ &\quad - \frac{h_c^j}{R^2}\Big\{\{\beta^j + \delta^j(1-\beta^j)\}(P_0^j + P_1^j D^j) - D^j\Big\}\{1-(1+Rt_1^j)e^{-Rt_1^j}\} \\ &\quad - \frac{h_c^j D^j}{R^2}\Big\{e^{-RT^j} - \{R(T^j - t_1^j)+1\}e^{-Rt_1^j}\Big\}\Big] \frac{e^{-\lambda T^j}}{1-e^{-(\lambda+R)T^j}}, \text{ (See Appendix } \mathbf{B}). \end{aligned}$$

4.3.2 Formulation for Last Cycle

In this case, the initial stock of the last cycle is zero and starts production with rate P^j. As production and rework are continue, inventory begins to pile up continuously after meeting demand with rate D^j. Production and reworking of last cycle stop at time $N^jT^j + t_1^j$. During the time $[N^jT^j, N^jT^j + t_1^j]$, the inventory rate increases with production rate of perfect item, rework rate of the imperfect item and decreases with demand rate. And during the time $[N^jT^j + t_1^j, (N^j+1)T^j\ ]$ inventory level decreases with demand rate. Hence, the differential equation describing the inventory level $q_L^j(t)$ in the interval $N^jT^j \le t \le (N^j+1)T^j$ is given by

$$\frac{dq_L^j(t)}{dt} = \begin{cases} \beta^j P^j + \delta^j(1-\beta^j)P^j - D^j, & N^jT^j \le t \le N^jT^j + t_1^j \\ -D^j, & N^jT^j + t_1^j \le t \le (N^j+1)T^j \end{cases}$$

subject to the boundary conditions: $q_L^j(N^jT^j) = 0$ and $q_L^j((N^j+1)T^j) = 0$.
The solution of above differential equation is given by

$$q_L^j(t) = \begin{cases} \Big[\{\beta^j + \delta^j(1-\beta^j)\}(P_0^j + P_1^j D^j) & \\ -D^j\Big](t - N^jT^j), & N^jT^j \le t \le N^jT^j + t_1^j \\ D^j\Big\{(N^j+1)T^j - t\Big\}, & N^jT^j + t_1^j \le t \le (N^j+1)T^j \end{cases} \tag{4.4}$$

The above equation indicate the amount of stock at any time during the last cycle. More-over, the parameter h present in the expression of last cycle is a random variable follows exponential distribution (as discuss in assumption (v)). For simplicity we consider two cases depending upon the cycle length.

Case-I: when $N^jT^j \leq h < N^jT^j + t_1^j$

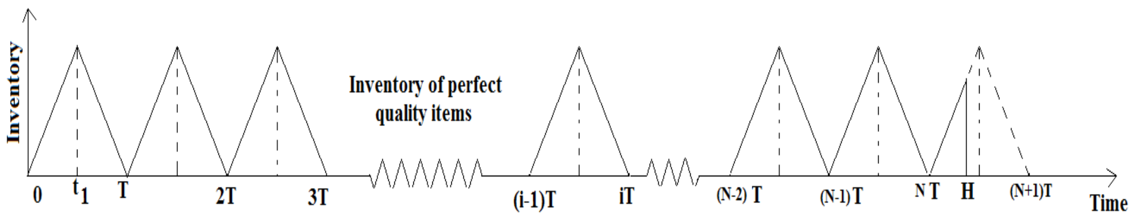

Figure 4.1: Graphical Representation of Inventory Model for case-I

As like earlier, the present value of production cost of the jth item for the last cycle is given by

$$PC_{L_1}^j = c_p^j[\int_{N^jT^j}^{h} P^j e^{-Rt}\,dt] = \frac{c_p^j}{R}(P_0^j + P_1^j D^j)(e^{-RN^jT^j} - e^{-Rh})$$

Present value of screening cost of the last cycle is given by

$$SC_{L_1}^j = c_{sr}^j[\int_{N^jT^j}^{h} Pe^{-Rt}\,dt] = \frac{c_{sr}^j}{R}(P_0^j + P_1^j D^j)(e^{-RN^jT^j} - e^{-Rh})$$

Present value of reworked cost of the last cycle is given by

$$RC_{L_1}^j = r_c^j[\int_{N^jT^j}^{h} \delta^j(1-\beta^j)P^j e^{-Rt}\,dt] = \frac{r_c^j}{R}\delta^j(1-\beta^j)(P_0^j + P_1^j D^j)(e^{-RN^jT^j} - e^{-Rh})$$

Present value of sales revenue for the last cycle is given by

$$SR_{L_1}^j = s^j\int_{N^jT^j}^{h} D^j e^{-Rt}\,dt = \frac{D^j s^j}{R}(e^{-RN^jT^j} - e^{-Rh})$$

Present value of holding cost of the inventory for the last cycle is given by

$$HC_{L_1}^j = h_c^j[\int_{N^jT^j}^{h} q_L^j(t)e^{-Rt}\,dt]$$
$$= \frac{h_c^j}{R^2}\Big[\Big\{\beta^j + \delta^j(1-\beta^j)\Big\}(P_0^j + P_1^j D^j) - D^j\Big]\Big[e^{-RN^jT^j} - \{1 + R(h - N^jT^j)\}e^{-Rh}\Big]$$

Case-II: when $N^jT^j + t_1^j \leq h < (N^j+1)T^j$

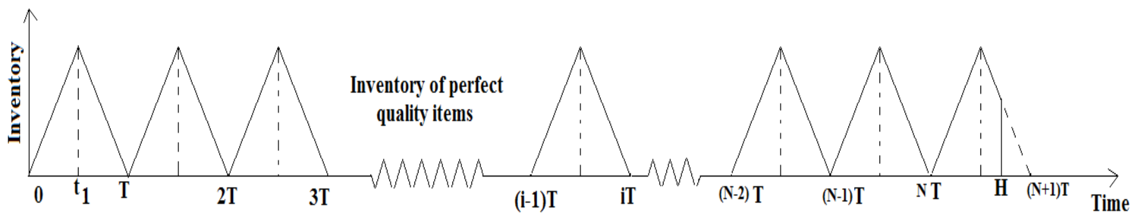

Figure 4.2: Graphical Representation of Inventory Model for case-II

Similarly, the present value of the production cost for the last cycle is given by

$$PC_{L_2}^j = c_p^j \int_{N^jT^j}^{N^jT^j+t_1^j} P^j e^{-Rt}\, dt = \frac{c_p^j}{R}(P_0^j + P_1^j D^j)(1-e^{-Rt_1^j})e^{-RN^jT^j}$$

Present value of screening cost of the last cycle is given by

$$SC_{L_2}^j = c_{sr}^j \int_{N^jT^j}^{N^jT^j+t_1^j} P^j e^{-Rt}\, dt = \frac{c_{sr}^j}{R}(P_0^j + P_1^j D^j)(1-e^{-Rt_1^j})e^{-RN^jT^j}$$

Present value of reworked cost of the last cycle is given by

$$RC_{L_2}^j = r_c^j \int_{N^jT^j}^{N^jT^j+t_1^j} \delta^j(1-\beta^j)P^j e^{-Rt}\, dt = \frac{r_c^j}{R}\delta^j(1-\beta^j)(P_0^j + P_1^j D^j)(1-e^{-Rt_1^j})e^{-RN^jT^j}$$

Present value of holding cost of the inventory for the last cycle is given by

$$\begin{aligned} HC_{L_2}^j &= h_c^j[\int_{N^jT^j}^{N^jT^j+t_1^j} q_L^j(t)e^{-Rt}\, dt + \int_{N^jT^j+t_1^j}^{h} q_L^j(t)e^{-Rt}\, dt] \\ &= \frac{h_c^j}{R^2}\Big[\{\beta^j + \delta^j(1-\beta^j)\}(P_0^j + P_1^j D^j) - D^j\Big]\Big\{1-(1+Rt_1^j)e^{-Rt_1^j}\Big\}e^{-RN^jT^j} \\ &\quad + \frac{h_c^j D^j}{R^2}[1 - R\{(N^j+1)T^j - h\}]e^{-Rh} + \frac{h_c^j D^j}{R^2}[(T^j - t_1^j)R - 1]e^{-RN^jT^j}.e^{-Rt_1^j} \end{aligned}$$

Present value of sales revenue for the last cycle is given by

$$SR_{L_2}^j = s^j \int_{N^jT^j}^{h} D^j e^{-Rt}\, dt = \frac{D^j s^j}{R}(e^{-RN^jT^j} - e^{-Rh})$$

Hence, expected production cost for the last cycle (see Appendix B) is given by

$$\begin{aligned} E[PC_L^j] &= \sum_{N^j=0}^{\infty} \int_{N^jT^j}^{(N^j+1)T^j} PC_L^j f(h)\, dh \\ &= \frac{c_p^j(P_0^j + P_1^j D^j)}{R\{1-e^{-(R+\lambda)T^j}\}}\Big[\frac{R}{R+\lambda}\{1-e^{-(R+\lambda)t_1^j}\} - (1-e^{-Rt_1^j})e^{-\lambda T^j})\Big] \end{aligned}$$

Expected screening cost for the last cycle (see Appendix B) is given by

$$\begin{aligned} E[SC_L^j] &= \sum_{N^j=0}^{\infty} \int_{N^jT^j}^{(N^j+1)T^j} SC_L^j f(h)\, dh \\ &= \frac{c_{sr}^j(P_0^j + P_1^j D^j)}{R\{1-e^{-(R+\lambda)T^j}\}} \Big[\frac{R}{R+\lambda}\{1-e^{-(R+\lambda)t_1^j}\} - (1-e^{-Rt_1^j})e^{-\lambda T^j} \Big] \end{aligned}$$

Expected reworked cost for the last cycle (see Appendix B) is given by

$$\begin{aligned} E[RC_L^j] &= \sum_{N^j=0}^{\infty} \int_{N^jT^j}^{(N^j+1)T^j} RC_L^j f(h)\, dh \\ &= \frac{\delta^j r_c^j(1-\beta^j)(P_0^j + P_1^j D^j)}{R\{1-e^{-(R+\lambda)T^j}\}} \Big[\frac{R}{R+\lambda}\{1-e^{-(R+\lambda)t_1^j}\} - (1-e^{-Rt_1^j})e^{-\lambda T^j} \Big] \end{aligned}$$

Expected holding cost for the last cycle is given by

$$\begin{aligned} &E[HC_L^j] = \sum_{N^j=0}^{\infty} \int_{N^jT^j}^{N^jT^j+t_1^j} HC_{L_1}^j f(h)\, dh + \sum_{N^j=0}^{\infty} \int_{N^jT^j+t_1^j}^{(N^j+1)T^j} HC_{L_2}^j f(h)\, dh \\ &= \frac{h_c^j}{R^2\{1-e^{-(R+\lambda)T^j}\}} \Big[\{\beta^j + \delta^j(1-\beta^j)\}(P_0^j + P_1^j D^j) - D^j \Big] \Big[(1-e^{-\lambda t_1^j}) \\ &+ \frac{\lambda}{R^2(R+\lambda)^2} \Big\{ \{R(R+\lambda)t_1^j + \lambda\} e^{-(R+\lambda)t_1^j} - \lambda \Big\} + \{1-(1+Rt_1^j)e^{-Rt_1^j}\}(e^{-\lambda t_1^j} - e^{-\lambda T^j}) \Big] \\ &- \frac{\lambda h_c^j D^j}{R^2(R+\lambda)^2\{1-e^{-(R+\lambda)T^j}\}} \Big[\lambda\{e^{-(R+\lambda)T^j} - e^{-(R+\lambda)t_1^j}\} + R(R+\lambda)e^{-(R+\lambda)t_1^j} \Big] \\ &+ \frac{h_c^j D^j}{R^2\{1-e^{-(R+\lambda)T^j}\}} \Big[\{(T^j - t_1^j)R - 1\}\{e^{-\lambda t_1^j} - e^{-\lambda T^j}\} e^{-Rt_1^j} \Big] \end{aligned}$$

Expected sales revenue from the last cycle (see Appendix B) is given by

$$\begin{aligned} E[SR_L^j] &= \sum_{N^j=0}^{\infty} \int_{N^jT^j}^{(N^j+1)T^j} SR_L^j f(h)\, dh \\ &= \frac{D^j s^j}{R\{1-e^{-(R+\lambda)T^j}\}} \Big[(1-e^{-\lambda T^j}) + \frac{\lambda}{R+\lambda}(e^{-(R+\lambda)T^j} - 1) \Big] \end{aligned}$$

4.3.3 Objective of the Proposed Model

Therefore, expected total profit in the last cycle for jth item is given by the positive differences of the costs from the sales revenue, i.e.,

$$E[TP_L^j(T^j)] = E[SR_L^j] - E[PC_L^j] - E[SC_L^j] - E[RC_L^j] - E[HC_L^j]$$

Now, total expected profit during the time horizon is given by adding the total expected profit of N^j complete cycles and expected total profit for last cycle and summing over the number of items. i.e.,

$$E[TP] \;=\; \sum_{j=1}^{M} E[TP^j(T^j)] + \sum_{j=1}^{M} E[TP_L^j(T^j)] \tag{4.5}$$

In the present chapter the decision manager with its limited space investigate the optimum decision variables T^j in such a manner that the total expected profit is maximum. Hence the problem becomes:

$$\text{Maximize} \quad E[TP] = \sum_{j=1}^{M} E[TP^j(T^j)] + \sum_{j=1}^{M} E[TP_L^j(T^j)] \tag{4.6}$$

$$\text{Subject to the constraint:} \quad \sum_{j=1}^{M} a^j Q^j \le B \tag{4.7}$$

4.4 Solution Procedure

The proposed model is numerically solved through following steps:

Step-1: Consider the minimum demand rates of the items D_0^j.

Step-2: Estimate the other demand parameters.

Step-3: Receive the other input parameters.

Step-4: Hypothetically, propose the available space.

Step-5: With the help of above data, decision variables T^1, T^2 are optimized through GAVP, described in section 2.4.4.

4.5 Numerical Illustration

To find the different input parameters of the demand function, we consider the market survey at "*Sarama Steel Furniture, Midnapore, Paschim Medinipur, W.B., India*", for two different items: steel almiraha and steel bed. The minimum demand be D_0^j estimated by considering the minimum demand for the year-2012, D_0^1=49 unit and D_0^2=47 unit.
The second term of demand expression $D_1^j(s^j)$, s_{min}^1=$Rs.72$ s_{min}^2=$Rs.75$. Other constant parameters A^j and s_{max}^j are estimated using the method curve fitting from the average daily demands of 7 months (during this months the selling prices are different).

Table 4.2: Market survey for estimating $D_1^1(s^1)$

$Markets \rightarrow$	Jan'12	Feb'12	Mar'12	Apr'12	May'12	Jun'12	Jul'12
$s^1(Rs.)$	88	89	90	91	92	93	94
$D^1(unit)$	55.8	54.6	53.1	52.9	52.3	51.5	51.3

Following Appendix B, the estimated values are: for the first item $A^1 = 10.20$, $s^1_{max} = Rs.\,98.30$, and similarly for the 2nd item $A^2 = 7$, $s^2_{max} = Rs.\,98.00$. Again, the demand parameter k^j present in the third term are also estimated from the daily market demand about 7 months (January, 2007 to July, 2007) (This survey is made in the period, when selling price is fixed at $Rs.\,91$) by increasing advertisement efforts month to month.

Table 4.3: Market survey for estimating $D_2^1(\nu^1)$

$Days \rightarrow$	1	2	3	4	5	6	7
ν^1	8	11	14	17	20	23	26
$D^1(unit)$	58.3	58.8	59.1	59.5	59.8	60.2	60.8

Following Appendix B, the estimated value: for the 1st item $k^1 = 6.97$ and similarly for the 2nd item $k^2 = 4.0$. Other input parameters are: $\lambda = 0.1009$, $R = 0.39$, $D^1 = 59.56\,unit$, $D^2 = 51.81\,unit$, $\beta^1 = 0.95$, $\beta^2 = 0.94$, $\delta^1 = 0.86$, $\delta^2 = 0.87$, $\eta_1 = 0.3$, $\eta_2 = 0.83$, $L^1 = 6$, $L^2 = 16$, $M^1 = 2$, $M^2 = 2.5$, $\rho_1 = .03$, $\rho_2 = .031$, $c_r^1 = Rs.\,34$, $c_r^2 = Rs.\,28$, $\nu^1 = 20$, $\nu^2 = 16$, $s^1 = Rs.\,91$, $s^2 = Rs.\,95$, $r_c^1 = Rs.\,2$, $r_c^2 = Rs.\,2.02$, $c_{sr}^1 = Rs.\,1.5$, $c_{sr}^2 = Rs.\,1.25$, $h_c^1 = Rs.\,2.10$, $h_c^2 = Rs.\,2.12$.
Now, using above input values we get the optimum expected profit which given computational result Table 4.4.

Table 4.4: Optimum results of the illustrated model

$ITEM$	T^j(unit)	t_1^j(unit)	Q^j(unit)	P^j(unit)	$E(PC^j)$(Rs.)	ETP(Rs.)
ITEM-1	7.60	3.67	234.08	124.20	37.72	6914.45
ITEM-2	4.53	1.97	132.85	120.31	31.19	

The graphical representation of the fitness value of ETP for different number of generations are given in the following figure:

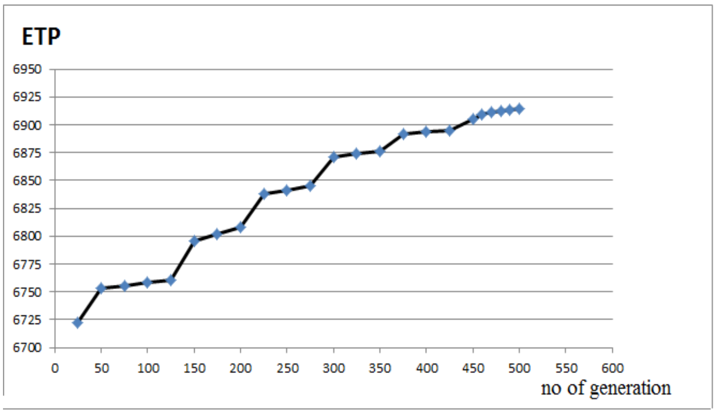

Figure 4.3: ETP vs no of generation

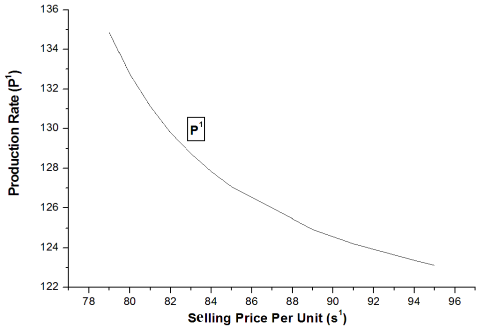

Figure 4.4: Selling price vs production rate

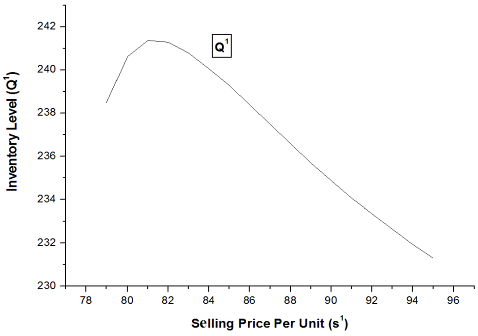

Figure 4.5: Selling price vs inventory level

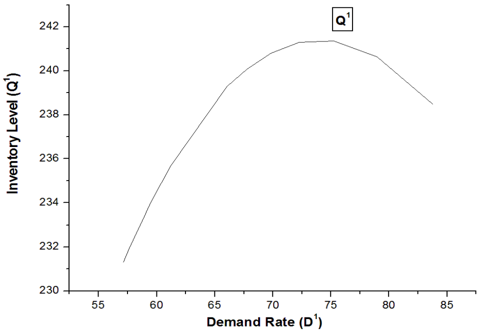

Figure 4.6: Demand rate vs inventory level

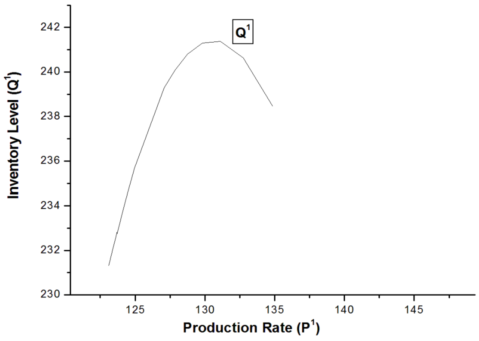

Figure 4.7: Production rate vs inventory level

4.5.1 Sensitivity Analysis

The change in the values of parameters may happen due to uncertainties in any decision-making situation. In order to examine the implications of these changes, the sensitivity analysis will be of great help in decision-making. The numerical example given in the preceding section, the sensitivity analysis of various parameters such as advertisement efforts (v^1, v^2), inflation effect (R) and mean (λ) of exponential distribution has been done. The optimal values of T^1 and T^2 along with the maximum expected total profit have been calculated for different values of v^1, v^2, R and λ. The results of sensitivity analysis are summarized in Table 4.5 and Table 4.6.

Table 4.5: Sensitivity analysis on advertisement parameters ν_1 and ν_2

Advertisement effort 1st item (ν_1)	Advertisement effort 2nd item (ν_2)	Demand rate 1st item (D^1)	Demand rate 2nd item (D^2)	Expected total profit (ETP)
18	14	59.522	51.783	6907.36
	15	59.522	51.800	6908.84
	16	59.522	51.814	6910.15
	17	59.522	51.827	6911.31
	18	59.522	51.839	6912.36
19	14	59.540	51.783	6909.62
	15	59.540	51.800	6911.09
	16	59.540	51.814	6912.41
	17	59.540	51.827	6913.57
	18	59.540	51.839	6914.62
20	14	59.557	51.783	6911.66
	15	59.557	51.800	6913.14
	16	59.557	51.814	6914.45
	17	59.557	51.827	6915.61
	18	59.557	51.839	6916.66
21	14	59.572	51.783	6913.51
	15	59.572	51.800	6914.99
	16	59.572	51.814	6916.30
	17	59.572	51.827	6917.47
	18	59.572	51.839	6918.51
22	14	59.585	51.783	6915.20
	15	59.585	51.800	6916.68
	16	59.585	51.814	6917.99
	17	59.585	51.827	6919.16
	18	59.585	51.839	6910.21

Table 4.6: Sensitivity analysis when R and λ changes

Percentage change in R	Percentage change in λ	Percentage change in $[T^1, T^2]$	Percentage change in expected total profit
-7.69	0	[0, +8.34]	+6.76
-2.56	0	[0, +2.63]	+2.19
0	0	[0, 0]	0
+2.56	0	[0, -2.5]	-2.14
+7.69	0	[0, -7.14]	-6.23
0	-0.20	[+0.2, 0]	-0.04
0	-0.30	[+0.3, 0]	-0.06
0	-0.40	[+0.4, 0]	-0.08
0	+1.09	[-1.08, 0]	+0.22
0	+2.08	[-2.04, 0]	+0.41

4.5.2 Discussion and Managerial Insights

The proposed model and its numerical character help the decision maker to make different managerial decisions. Here, the demands of the items are affected by more than one influence (advertisement and selling price) and the affections are find-out by method of estimation. From Table 4.5, it reveals that with the frequency of advertisement the market demand varies proportionally, which gives more profit for more advertisement. In numerical point of view, the following decisions can also be made, which are also reflected from the figures
(i) Figure 4.4 shows that higher selling price implies lower production rate.
(ii) Figure 4.5, 4.6 and 4.7 indicate that inventory is a convex function of selling price, demand rate and production rate respectively.
Significant management implications for the practical application of the proposed approach are as follows. The obtained optimal order quantity is a crisp and precise solution, although the inventory problem is stochastic. This is important to decision-makers since one of their major concerns is how to find the precise target values for order quantities when some data are uncertain in terms of stochastic. Moreover, the proposed research work lights to the newly established companies with considerable input parameters.

4.5.3 Comparison between the Demands by ANOVA Test

Comparison test of the demands of the 1st item present in Table 4.2 and Table 4.3 with constant demand $D_0^1 = 49$ by ANOVA test. For convenience, let $X_1(i) = D_0^1$, $X_2(i) = D^1(i)$(presented

in Table 4.2) and $X_3(i) = D^1(i)$(presented in Table 4.3), for $i = 1, 2, \ldots, 7$.
Then, the corresponding means are, $\bar{X}_1$=49, $\bar{X}_2 = 53.07$ and $\bar{X}_3 = 59.5$.
The total sum of squares be $SS_t = 412.73$, with degree of freedom $df_t = 20$.
Between groups sum of squares be $SS_b = 329.36$, with degree of freedom $df_b = 2$.
And within groups sum squares be SS_w= 20.37, with degree of freedom $df_w = 18$.
Therefore, $F = \frac{s_b^2}{s_w^2}$ with (k-1, N-k)d.f where $s_b^2 = \frac{SSB}{k-1}$ and $s_w^2 = \frac{SSW}{N-k}$ is given in the following Table 4.7.

Table 4.7: Table for different demand pattern ANOVA test

Sources of variation	Sums of squares	df	Variances	F
Between groups	329.36	2	196.18	173.31
Within groups	20.37	18	1.13	
Total	349.73	20		

The critical F values (df=2, 18) are quoted below.

$$F_{.05(2,18)} = 3.55,\ F_{.01(2,18)} = 5.61$$

As the computed F is found to be higher than the critical F for 0.01 level also, the computed F is significant beyond the 0.01 level ($P < 0.01$). Hence, there is a significant added treatment component between the groups.

4.5.4 Practical Implications

A manufacturing system may be illustrated as follows: Let an automobile industrial company produces two different types of vehicles. The company has a showroom of fixed space to store the vehicles (say, 500 acre land). Each of the items has its different demand rate and other parameters. The decision managers of the company decide that how much of each quantity is produced? and what will be the length or frequency of the production cycle ? In such a real life problem, the present model can be implemented.

4.6 Conclusion

For the first time, this model presents a production inventory model for multi-item with constant rate of reworking the defective items in finite time horizon employing the net present value in the objective function. Moreover, the time horizon is randomly distributed. The demands of the items have a promotional effort due to the selling price of the items and frequency of the advertisement. The proposed model provides an optimal cycle length in the random time period, production quantity on the basis of total expected profit. The described model is optimized through the population varying genetic algorithm. Finally, numerical examples are considered through a market survey.

Chapter 5

A deteriorating manufacturing system considering inspection errors with discount and warranty period dependent demand

5.1 Introduction

The classical economic production quantity (EPQ) models assume that the production system is free of failures and all items produced are perfect [194]. Many research efforts have been made to extend the classical EPQ model by loosening various assumptions. In real world manufacturing system, the quality of the product is not always perfect but it is dependent on the product design and stability of the production process. However, the production process starts in an in-control state by producing perfect items, and then it may become out-of-control state by producing mixture of perfect and imperfect items due to deterioration of machinery system as well as other factors. Porteus [166] was one of the researchers to consider the situation where the production process may shift from an ‘in-control state to an ‘out-of-control state with a given probability. Rosenblatt and Lee [181] considered an EPQ model that deals with imperfect quality. They assumed that at some random point in time the process might shift from an in-control to an out-of-control state, and a fixed percentage of defective items are produced. Liao [135] studied imperfect production processes that require maintenance. They considered two states, namely the in-control and the out-of-control states of the production process. It is seen that the maintenance process improves a production system to a perfect state.

Most of researchers on the EPQ models with imperfect production process considered the inspection process for searching the defective items to be perfect i.e., error-free. But in general, this assumption is not true in real business world. Practically, the inspection process is not error free due to different types of factors related to machine and human factors in the system. It may be a possibility of Type I error (falsely rejecting non-defective items) or Type II error (falsely accepting defective items) in any industry. Salameh and Jaber [184] studied a joint lot sizing and inspection policy for an EOQ model when a random proportion of the units in a lot are defective. They assumed a 100% screening process with no human error. Raouf et al. [172] is one of the researchers of inspection policies who showed for the first time that inspection process is not entirely correct as it is controlled by human. Raz and Bricker [174] considered inspection errors during screening in an production process. Rentoul et al. [175] studied several ways of inspection errors in manufacturing system which are made by comparing inspection points with a solid model of the desired component. Wang and Sheu [212] considered inspection errors on an optimal production, inspection and maintenance policy. An inventory model in an imperfect production process with the preventive maintenance and inspection errors is considered by Darwish and Ben-Daya [47]. Duffuaa and Khan [66] studied the optimal inspection policy under different kinds of misclassifications. Wang [215] considered inspection policy with two types of inspection errors to accept the economic production quantity for real world applications. Khan et al. [112] generalized Salameh and Jaber's [184] model by considering the imperfect screening process and adopting the approach of Raouf et al. [172] to depict the misclassifications. Hsu and Hsu [94] developed an economic order quantity (EOQ) model with imperfect quality items, inspection errors, shortages and sales returns.

Any manufacturing system produces perfect as well as defective items. A reworking cost is allowed to make the defective items as new as the perfect one through some reworks. This rework policy plays an important role in eliminating waste and affecting the cost of manufacturing. Therefore, determining the optimal lot size in a system that allows for rework is a valuable objective for maximizing profit. Hayek and Salameh [97] and Chiu [38] studied the determination of optimal production lot size with reworking of defective items. Flappera and Teunterb [73] showed how reworking plans could reduce the costs keeping the environment from disposal pollution. Chen [30] and Chiu et al. [38] proposed a more general model that allowed a certain proportion of reworked units to be scrapped.

In practice, the manufacturer and retailer usually offer a warranty for all selling items for the specific warranty period due to increasing the selling rate and reliability of product. Warranty period of a product is a duration in which a purchased product provides satisfactory performance to the customer. If any purchased product failed to work within its warranty period, then the servicing center replaces it with a new item or repair the product by replacing one or more parts. Due to this reason, a manufacturer considers warranty cost if there exists free-warranty on selling items in the warranty period. Wang and Sheu [212] investigated the imperfect production model with a free warranty for the discrete unit item. Wang [213] studied the production process problem to consider the sold product with free warranty. Yeh et al. [235] developed a production inventory model considering the free warranty and derived the optimal production cycle time.

Recently, the most of the inventory models have generally considered the various demand functions for the business period, such as constant, stock-dependent, time dependent, selling price dependent etc. But in real life, the assumption is not always true in general. The demand may vary with selling price discount and warranty period of the product. The retailers and users are both lured by the displays, advertisements, selling price discount and warranty period. For example, mobile, computers etc., lead to greater sales for selling price discount and warranty period. From previous researches, inventory models incorporating warranty-dependent demand have been widely studied. Chung [44] considered the economic production quantity (EPQ) model with the warranty period-dependent demand, effects on inspection scheduling and supply chain replenishment policy. Nevertheless, the integration issue in two entities, both the warranty period and selling price discount effect on demand and sales is ignored in above researches. **First time in this model, we consider a new type of selling rate which depends on both selling price discount and warranty period**. The detailed comparative statement of the proposed model with the existing literature has been given in Table 5.1.

Table 5.1: Summary of related literature for EPQ/EOQ models

Author(s)	Production inventory model	Imperfect items inventory	Inspection error	Time (τ) which shifts in-control to out-control	Rework of imperfect item	Warranty Policy	Warranty policy and discount dependent demand
Chakraborty and Giri [17]	√	√		√		√	
Chung [44]	√	√	√				
Hayek and Salameh [97]	√	√			√		
Hsu and Hsu [94]		√	√				
Khan et al. [112]		√	√				
Liao [135]	√	√		√			
Ma et al. [146]	√	√		√			
Rosenblatt and Lee [181]	√	√					
Taheri-Tolgari et al. [199]	√	√	√		√		
Wang and Sheu [212]	√	√	√	√			
Wang [213]	√	√	√			√	
Yeh et al. [235]	√	√				√	
Present model	√	√	√	√	√	√	√

In this chapter, we consider imperfect production inventory model in which the production process shifts from in-control state to out-of-control state at any random time. Also we consider discount and warranty period dependent selling rate with replenishment policy. **Here, we develop a new type inspection errors to inspect the imperfect items.**

Type I error has possibilities of two types such as (i) when a non-defective items are rejected and (ii) when a non-defective items are submitted in reworked cell. Here, rejecting a non-defective item manufacturer loses the good one. For the second case, there is no loss of manufacturer since it will be detected as a non-defective in the reworked cell again. Simultaneously, Type II error is also considered in this model where the acceptance of the defective item has been considered as a non-defective item and due to that it has a risk to a customer. Due to Type-I inspection error, a conforming item may be falsely selected as defective item. As a result, it may be put into (i) rejected item's cell, where the item is treated as rejected or (ii) reworked item's cell, where the item is prepared for reworking, but without any rework the item is transferred to the conforming item cell. Due to Type-II inspection error, a defective item (either rejected or reworkable item) may be falsely accepted as non-defective item. As a result (i) a rejected item may be sold to the customer as a non-defective item and obviously it is returned back from the market. Then a new non-defective item be replaced instead of the rejected item to the customer by the manufacturer or (ii) a reworkable item may be sold to the customer as a non-conforming item without any necessary rework and it is also returned back to the manufacturer from the market. Then after some necessary reworks, the manufacturer returns back to the customer with full satisfaction.

5.2 Notations and Assumptions

The following notations and assumptions have been used to develop the model.

5.2.1 Notations

The following notations are used throughout the entire chapter.

P	:	Production rate.
D	:	Selling rate of manufacturer/demand rate customer.
η	:	Selling price discount parameter.
ρ	:	Effective parameter of demand on warranty period.
τ	:	Random time with mean $\frac{1}{\lambda}$ after which the system shifts from an "in-control" state to an "out-of-control" state for manufacturer.
θ_1	:	Percentage of defective items produced in in-control state.
θ_1	:	Percentage of defective items produced in out-of-control state ($\theta_1 < \theta_2$).
δ	:	Probability of rework rate of defective units per unit time.
c_p	:	Production cost per unit.
c_{sr}	:	Screening cost per unit item.
h_c	:	The inventory holding cost per unit time for products in production center.
c_w	:	Average warranty cost for selling item.
s	:	Selling price per unit item sold for perfect quality.
c_r	:	Average reworking cost per unit item for manufacturer.

c_d	:	Disposal cost per unit.
A	:	$= (A_0 + \frac{K}{Pt_1})$, set up cost of retailer.
c_v	:	Development cost.
m_1	:	Probability of a Type I error (classifying a non- defective item as defective).
m_2	:	Probability of a Type II error (classifying a defective item as non-defective).
$f(\tau)$	:	Probability density function of τ.
$\phi(\delta)$	:	Probability density function of δ.
$\phi(m_1)$	:	Probability density function of m_1.
$\phi(m_2)$	:	Probability density function of m_2.
c_a	:	The cost of accepting a defective item, where $c_a = c_t + c_l$.
t_1	:	Production period.
t_w	:	Warranty period of selling item.
T	:	Total business period.

5.2.2 Assumptions

The proposed model is based on the following assumptions.

(i) In a production system, a manufacturer produces a mixture of defective and non-defective quality items and some portion of defective items are reworked at a cost.

(ii) In any production system, it is seen that initially in the production cycle, the production process is in an in-control state, because every factors associated with the system are fresh. But, due to continuous running of system these factors gradually losses their perfectness. So, after some time (τ) of production, the production process may shift from the in-control state to out-control state. The time (τ) is an exponential distributed with a finite mean.

(iii) When the production process shifts to out-control state then this shift continues upto the end of the production period. The production process will be reset to an in-control state at the beginning of next production period.

(iv) According to assumption (ii), every one connected to the production has capability to give its perfection to produce an item perfectly. So, during the period $(0, \tau)$, obviously the number of imperfect items is small in amount but after this period it will be high. Therefore, it is assumed that the defective rate (θ_1) in in-control state is less than the defective rate (θ_2) in out-control state and is given by

$$\theta = \begin{cases} \theta_1, & 0 \le t \le \tau \\ \theta_2, & \tau \le t \le t_1 \end{cases} \tag{5.1}$$

where θ_1 and θ_2 $(\theta_1 < \theta_2)$ be the percentages of defective items to be produced in in-control state and out-of-control state respectively. Here the defective rates θ_1 and θ_2 are considered to be distributed uniformly with a finite mean and variance.

(v) In any production system, it is seen that every produced item may not be 100% perfect due to different factors involved in the system such as machine, raw-materials, unexpertised labour etc. So, there is necessary to inspect each item after production to check whether the item is perfect or not. But, after inspection there may exist some possibility that a perfect item is treated as an imperfect item and an imperfect item may be considered as a perfect item which are known as type-I error and type-II error respectively. According to the production process proposed in the system, the defective items are send to the reworked cell to convert into the non-defective items but, under type-I error an item may be truly non-defective or defective. If it is non-defective then it is send to the inventory of non-defective items after checking from the reworked cell. Other hand, if it is defective then it is reworked on item and then send to the inventory to the non-defective items, otherwise it is rejected completely from the rework cell. Again in case of type-II error, an defective item is delivered as a non-defective item to the customers. So after checking it by customer, it is send back to the manufacturer. Therefore under type-II error the manufacturer is compelled to bear an extra cost as a miss-classification cost.

(vi) We consider the warranty cost (c_w) per unit item is not constant, it is depend on the production period (t_w) and is given by

$$c_w = a + bt_w \tag{5.2}$$

(vii) Selling price (s) per unit item sold for non-defective item is not fixed always, we consider

$$s = \begin{cases} s_0 - \eta s_0, & 0 \leq t \leq t_1 \\ s_0, & t_1 \leq t \leq T \end{cases} \tag{5.3}$$

where η is the discounts percentage of selling price .

$(viii)$ Production rate is constant.

(ix) In this model, we assume that the selling rate which dependent both selling price discount and warranty period due to increasing selling rate and reliability of product. The selling rate is defined as

$$D = \begin{cases} (D_0 + \rho t_w)e^{k\eta}, & 0 \leq t \leq t_1 \\ D_0 + \rho t_w, & t_1 \leq t \leq T \end{cases} \tag{5.4}$$

where k is the effective parameter of demand rate on discount and ρ is the effective parameter of demand rate on warranty period.

(x) Due to long run production process and increasing the duration of in-control state, refine production methods and reduce production costs we consider devolvement cost (c_v) as the form $c_v = f(\tau)$, where

$$f(\tau) = \begin{cases} B_0, & 0 \leq t \leq \tau \\ B_0 + B_1(t-\tau)e^{k_1 \frac{v_{max}-v}{v-v_{min}}}, & \tau \leq t \leq t_1 \end{cases} \tag{5.5}$$

(xi) Production period (t_1) and warranty period (t_w) are decision variables.

5.3 Mathematical Formulation of the Proposed Model

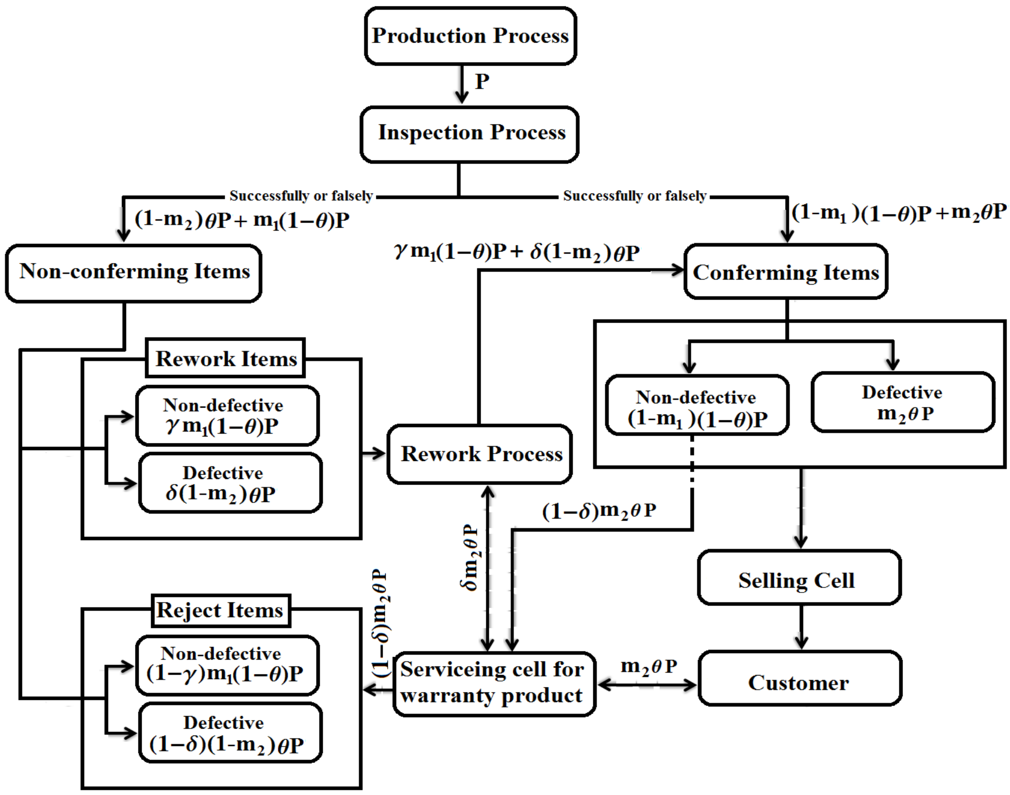

Figure 5.1: Schematic representation of the production inventory model

This model considers a supply chain system between manufacturer and customer for single type of products such as mobiles, in which the qualities of the production process and inspection process are not perfect. In this manufacturing system, it is considered that production, inspections and reworked processes are performed simultaneously. Here production is started at a rate of P from the beginning and it continues upto the end of the production run, t_1. During the whole production period all produced items are inspected at the rate of P. Initially, the production system starts from in-control state and continues to any random time, τ from which in-control state shifts to out-of-control state and it stays in out-of-control state until the end of the production-run, t_1. According to assumption (iv), the probability of the number of defective items in in-control state is less than the probability of the number of defective items in out-of-control state.

Further, since the inspection process is not perfect, hence it generates both Type-I and Type-II inspection errors. During this process there is possibilities separate some non-defective products as defective of amount $m_1(1-\theta_1)P$ in in-control state and of amount $m_1(1-\theta_2)P$ in out-of-control state. On the other hand, it classifies some defective items as non-defective of amount $m_2\theta_1 P$ in in-control state and $m_2\theta_2 P$ in out-of-control state. Here for sorting the items an inspection cost (c_{sr}) has been considered. After the inspection of the product the portion δ of defective items are send in rework cell to covert it non-defective item as fixed cost r_c. Then all confirming items are sent to the market with warranty period t_w and warranty cost c_w. Figure 5.1 shows the relationship among production, inspection, selling and rework processes. The consumption process continues at the customer demand rate (D) according to the assumption (ix) until the end of business cycle time T. Since the time τ is random i.e., the time for which the system goes from in-control to out-control, depending on the position of τ with respect to the end of the production, the model has two different cases such as **Case** I**:** $0 < \tau < t_1$**;** and **Case** II**:** $t_1 \leq \tau < \infty$**;** which are discussed as follows:

5.3.1 Case I: when $0 < \tau < t_1$ i.e., the "out-of-control" state to be occurred during the production-run time

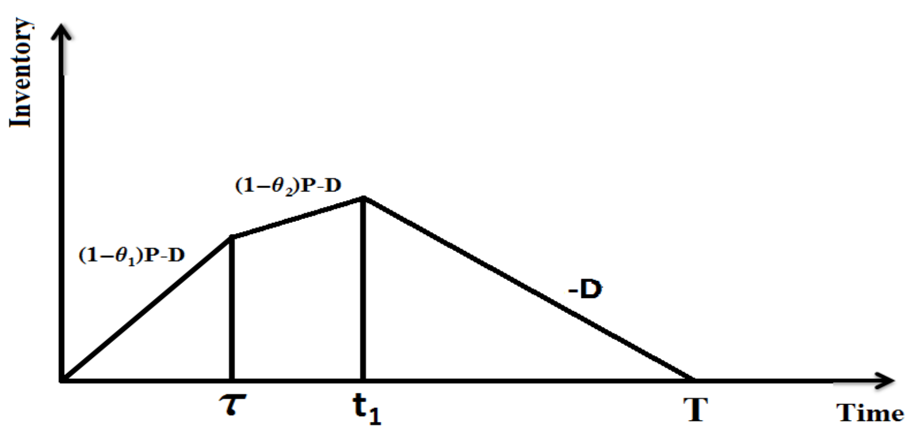

Figure 5.2: Graphical representation of inventory of selling item

In this case, the production period $[0, t_1]$ can be divided into two sub-intervals such as $[0, \tau]$ and $[\tau, t_1]$. During the time interval $[0, \tau]$, the production process is in in-control state and in $[\tau, t_1]$ the process is in out-of-control state. Throughout the time interval $[0, \tau]$, the amount of non-defective items, defective items and reworked items are $(1-\theta_1)P\tau$, $\theta_1 P\tau$ and $\delta\theta_1 P\tau$ respectively. Also on $[\tau, t_1]$, the amount of non-defective items, defective items and reworked items are $(1-\theta_2)P(t_1-\tau)$, $\theta_2 P(t_1-\tau)$ and $\delta\theta_2 P(t_1-\tau)$ respectively. During the inspection period $[0, \tau]$, the inspectors accept defective items the amount of $\theta_1 P\tau$ in which the amount of falsely accepted defective items and falsely rejected non-defective items are $m_2\theta_1 P\tau$ and $(1-\gamma)m_1(1-\theta_1)P\tau$ respectively. Also, the inspection period $[\tau, t_1]$, inspectors accept the defective items of the amount $\theta_2 P(t_1-\tau)$ in which the amount of falsely accepted defective items and falsely reject amount of non-defective items are $m_2\theta_2 P(t_1-\tau)$ and $(1-\gamma)m_1(1-\theta_2)P(t_1-\tau)$ respectively. During the period $[0, t_1]$, the inventory level increases due to excess production after fulfill the customer demand upto time $t = t_1$ at which the inventory level reaches at maximum. Therefor the behavior of the inventory level during the interval $[0, \tau]$ and $[\tau, t_1]$ are given by

$$\begin{aligned} I_1(t) &= [\{1-(1-\gamma)m_1\}(1-\theta_1)P + \{1-(1-\delta)(1-m_2)\}\theta_1 P \\ &\quad -(D_0+\rho t_w)e^{k\eta}]t, \;\; 0 \le t \le \tau \\ \text{and } I_2(t) &= [\{1-(1-\gamma)m_1\}(1-\theta_2)P + \{1-(1-\delta)(1-m_2)\}\theta_2 P \\ &\quad -(D_0+\rho t_w)e^{k\eta}](t-\tau) + [\{1-(1-\gamma)m_1\}(1-\theta_1)P \\ &\quad +\{1-(1-\delta)(1-m_2)\}\theta_1 P - (D_0+\rho t_w)e^{k\eta}]\tau, \;\; \tau \le t \le t_1 \end{aligned}$$

Then during the period $[t_1, T]$ the inventory level decline due to meeting customer demand and it reaches zero at T. Therefor the behavior of the inventory level during the interval $[t_1, T]$ is given by

$$I_3(t) = D(T-t) = (D_0+\rho t_w)(T-t), \;\; t_1 \le t \le T$$

Lemma 5.1. *When $0 < \tau \le t_1$, in a manufacturing system the business period (T) must satisfy the following relation in terms of production rate (P), demand rate (D), warranty period (t_w) and production period (t_1)*

$$\begin{aligned} T &= \frac{1}{(D_0+\rho t_w)}\Big[\Big\{\{1-(1-\gamma)m_1\}(1-\theta_2)P + \{1-(1-\delta)(1-m_2)\}\theta_2 P + (D_0 \\ &\quad +\rho t_w)(1-e^{k\eta})\Big\}t_1 + \Big\{\{1-(1-\gamma)m_1\} + \{1-(1-\delta)(1-m_2)\}\Big\}(\theta_2-\theta_1)P\tau\Big] \end{aligned}$$

Proof. Using the continuity condition of $I_2(t)$ and $I_3(t)$ at $t = t_1$ a relation is obtain as

$$\begin{aligned} &[\{1-(1-\gamma)m_1\}(1-\theta_2)P + \{1-(1-\delta)(1-m_2)\}\theta_2 P - (D_0+\rho t_w)e^{k\eta}]t_1 \\ &+[\{1-(1-\gamma)m_1\} + \{1-(1-\delta)(1-m_2)\}](\theta_2-\theta_1)P\tau = (D_0+\rho t_w)(T-t_1) \\ &\text{i.e., } (D_0+\rho t_w)T = [\{1-(1-\gamma)m_1\}(1-\theta_2)P + \{1-(1-\delta)(1-m_2)\}\theta_2 P \\ &+(D_0+\rho t_w)(1-e^{k\eta})]t_1 + [\{1-(1-\gamma)m_1\} + \{1-(1-\delta)(1-m_2)\}](\theta_2-\theta_1)P\tau \\ &\text{i.e., } T = \frac{1}{(D_0+\rho t_w)}\Big[\Big\{\{1-(1-\gamma)m_1\}(1-\theta_2)P + \{1-(1-\delta)(1-m_2)\}\theta_2 P \\ &+(D_0+\rho t_w)(1-e^{k\eta})\Big\}t_1 + \Big\{\{1-(1-\gamma)m_1\} + \{1-(1-\delta)(1-m_2)\}\Big\}(\theta_2-\theta_1)P\tau\Big] \end{aligned}$$

Now, the proof is complete. □

Now, we derive the holding cost, manufacturing cost, rework cost, setup cost, inspection cost, return cost, penalty cost, development cost and warranty cost in one cycle as follows.

Holding cost:
During the period $[0, T]$, the holding cost is given by

$$\begin{aligned} HC &= h_c\Big[\int_0^{\tau} I_1(t)dt + \int_{\tau}^{t_1} I_2(t)dt + \int_{t_1}^{T} I_3(t)dt\Big] \\ &= \frac{h_c}{2}\Big[\Big\{\{1-(1-\gamma)m_1\}(1-\theta_1)P + \{1-(1-\delta)(1-m_2)\}\theta_1 P\Big\}(2t_1\tau - \tau^2) \\ &\quad + \Big\{\{1-(1-\gamma)m_1\}(1-\theta_2)P + \{1-(1-\delta)(1-m_2)\}\theta_2 P\Big\}(t_1-\tau)^2 \\ &\quad -(D_0+\rho t_w)e^{k\eta}t_1^2 + (D_0+\rho t_w)(T-t_1)^2\Big] \end{aligned}$$

Manufacturing, inspection and reworked cost:
During the period $[0, t_1]$, total manufacturing, inspection and reworked cost is given by
$PC = (c_p + c_s)Pt_1 + c_r\delta(1-m_2)\{\theta_1 P\tau + \theta_2 P(t_1-\tau)\}$
Setup cost=$A_0 + \frac{K}{PT}$
Return and penalty cost:
The return cost including communication and reverse logistics per unit (C_t), and penalty cost per unit (c_l), due to inspection errors during the period $[0, T]$ is given by
$RC = (c_t + c_l)m_2\{\theta_1 P\tau + \theta_2 P(t_1-\tau)\}$
Inspection error cost (or misclassification cost):
During the period $[0, t_1]$, inspectors accept the amount of $\theta_1 Pt_1$ defective items in which falsely accepted amount of defective items and falsely reject amount of non-defective items are $m_2\{\theta_1 P\tau + \theta_2 P(t_1-\tau)\}$ and $m_1\{(1-\theta_1)P\tau + (1-\theta_2)P(t_1-\tau)\}$ respectively. Therefore the inspection error cost is given by,
$IEC = s(1-\gamma)m_1\{(1-\theta_1)P\tau + (1-\theta_2)P(t_1-\tau)\}$
Development cost:
During the period $[0, t_1]$, the development cost is given by

$$c_v = \Big[\int_0^{\tau} B_0 dt + \int_{\tau}^{t_1}\{B_0 + B_1(t-\tau)e^{k\frac{v_{max}-v}{v-v_{min}}}\}dt\Big] = B_0 t_1 + \frac{B_1}{2}(t_1-\tau)^2 e^{k_1\frac{v_{max}-v}{v-v_{min}}}$$

Revenue from serviceable items:
During the period $[0, T]$, the amount of serviceable items (i.e., falsely accepted defective and successfully accepted non-defective items) is $[(1-m_1)\{(1-\theta_1)P\tau + (1-\theta_2)P(t_1-\tau)\} + m_2\{\theta_1 P\tau + \theta_2 P(t_1-\tau)\}]$ at a unit selling price of s, so the sales revenue is $s[(1-m_1)\{(1-\theta_1)P\tau + (1-\theta_2)P(t_1-\tau)\} + m_2\{\theta_1 P\tau + \theta_2 P(t_1-\tau)\}]$. Again, the amount of defective items is returned from the customers' due to Type-II inspection errors is $m_2\{\theta_1 P\tau + \theta_2 P(t_1-\tau)\}$ for refunds at its full unit price s or replace by non-defective items, it incurs revenue loss which is $sm_2\{\theta_1 P\tau + \theta_2 P(t_1-\tau)\}$. Further, the amount of returned non-defective items from rework cell due to inspection errors is $m_1\gamma\{(1-\theta_1)P\tau + (1-\theta_2)P(t_1-\tau)\}$ and the manufacturer obtained sales revenue $sm_1\gamma\{(1-\theta_1)P\tau + (1-\theta_2)P(t_1-\tau)\}$. Also the amount of reworked serviceable items is $\delta(1-m_2)\{\theta_1 P\tau + \theta_2 P(t_1-\tau)\}$ at the same unit selling price of s and obtained sales revenue $s\delta(1-m_2)\{\theta_1 P\tau + \theta_2 P(t_1-\tau)\}$. Thus, the total sales revenues during

the interval $[0, T]$ is given by

$$\begin{aligned} R &= s[\{1-(1-\gamma)m_1\}\{(1-\theta_1)P\tau+(1-\theta_2)P(t_1-\tau)\} \\ &\quad +\{\delta+(1-\delta)m_2\}\{\theta_1 P\tau+\theta_2 P(t_1-\tau)\}] \end{aligned}$$

The warranty cost:
During the period $[0, T]$, the warranty cost is given by

$$\begin{aligned} WC &= c_w[\{1-(1-\gamma)m_1\}\{(1-\theta_1)P\tau+(1-\theta_2)P(t_1-\tau)\} \\ &\quad +\{\delta+(1-\delta)m_2\}\{\theta_1 P\tau+\theta_2 P(t_1-\tau)\}] \end{aligned}$$

Therefore, the profit function of production system is given by during the time period $0 < \tau < t_1$

$$\begin{aligned} \pi_1(t_1, t_w) &= (s-c_w)[\{1-(1-\gamma)m_1\}\{(1-\theta_1)P\tau+(1-\theta_2)P(t_1-\tau)\}+\{\delta+(1-\delta)m_2\} \\ &\quad \times\{\theta_1 P\tau+\theta_2 P(t_1-\tau)\}]-[(c_p+c_s)Pt_1+c_r\delta(1-m_2)\{\theta_1 P\tau+\theta_2 P(t_1-\tau)\}] \\ &\quad -\frac{h_c}{2}\Big[\Big\{\{1-(1-\gamma)m_1\}(1-\theta_1)P+\{1-(1-\delta)(1-m_2)\}\theta_1 P\Big\}(2t_1\tau-\tau^2) \\ &\quad +\Big\{\{1-(1-\gamma)m_1\}(1-\theta_2)P+\{1-(1-\delta)(1-m_2)\}\theta_2 P\Big\}(t_1-\tau)^2 \\ &\quad -(D_0+\rho t_w)e^{k\eta}t_1^2+(D_0+\rho t_w)(T-t_1)^2\Big]-(A_0+\frac{K}{Pt_1}) \\ &\quad -(c_t+c_l)m_2\{\theta_1 P\tau+\theta_2 P(t_1-\tau)\}-B_0 t_1-\frac{B_1}{2}(t_1-\tau)^2 e^{k_1\frac{v_{max}-v}{v-v_{min}}} \end{aligned}$$

The expected profit function during the time period $0 < \tau < t_1$ **is given by**

$$\begin{aligned} &E[\pi_1(t_1, t_w)] = (s-c_w)\Big[\{1-E[m_1(1-\gamma)]\}\{(1-\theta_1)PE[\tau]+(1-\theta_2)PE[(t_1-\tau)]\} \\ &+E[\{m_2+\delta(1-m_2)\}]\{\theta_1 PE[\tau]+\theta_2 PE[(t_1-\tau)]\}\Big]-(A_0+\frac{K}{Pt_1})\int_0^{t_1} f(\tau)\,d\tau \\ &-\Big[(c_p+c_s)Pt_1+c_r E[\delta(1-m_2)]\{\theta_1 PE[\tau]+\theta_2 PE[(t_1-\tau)]\}\Big] \\ &-\frac{h_c}{2}\Big[\Big\{\{1-E[m_1(1-\gamma)]\}(1-\theta_1)P+\{1-E[(1-\delta)(1-m_2)]\}\theta_1 P\Big\}(2t_1E[\tau]-E[\tau^2]) \\ &+\Big\{\{1-E[(1-\gamma)m_1]\}(1-\theta_2)P+\{1-E[(1-\delta)(1-m_2)]\}\theta_2 P\Big\}E[(t_1-\tau)^2] \\ &+\Big\{(D_0+\rho t_w)(T-t_1)^2-(D_0+\rho t_w)e^{k\eta}t_1^2\Big\}\int_0^{t_1} f(\tau)\,d\tau\Big]-(c_t+c_l)E[m_2]\{\theta_1 PE[\tau] \\ &+\theta_2 PE[(t_1-\tau)]\}-B_0 t_1\int_0^{t_1} f(\tau)\,d\tau-\frac{B_1}{2}e^{k\frac{v_{max}-v}{v-v_{min}}}\int_0^{t_1}(t_1-\tau)^2 f(\tau)\,d\tau \end{aligned}$$

$$= (s-c_w)\Big[\{1-E[m_1(1-\gamma)]\}\{(1-\theta_1)P\lambda t_1^2+(1-\theta_2)P\frac{\lambda t_1^2}{2}\}+E[\{m_2+\delta(1-m_2)\}]$$
$$\times\{\theta_1P\lambda t_1^2+\theta_2P\frac{\lambda t_1^2}{2}\}\Big]-\Big[(c_p+c_s)Pt_1+c_rE[\delta(1-m_2)]\{\theta_1P\lambda t_1^2+\theta_2P\frac{\lambda t_1^2}{2}\}\Big]$$
$$-\frac{h_c}{2}\Big[\Big\{\{1-E[m_1(1-\gamma)]\}(1-\theta_1)P+\{1-E[(1-\delta)(1-m_2)]\}\theta_1P\Big\}(2\lambda t_1^3-\frac{1}{2}\lambda^2t_1^4)$$
$$+\Big\{\{1-E[(1-\gamma)m_1]\}(1-\theta_2)P+\{1-E[(1-\delta)(1-m_2)]\}\theta_2P\Big\}\frac{1}{3}\lambda t_1^3$$
$$+\Big\{(D_0+\rho t_w)(T-t_1)^2-(D_0+\rho t_w)e^{k\eta}\Big\}\lambda t_1\Big]-(A_0+\frac{K}{Pt_1})\lambda t_1$$
$$-(c_t+c_l)E[m_2]\{\theta_1P\lambda t_1^2+\theta_2P\frac{\lambda t_1^2}{2}\}-B_0\lambda t_1^2-\frac{B_1\lambda t_1^3}{6}e^{k_1\frac{v_{max}-v}{v-v_{min}}}$$

See Appendix C, approximating the function $exp(-\lambda t_1)$ for its expansion.

5.3.2 Case II: when $\tau \geq t_1$, the "out-of-control" state not to be occurred with in the production-run time

In this case, during the production period $[0, t_1]$, the production process does not occurs "out-of-control" state, i.e., the whole production process is in "in-control" state. Through-out the time interval $[0, t_1]$, the amount of non-defective items, defective items and reworked items are $(1-\theta_1)Pt_1$, $\theta_1 Pt_1$ and $\delta\theta_1 Pt_1$ respectively.

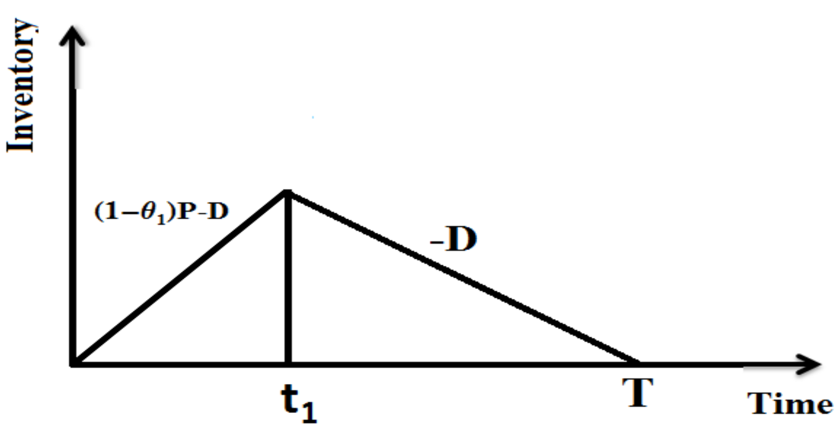

Figure 5.3: Graphical representation of inventory model of perfect quality item

During the inspection period $[0, t_1]$, the inspectors accept defective items of amount $\theta_1 Pt_1$ in which the amount of falsely accepted defective items and falsely rejected non-defective items are $m_2\theta_1 Pt_1$ and $(1-\gamma)m_1(1-\theta_1)Pt_1$ respectively.
During the period $[0, t_1]$, the inventory level increases due to production after fulfill the

customer demand upto time $t = t_1$ at which the inventory level reaches at maximum. Therefore, the behavior of the inventory level during the interval $[0, t_1]$ is given by

$$\begin{aligned} I_1(t) &= [\{1-(1-\gamma)m_1\}(1-\theta_1)P + \{1-(1-\delta)(1-m_2)\}\theta_1 P \\ &\quad -(D_0+\rho t_w)e^{k\eta}]t, \;\; 0 \le t \le t_1 \end{aligned}$$

Then during the period $[t_1, T]$, the inventory level declines due to meet the customer demand and it reaches zero at T. Therefore the behavior of the inventory level during the interval $[t_1, T]$ is given by

$$I_2(t) = D(T-t) = (D_0+\rho t_w)(T-t), \;\; t_1 \le t \le T$$

Lemma 5.2. *When $0 < \tau < t_1$, in a manufacturing system the business period (T) must satisfy the following relation in terms of production rate (P), demand rate (D), warranty period (t_w) and production period (t_1) is given by*

$$T = \frac{1}{(D_0+\rho t_w)}\Big[\{1-(1-\gamma)m_1\}(1-\theta_1) + \{1-(1-\delta)(1-m_2)\}\theta_1\Big]Pt_1 + (1-e^{k\eta})t_1$$

Proof. From the continuity condition of $I_1(t)$ and $I_2(t)$ at $t = t_1$ a relation is obtain as

$$\begin{aligned} &[\{1-(1-\gamma)m_1\}(1-\theta_1)P + \{1-(1-\delta)(1-m_2)\}\theta_1 P - (D_0+\rho t_w)e^{k\eta}]t_1 \\ &\qquad = (D_0+\rho t_w)(T-t_1) \\ &\text{i.e., } T = \frac{1}{(D_0+\rho t_w)}\Big[\{1-(1-\gamma)m_1\}(1-\theta_1) + \{m_2+\delta(1-m_2)\}\theta_1\Big]Pt_1 + (1-e^{k\eta})t_1 \end{aligned}$$

Now, the proof is complete. □

Now, we derive the holding cost, manufacturing cost, rework cost, inspection cost, setup cost, return cost, penalty cost, development cost and warranty cost in one cycle as follows.

Holding cost:

During the period $[0, T]$, the holding cost is given by

$$\begin{aligned} HC &= h_c\Big[\int_0^{t_1} I_1(t)dt + \int_{t_1}^{T} I_2(t)dt\Big] \\ &= \frac{h_c}{2}\Big[\Big\{\{1-(1-\gamma)m_1\}(1-\theta_1)P + \{1-(1-\delta)(1-m_2)\}\theta_1 P\Big\}t_1^2 \\ &\quad +(D_0+\rho t_w)(T^2-2t_1T)\Big] \end{aligned}$$

Manufacturing, inspection and rework cost:

During the period $[0, t_1]$, total manufacturing, inspection and reworked cost is given by
$PC = \{c_p + c_s + c_r\delta(1-m_2)\theta_1\}Pt_1$

Setup cost=$A_0 + \frac{K}{PT}$
Return and penalty cost:
The return cost including communication and reverse logistics per unit (C_t), and penalty cost per unit (c_l), due to inspection errors during the period $[0, T]$ is given by
$RC = (c_t + c_l)m_2\theta_1 Pt_1$
Inspection error cost (or misclassification cost):
During the period $[0, t_1]$, inspectors accepts the amount of defective items $\theta_1 Pt_1$ in which falsely accepted amount of defective items and falsely reject amount of non-defective items are $m_2\theta_1 Pt_1$ and $m_1(1-\theta_1)Pt_1$ respectively.
Therefore the inspection error cost is given by,
$IEC = s(1-\gamma)m_1(1-\theta_1)Pt_1$
Development cost:
During the period $[0, t_1]$, the development cost is given by

$$c_v = \int_0^{t_1} B_0 dt = B_0 t_1$$

Revenue from serviceable items:
During the period $[0, T]$, the amount of serviceable items (i.e., falsely accepted defective and successfully accepted non-defective items) is $[(1-m_1)(1-\theta_1)Pt_1 + m_2\theta_1 Pt_1]$ at a unit selling price of s, so the sales revenue is $s[(1-m_1)(1-\theta_1)Pt_1 + m_2\theta_1 Pt_1]$. Again, amount of defective items to be returned from the customers' due to Type-II inspection errors is $m_2\theta_1 Pt_1$ and refunds at its full unit price s or replace by non-defective items, it incurs revenue loss which is $sm_2\theta_1 Pt_1$. Further, the amount of returned non-defective items from rework cell due to inspection error is $m_1\gamma(1-\theta_1)Pt_1$ and the manufacturer obtains sales revenue $sm_1\gamma(1-\theta_1)Pt_1$. Also the amount of reworked serviceable items is $\delta(1-m_2)\theta_1 Pt_1$ which is sold at the same unit selling price of s and obtains sales revenue $s\delta(1-m_2)\theta_1 Pt_1$. Thus, the total sales revenues during the interval $[0, T]$ is given by

$$R = s\Big[\{1-(1-\gamma)m_1\}(1-\theta_1)Pt_1 + \{m_2 + \delta(1-m_2)\}\theta_1 Pt_1\Big]$$

The warranty cost:
During the period $[0, T]$, the warranty cost is given by

$$WC = c_w\Big[\{1-(1-\gamma)m_1\}(1-\theta_1)Pt_1 + \{m_2 + \delta(1-m_2)\}\theta_1 Pt_1\Big]$$

Now, we derive the equations for the present worth costs of the holding, setup, inspection, return, penalty, manufacturing, and rework for one cycle as follows:
Therefore, the profit function of production system is given by

$$\begin{aligned}
\pi_2(t_1, t_w) &= (s - c_w)[\{1-(1-\gamma)m_1\}(1-\theta_1)Pt_1 + \{\delta + (1-\delta)m_2\}\theta_1 Pt_1] \\
&\quad -\{c_p + c_s + c_r\delta(1-m_2)\theta_1\}Pt_1 - \frac{h_c}{2}\Big[\Big\{\{1-(1-\gamma)m_1\}(1-\theta_1)Pt_1 \\
&\quad +\{\delta + (1-\delta)m_2\}\theta_1 Pt_1 - (D_0 + \rho t_w)e^{k\eta}\Big\}t_1^2 + (D_0 + \rho t_w)(T - t_1)^2\Big] \\
&\quad -(A_0 + \frac{K}{Pt_1}) - \{(c_t + c_l)m_2\theta_1 Pt_1 - s(1-\gamma)m_1(1-\theta_1)Pt_1\} - B_0 t_1
\end{aligned}$$

Therefore, the expected profit during the time period $t_1 < \tau < T$ is given by

$$E[\pi_2(t_1,t_w)] = \Big[(s-c_w)\Big\{\{1-E[m_1(1-\gamma)]\}(1-\theta_1)+E[\{\delta+(1-\delta)m_2\}]\theta_1\Big\}$$
$$-\{c_p+c_s+c_rE[\delta(1-m_2)]\theta_1\}\Big]Pt_1\int_{t_1}^{\infty} f(\tau)\,d\tau - \frac{h_c}{2}\Big[\Big\{E[\{1-(1-\gamma)m_1\}](1-\theta_1)Pt_1$$
$$+E[\{\delta+(1-\delta)m_2\}]\theta_1 Pt_1-(D_0+\rho t_w)e^{k\eta}\Big\}t_1^2+(D_0+\rho t_w)(T-t_1)^2\Big]\int_{t_1}^{\infty} f(\tau)\,d\tau$$
$$-\Big[\{(c_t+c_l)E[m_2]\theta_1 Pt_1+c_rE[(1-\delta)m_1](1-\theta_1)Pt_1\}+(A_0+B_0t_1+\frac{K}{Pt_1})\Big]\int_{t_1}^{\infty} f(\tau)\,d\tau$$
$$=(s-c_w)\Big[\{1-E[m_1(1-\gamma)](1-\theta_1)Pt_1+E[\{\delta+(1-\delta)m_2\}]\theta_1 Pt_1\Big](1-\lambda t_1)$$
$$-\{c_p+c_s+c_rE[\delta(1-m_2)]\theta_1\}Pt_1(1-\lambda t_1)-\frac{h_c}{2}\Big[\Big\{E[\{1-(1-\gamma)m_1\}](1-\theta_1)Pt_1$$
$$+E[\{\delta+(1-\delta)m_2\}]\theta_1 Pt_1-(D_0+\rho t_w)e^{k\eta}\Big\}t_1^2+(D_0+\rho t_w)(T-t_1)^2\Big](1-\lambda t_1)$$
$$-\Big[\{(c_t+c_l)E[m_2]\theta_1 Pt_1+c_rE[(1-\delta)m_1](1-\theta_1)Pt_1\}+(A_0+B_0t_1+\frac{K}{Pt_1})\Big](1-\lambda t_1)$$

Now combining Case I and Case II, we have, the expected total profit during whole business period T is given by

$$E[\pi(t_1,t_w)] = (s-c_w)\Big[\{1-E[m_1(1-\gamma)]\}\{(1-\theta_1)Pt_1+(1-\theta_2)P\frac{\lambda t_1^2}{2}\}$$
$$+E[m_2+\delta(1-m_2)]\{\theta_1+\frac{\theta_2}{2}\lambda t_1\}Pt_1\Big]-\Big[(c_p+c_s)+c_rE[\delta(1-m_2)]\{\theta_1+\frac{\theta_2}{2}\lambda t_1\}\Big]Pt_1$$
$$-\frac{h_c}{2}\Big[\{E[1-(1-\gamma)m_1](1-\theta_1)P+E[\delta+(1-\delta)m_2]\theta_1 P\}(t_1^2+\lambda t_1^3-\frac{\lambda^2 t_1^4}{2})$$
$$+\frac{1}{3}\Big\{E[1-(1-\gamma)m_1](1-\theta_2)P+E[\delta+(1-\delta)m_2]\theta_2 P\Big\}\lambda t_1^3-(D_0+\rho t_w)e^{k\eta}t_1^2$$
$$+(D_0+\rho t_w)(T-t_1)^2\Big]-(c_t+c_l)E[m_2]\{\theta_1+\frac{\theta_2}{2}\lambda t_1\}Pt_1-c_rE[(1-\delta)m_1]\{(1-\theta_1)$$
$$+(1-\theta_2)\frac{\lambda t_1}{2}\}Pt_1-B_0t_1-\frac{B_1\lambda t_1^3}{6}e^{k_1\frac{v_{max}-v}{v-v_{min}}}-(A_0+\frac{K}{Pt_1})$$

Lemma 5.3. *In the manufacturing system the business period (T) must satisfy the following relation in terms of production rate (P), demand rate (D), warranty period (t_w) and production period (t_1) as follows*

$$T = \frac{1}{(D_0+\rho t_w)}\Big[\Big\{\{1-E[(1-\gamma)m_1]\}(1-\theta_1)P+\{1-E[(1-\delta)(1-m_2)]\}$$
$$\times\{2(\theta_2-\theta_1)\lambda t_1+\theta_1\}P+(D_0+\rho t_w)(1-e^{k\eta})\Big\}t_1\Big]$$

Proof. From Lemma 5.1, the expected value of T in $0 < \tau \leq t_1$ is given by

$$\begin{aligned}\int_0^{t_1} T\, f(\tau)\, d\tau \;=\; & \frac{1}{(D_0+\rho t_w)}\Big[\Big\{\{1-E[(1-\gamma)m_1]\}(1-\theta_2)P+(D_0+\rho t_w)(1-e^{k\eta}) \\ & +\{1-E[(1-\delta)(1-m_2)]\}\theta_2 P\Big\}t_1\int_0^{t_1} f(\tau)\, d\tau+\Big\{\{1-E[](1-\gamma)m_1]\} \\ & +\{1-E[(1-\delta)(1-m_2)]\}\Big\}(\theta_2-\theta_1)P\int_0^{t_1}\tau\, f(\tau)\, d\tau\Big] \qquad (5.6)\end{aligned}$$

Again, from Lemma 5.2, the expected value of T in $t_1 < \tau \leq \infty$ is given by

$$\begin{aligned}\int_{t_1}^{\infty} T\, f(\tau)\, d\tau \;=\; & \frac{1}{(D_0+\rho t_w)}[\{1-E[(1-\gamma)m_1]\}(1-\theta_1)P+(D_0+\rho t_w)(1-e^{k\eta}) \\ & +\{1-E[(1-\delta)(1-m_2)]\}\theta_1 P]t_1\int_{t_1}^{\infty} f(\tau)\, d\tau \qquad (5.7)\end{aligned}$$

Combining (5.6) and (5.7), we have

$$\begin{aligned}T \;=\; & \frac{1}{(D_0+\rho t_w)}\Big[\Big\{\{1-E[(1-\gamma)m_1]\}(1-\theta_1)P+\{1-E[(1-\delta)(1-m_2)]\} \\ & \times\{2(\theta_2-\theta_1)\lambda t_1+\theta_1\}P+(D_0+\rho t_w)(1-e^{k\eta})\Big\}t_1\Big]\end{aligned}$$

Now, the proof is complete. □

Hence, the average total expected profit is given by

$$\begin{aligned}& AEP(t_1,t_w)=\frac{(s-a-bt_w)}{T}\Big[\{1-E[m_1(1-\gamma)]\}\{(1-\theta_1)+(1-\theta_2)\frac{\lambda t_1}{2}\} \\ & +E[m_2+\delta(1-m_2)]\{\theta_1+\theta_2\frac{\lambda t_1}{2}\}\Big]Pt_1-\frac{Pt_1}{T}\Big[(c_p+c_s)+c_r E[\delta(1-m_2)]\{\theta_1+\theta_2\frac{\lambda t_1}{2}\}\Big] \\ & -\frac{h_c}{2T}\Big[\{E[1-(1-\gamma)m_1](1-\theta_1)P+E[\delta+(1-\delta)m_2]\theta_1 P\}(t_1^2+\lambda t_1^3-\frac{\lambda^2 t_1^4}{2}) \\ & +\frac{1}{3}\Big\{E[1-(1-\gamma)m_1](1-\theta_2)P+E[\delta+(1-\delta)m_2]\theta_2 P\Big\}\lambda t_1^3-(D_0+\rho t_w)e^{k\eta}t_1^2 \\ & +(D_0+\rho t_w)(T-t_1)^2\Big]-(c_t+c_l)E[m_2]\{\theta_1 t_1+\theta_2\frac{\lambda t_1^2}{2}\}\frac{P}{T}-\frac{c_r}{T}E[(1-\delta)m_1] \\ & \times\{(1-\theta_1)Pt_1+(1-\theta_2)P\frac{\lambda t_1^2}{2}\}-\frac{B_0 t_1}{T}-\frac{B_1\lambda t_1^3}{6T}e^{k_1\frac{v_{max}-v}{v-v_{min}}}-\frac{1}{T}(A_0+\frac{K}{Pt_1}) \qquad (5.8)\end{aligned}$$

By taking the first derivatives of $AEP(t_1, t_w)$ with respect to t_w, we have

$$\frac{\partial}{\partial t_w}\{AEP(t_1,t_w)\} = -\frac{b}{T}\Big[\{1-E[m_1(1-\gamma)]\}\{(1-\theta_1)Pt_1 + (1-\theta_2)P\frac{\lambda t_1^2}{2}\}$$
$$+E[m_2+\delta(1-m_2)]\{\theta_1+\theta_2\frac{\lambda t_1}{2}\}Pt_1\Big] + \frac{h_c}{2T}\Big[\rho e^{k\eta}(t_1^2+\frac{\lambda^2 t_1^4}{2}) - \rho(T-t_1)^2\Big]$$
$$-\frac{(s-a-bt_w)}{T^2}\Big[\{1-E[m_1(1-\gamma)]\}\{(1-\theta_1)Pt_1+(1-\theta_2)P\frac{\lambda t_1^2}{2}\} + E[m_2$$
$$+\delta(1-m_2)]\{\theta_1 Pt_1+\theta_2 P\frac{\lambda t_1^2}{2}\}\Big]\frac{\partial T}{\partial t_w} + \frac{1}{T^2}\Big[(c_p+c_s)Pt_1 + c_r E[\delta(1-m_2)]\{\theta_1 Pt_1$$
$$+\theta_2 P\frac{\lambda t_1^2}{2}\}\Big]\frac{\partial T}{\partial t_w} + \frac{h_c}{2T^2}\Big[\{(1-\theta_1)P + E[\delta]\theta_1 P - (D_0+\rho t_w)e^{k\eta}\}(t_1^2 - \lambda t_1^3 + \frac{\lambda^2 t_1^4}{2})$$
$$+\{(1-\theta_2)P + E[\delta]\theta_2 P - (D_0+\rho t_w)e^{k\eta}\}\lambda t_1^3 + (D_0+\rho t_w)(T-t_1)^2\Big]\frac{\partial T}{\partial t_w}$$
$$+\frac{1}{T^2}(c_t+c_l)E[m_2]\{\theta_1 Pt_1+\theta_2 P\frac{\lambda t_1^2}{2}\}\frac{\partial T}{\partial t_w} + \frac{c_r}{T^2}E[(1-\delta)m_1]\{(1-\theta_1)Pt_1$$
$$+(1-\theta_2)P\frac{\lambda t_1^2}{2}\}\frac{\partial T}{\partial t_w} + \frac{B_1\lambda t_1^3}{6T^2}\frac{\partial T}{\partial t_w}e^{k_1\frac{v_{max}-v}{v-v_{min}}} + \frac{1}{T^2}(A_0+B_0 t_1+\frac{K}{Pt_1})\frac{\partial T}{\partial t_w}$$

Taking the second derivatives of $AEP(t_1, t_w)$ with respect to t_w, we have

$$\frac{\partial^2}{\partial t_w^2}\{AEP(t_1,t_w)\} = \frac{2b}{T^2}\Big[\{1-E[m_1(1-\gamma)]\}\{(1-\theta_1)Pt_1+(1-\theta_2)P\frac{\lambda t_1^2}{2}\}$$
$$+E[m_2+\delta(1-m_2)]\{\theta_1+\theta_2\frac{\lambda t_1}{2}\}Pt_1\Big]\frac{\partial T}{\partial t_w} + \frac{h_c}{T^2}\Big[\rho e^{k\eta}(t_1^2+\frac{\lambda^2 t_1^4}{2}) - \rho(T-t_1)^2\Big]$$
$$+(s-a-bt_w)\Big[\{1-E[m_1(1-\gamma)]\}\{(1-\theta_1)Pt_1+(1-\theta_2)P\frac{\lambda t_1^2}{2}\} + E[m_2+\delta(1$$
$$-m_2)]\{\theta_1 Pt_1+\theta_2 P\frac{\lambda t_1^2}{2}\}\Big]\Big\{\frac{2}{T^3}(\frac{\partial T}{\partial t_1})^2 - \frac{1}{T^2}\frac{\partial^2 T}{\partial t_1^2}\Big\} + \Big[(c_p+c_s)Pt_1 + c_r E[\delta(1-m_2)]$$
$$\times\{\theta_1 Pt_1+\theta_2 P\frac{\lambda t_1^2}{2}\}\Big]\Big\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1^2} - \frac{2}{T^3}(\frac{\partial T}{\partial t_1})^2\Big\} + \frac{h_c}{2}\Big[\{(1-\theta_1)P + E[\delta]\theta_1 P - (D_0$$
$$+\rho t_w)e^{k\eta}\}(t_1^2-\lambda t_1^3+\frac{\lambda^2 t_1^4}{2}) + \{(1-\theta_2)P + E[\delta]\theta_2 P\}\lambda t_1^3 + (D_0+\rho t_w)\{(T-t_1)^2$$
$$-e^{k\eta}\lambda t_1^3\}\Big]\Big\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1^2} - \frac{2}{T^3}(\frac{\partial T}{\partial t_1})^2\Big\} + \Big[(c_t+c_l)E[m_2]\{\theta_1 Pt_1+\theta_2 P\frac{\lambda t_1^2}{2}\}$$
$$+\frac{c_r}{T^2}E[(1-\delta)m_1]\{(1-\theta_1)Pt_1+(1-\theta_2)P\frac{\lambda t_1^2}{2}\}\Big]\Big\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1^2} - \frac{2}{T^3}(\frac{\partial T}{\partial t_1})^2\Big\}$$
$$+\frac{B_1\lambda t_1^3}{6T^2}\Big\{\frac{\partial^2 T}{\partial t_1^2} - \frac{2}{T^3}(\frac{\partial T}{\partial t_1})^2\Big\}e^{k_1\frac{v_{max}-v}{v-v_{min}}} + (A_0+B_0 t_1+\frac{K}{Pt_1})\Big\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1^2} - \frac{2}{T^3}(\frac{\partial T}{\partial t_1})^2\Big\}$$

By taking the first derivatives of $AEP(t_1, t_w)$ with respect to t_1, we have

$$\frac{\partial}{\partial t_1}\{AEP(t_1,t_w)\} = \frac{(s-a-bt_w)}{T}\Big[\{1-E[m_1(1-\gamma)]\}\{(1-\theta_1)P+(1-\theta_2)P\lambda t_1\}$$
$$+E[m_2+\delta(1-m_2)]\{\theta_1 P+\theta_2 P\lambda t_1\}\Big] - \frac{1}{T}\Big[(c_p+c_s)P+c_r E[\delta(1-m_2)]\{\theta_1 P$$
$$+\theta_2 P\lambda t_1\}\Big] - \frac{h_c}{2T}\Big[\{(1-\theta_1)P+E[\delta]\theta_1 P-(D_0+\rho t_w)e^{k\eta}\}(2t_1-3\lambda t_1^2+2\lambda^2 t_1^3)$$
$$+3\{(1-\theta_2)P+E[\delta]\theta_2 P-(D_0+\rho t_w)e^{k\eta}\}\lambda t_1^2-2(D_0+\rho t_w)(T-t_1)\Big]$$
$$-\frac{1}{T}(c_t+c_l)E[m_2]\{\theta_1 P+\theta_2 P\lambda t_1\}-\frac{c_r}{T}E[(1-\delta)m_1]\{(1-\theta_1)+(1-\theta_2)\lambda t_1\}P$$
$$-\frac{B_0}{T}-\frac{B_1\lambda t_1^2}{2T}e^{k\frac{v_{max}-v}{v-v_{min}}}+\frac{K}{PTt_1^2}-\frac{(s-a-bt_w)}{T^2}\Big[\{1-E[m_1(1-\gamma)]\}$$
$$\times\{(1-\theta_1)Pt_1+(1-\theta_2)P\frac{\lambda t_1^2}{2}\}+E[m_2+\delta(1-m_2)]\{\theta_1 Pt_1+\theta_2 P\frac{\lambda t_1^2}{2}\}\Big]\frac{\partial T}{\partial t_1}$$
$$+\frac{1}{T^2}\Big[(c_p+c_s)Pt_1+c_r E[\delta(1-m_2)]\{\theta_1 Pt_1+\theta_2 P\frac{\lambda t_1^2}{2}\}\Big]\frac{\partial T}{\partial t_1}$$
$$+\frac{h_c}{2T^2}\Big[\{(1-\theta_1)P+E[\delta]\theta_1 P-(D_0+\rho t_w)e^{k\eta}\}(t_1^2-\lambda t_1^3+\frac{\lambda^2 t_1^4}{2})$$
$$+\{(1-\theta_2)P+E[\delta]\theta_2 P-(D_0+\rho t_w)e^{k\eta}\}\lambda t_1^3+(D_0+\rho t_w)(T-t_1)^2\Big]\frac{\partial T}{\partial t_1}$$
$$+\frac{1}{T^2}(c_t+c_l)E[m_2]\{\theta_1 Pt_1+\theta_2 P\frac{\lambda t_1^2}{2}\}\frac{\partial T}{\partial t_1}+\frac{c_r}{T^2}E[(1-\delta)m_1]\{(1-\theta_1)Pt_1$$
$$+(1-\theta_2)P\frac{\lambda t_1^2}{2}\}\frac{\partial T}{\partial t_1}+\frac{B_1\lambda t_1^3}{6T^2}\frac{\partial T}{\partial t_1}e^{k_1\frac{v_{max}-v}{v-v_{min}}}+\frac{1}{T^2}(A_0+B_0 t_1+\frac{K}{Pt_1})\frac{\partial T}{\partial t_1}$$

Taking the second derivatives of $AEP(t_1, t_w)$ with respect to t_1, we have

$$\frac{\partial^2}{\partial t_1^2}\{AEP(t_1,t_w)\} = \frac{(s-a-bt_w)}{T}\Big[\{1-E[m_1(1-\gamma)]\}(1-\theta_2)P\lambda$$
$$+E[m_2+\delta(1-m_2)]\theta_2 P\lambda\Big]-\frac{c_r}{T}E[\delta(1-m_2)]\theta_2 P\lambda$$
$$-\frac{h_c}{2T}\Big[\{(1-\theta_1)P+E[\delta]\theta_1 P-(D_0+\rho t_w)e^{k\eta}\}(2-6\lambda t_1+6\lambda^2 t_1^2)$$
$$+6\{(1-\theta_2)P+E[\delta]\theta_2 P-(D_0+\rho t_w)e^{k\eta}\}\lambda t_1+2(D_0+\rho t_w)\Big]$$
$$-\frac{1}{T}(c_t+c_l)E[m_2]\theta_2 P\lambda-\frac{c_r}{T}E[(1-\delta)m_1](1-\theta_2)P\lambda+\frac{B_1\lambda t_1^2}{T}e^{k_1\frac{v_{max}-v}{v-v_{min}}}-\frac{2K}{PTt_1^3}$$
$$-\frac{2(s-a-bt_w)}{T^2}\Big[\{1-E[m_1(1-\gamma)]\}\{(1-\theta_1)P+(1-\theta_2)P\lambda t_1\}+E[m_2+\delta(1-m_2)]$$
$$\times\{\theta_1 P+\theta_2 P\lambda t_1\}\Big]\frac{\partial T}{\partial t_1}+\frac{2}{T^2}\Big[(c_p+c_s)P+c_r E[\delta(1-m_2)]\{\theta_1 P+\theta_2 P\lambda t_1\}\Big]\frac{\partial T}{\partial t_1}$$

$$
\begin{aligned}
&+\frac{h_c}{T^2}\Big[\{(1-\theta_1)P+E[\delta]\theta_1P-(D_0+\rho t_w)e^{k\eta}\}(2t_1-3\lambda t_1^2+2\lambda^2t_1^3)\\
&+3\{(1-\theta_2)P+E[\delta]\theta_2P-(D_0+\rho t_w)e^{k\eta}\}\lambda t_1^2-2(D_0+\rho t_w)(T-t_1)\Big]\frac{\partial T}{\partial t_1}\\
&+\frac{2}{T^2}(c_t+c_l)E[m_2]\{\theta_1P+\theta_2P\lambda t_1\}\frac{\partial T}{\partial t_1}+\frac{2c_r}{T^2}E[(1-\delta)m_1]\{(1-\theta_1)P\\
&+(1-\theta_2)P\lambda t_1\}\frac{\partial T}{\partial t_1}+\frac{2B_0}{T^2}\frac{\partial T}{\partial t_1}+\frac{B_1\lambda t_1^2}{2T^2}\frac{\partial T}{\partial t_1}e^{k\frac{v_{max}-v}{v-v_{min}}}+\frac{2K}{PT^2t_1^2}\frac{\partial T}{\partial t_1}\\
&+(s-a-bt_w)\Big[\{1-E[m_1(1-\gamma)]\}\{(1-\theta_1)Pt_1+(1-\theta_2)P\frac{\lambda t_1^2}{2}\}\\
&+E[m_2+\delta(1-m_2)]\{\theta_1Pt_1+\theta_2P\frac{\lambda t_1^2}{2}\}\Big]\Big\{\frac{2}{T^3}(\frac{\partial T}{\partial t_1})^2-\frac{1}{T^2}\frac{\partial^2T}{\partial t_1^2}\Big\}\\
&+\Big[(c_p+c_s)Pt_1+c_rE[\delta(1-m_2)]\{\theta_1Pt_1+\theta_2P\frac{\lambda t_1^2}{2}\}\Big]\Big\{\frac{1}{T^2}\frac{\partial^2T}{\partial t_1^2}-\frac{2}{T^3}\frac{\partial T}{\partial t_1}\Big\}\\
&+\frac{h_c}{2}\Big[\{(1-\theta_1)P+E[\delta]\theta_1P-(D_0+\rho t_w)e^{k\eta}\}(t_1^2-\lambda t_1^3+\frac{\lambda^2t_1^4}{2})\\
&+\{(1-\theta_2)P+E[\delta]\theta_2P-(D_0+\rho t_w)e^{k\eta}\}\lambda t_1^3+(D_0+\rho t_w)(T-t_1)^2\Big]\\
&\times\Big\{\frac{1}{T^2}\frac{\partial^2T}{\partial t_1^2}-\frac{2}{T^3}\frac{\partial T}{\partial t_1}\Big\}+(c_t+c_l)E[m_2]\{\theta_1Pt_1+\theta_2P\frac{\lambda t_1^2}{2}\}\frac{\partial T}{\partial t_1}\Big\{\frac{1}{T^2}\frac{\partial^2T}{\partial t_1^2}-\frac{2}{T^3}\frac{\partial T}{\partial t_1}\Big\}\\
&+c_rE[(1-\delta)m_1]\{(1-\theta_1)Pt_1+(1-\theta_2)P\frac{\lambda t_1^2}{2}\}\Big\{\frac{1}{T^2}\frac{\partial^2T}{\partial t_1^2}-\frac{2}{T^3}\frac{\partial T}{\partial t_1}\Big\}\\
&+\frac{B_1\lambda t_1^3}{6}\Big\{\frac{1}{T^2}\frac{\partial^2T}{\partial t_1^2}-\frac{2}{T^3}\frac{\partial T}{\partial t_1}\Big\}e^{k_1\frac{v_{max}-v}{v-v_{min}}}+(A_0+B_0t_1+\frac{K}{Pt_1})\Big\{\frac{1}{T^2}\frac{\partial^2T}{\partial t_1^2}-\frac{2}{T^3}\frac{\partial T}{\partial t_1}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial^2}{\partial t_1\partial t_w}\{AEP(t_1,t_w)\}=-\frac{b}{T}\Big[\{1-E[m_1(1-\gamma)]\}\{(1-\theta_1)P+(1-\theta_2)P\lambda t_1\}\\
&+E[m_2+\delta(1-m_2)]\{\theta_1P+\theta_2P\lambda t_1\}\Big]+\frac{h_c}{2T}\Big[\rho e^{k\eta}(2t_1+2\lambda^2t_1^3)+2\rho(T-t_1)\Big]\\
&+\frac{b}{T^2}\Big[\{1-E[m_1(1-\gamma)]\}\{(1-\theta_1)Pt_1+(1-\theta_2)P\frac{\lambda t_1^2}{2}\}+E[m_2+\delta(1-m_2)]\\
&\times\{\theta_1Pt_1+\theta_2P\frac{\lambda t_1^2}{2}\}\Big]\frac{\partial T}{\partial t_1}-\frac{h_c}{2T^2}\Big[\rho e^{k\eta}(t_1^2+\frac{\lambda^2t_1^4}{2})-\rho(T-t_1)^2\Big]\frac{\partial T}{\partial t_1}-\frac{(s-a-bt_w)}{T^2}\\
&\times\Big[\{1-E[m_1(1-\gamma)]\}\{(1-\theta_1)P+(1-\theta_2)P\lambda t_1\}+E[m_2+\delta(1-m_2)]\\
&\times\{\theta_1P+\theta_2P\lambda t_1\}\Big]\frac{\partial T}{\partial t_w}+\frac{1}{T^2}\Big[(c_p+c_s)P+c_rE[\delta(1-m_2)]\{\theta_1P+\theta_2P\lambda t_1\}\Big]\frac{\partial T}{\partial t_w}\\
&+\frac{h_c}{2T^2}\Big[\{(1-\theta_1)P+E[\delta]\theta_1P-(D_0+\rho t_w)e^{k\eta}\}(2t_1-3\lambda t_1+2\lambda^2t_1^3)\\
&+\{(1-\theta_2)P+E[\delta]\theta_2P-(D_0+\rho t_w)e^{k\eta}\}3\lambda t_1^2-2(D_0+\rho t_w)(T-t_1)\Big]\frac{\partial T}{\partial t_w}
\end{aligned}
$$

$$+\frac{1}{T^2}(c_t+c_l)E[m_2]\{\theta_1 P+\theta_2 P\lambda t_1\}\frac{\partial T}{\partial t_w}+\frac{c_r}{T^2}E[(1-\delta)m_1]\{(1-\theta_1)P$$
$$+(1-\theta_2)P\lambda t_1\}\frac{\partial T}{\partial t_w}+\frac{B_1\lambda t_1^2}{2T^2}\frac{\partial T}{\partial t_w}e^{k_1\frac{v_{max}-v}{v-v_{min}}}+\frac{1}{T^2}(B_0-\frac{K}{Pt_1^2})\frac{\partial T}{\partial t_w}$$
$$+(s-a-bt_w)\Big[\{1-E[m_1(1-\gamma)]\}\{(1-\theta_1)Pt_1+(1-\theta_2)P\frac{\lambda t_1^2}{2}\}$$
$$+E[m_2+\delta(1-m_2)]\{\theta_1 Pt_1+\theta_2 P\frac{\lambda t_1^2}{2}\}\Big]\Big\{\frac{2}{T^3}\frac{\partial T}{\partial t_1}\frac{\partial T}{\partial t_w}-\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1\partial t_w}\Big\}$$
$$+\Big[(c_p+c_s)Pt_1+c_rE[\delta(1-m_2)]\{\theta_1 Pt_1+\theta_2 P\frac{\lambda t_1^2}{2}\}\Big]\Big\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1\partial t_w}-\frac{2}{T^3}\frac{\partial T}{\partial t_1}\frac{\partial T}{\partial t_w}\Big\}$$
$$+\frac{h_c}{2}\Big[\{(1-\theta_1)P+E[\delta]\theta_1 P-(D_0+\rho t_w)e^{k\eta}\}(t_1^2-\lambda t_1^3+\frac{\lambda^2 t_1^4}{2})$$
$$+\{(1-\theta_2)P+E[\delta]\theta_2 P-(D_0+\rho t_w)e^{k\eta}\}\lambda t_1^3+(D_0+\rho t_w)(T-t_1)^2\Big]$$
$$\times\Big\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1\partial t_w}-\frac{2}{T^3}\frac{\partial T}{\partial t_1}\frac{\partial T}{\partial t_w}\Big\}+(c_t+c_l)E[m_2]\{\theta_1 Pt_1+\theta_2 P\frac{\lambda t_1^2}{2}\}$$
$$\times\Big\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1\partial t_w}-\frac{2}{T^3}\frac{\partial T}{\partial t_1}\frac{\partial T}{\partial t_w}\Big\}+\frac{B_1\lambda t_1^3}{6}\Big\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1\partial t_w}-\frac{2}{T^3}\frac{\partial T}{\partial t_1}\frac{\partial T}{\partial t_w}\Big\}e^{k\frac{v_{max}-v}{v-v_{min}}}$$
$$+c_rE[(1-\delta)m_1]\{(1-\theta_1)Pt_1+(1-\theta_2)P\frac{\lambda t_1^2}{2}\}\Big\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1\partial t_w}-\frac{2}{T^3}\frac{\partial T}{\partial t_1}\frac{\partial T}{\partial t_w}\Big\}$$
$$+(A_0+B_0t_1+\frac{K}{Pt_1})\Big\{\frac{1}{T^2}\frac{\partial^2 T}{\partial t_1\partial t_w}-\frac{2}{T^3}\frac{\partial T}{\partial t_1}\frac{\partial T}{\partial t_w}\Big\}$$

Now the average expected profit (AEP) is a function of two independent variables t_1 and t_w. Here it is considered that $\frac{\partial}{\partial t_1}(AEP(t_1,t_w))=F(t_1,t_w)$ and $\frac{\partial}{\partial t_w}(AEP(t_1,t_w))=G(t_1,t_w)$. Due to complexity of the equations, $F(t_1,t_w)=0$ and $G(t_1,t_w)=0$, it is not possible to show the existence of the solution analytically. Now it is supposed that there exists at least one positive point (t_1^r,t_w^r) for which $F(t_1^r,t_w^r)=0$ and $G(t_1^r,t_w^r)=0$ for some parametric values involved in the system.
Let at (t_1^r,t_w^r), $\frac{\partial F}{\partial t_1}=\Delta_1$, $\frac{\partial G}{\partial t_w}=\Delta_2$ and $\frac{\partial F}{\partial t_w}=\Delta_3$.

Lemma 5.4. *The maximum average profit $AEP(t_1^r,t_w^r)$ exist if $\Delta_1\Delta_2>\Delta_3^2$, $\Delta_1<0$ and $\Delta_2<0$.*

Proof. Now, from the optimization calculus, it is known that a function of two variables, $\phi(u,v)$ is maximum at the stationary point (a,b) if $\frac{\partial^2}{\partial u^2}(\phi(a,b))$ $\frac{\partial^2}{\partial v^2}(\phi(a,b))$ - $\{\frac{\partial^2}{\partial u\partial v}(\phi(a,b))\}^2>0$, $\frac{\partial^2}{\partial u^2}(\phi(a,b))<0$ and $\frac{\partial^2}{\partial v^2}(\phi(a,b))<0$.
The $1st$ condition for the existence of maximum value of $AEP(t_1,t_w)$ at the point (t_1^r,t_w^r) is $\frac{\partial^2}{\partial t_1^2}(AEP(t_1^r,t_w^r))$ $\frac{\partial^2}{\partial t_w^2}(AEP(t_1^r,t_w^r))$-$\{\frac{\partial^2}{\partial t_1\partial t_w}(AEP(t_1^r,t_w^r))\}^2>0$.
i.e., $\frac{\partial}{\partial t_1}(F(t_1^r,t_w^r))$ $\frac{\partial}{\partial t_w}(G(t_1^r,t_w^r))$-$\{\frac{\partial}{\partial t_w}(F(t_1^r,t_w^r))\}^2$ $>$ 0, since

$\frac{\partial}{\partial t_1}(AEP(t_1,t_w)) = F(t_1,t_w)$ and $\frac{\partial}{\partial t_w}(AEP(t_1,t_w)) = G(t_1,t_w)$.
i.e., $\Delta_1\Delta_2 > \Delta_3^2$, since $\frac{\partial}{\partial t_1}F(t_1^r,t_w^r) = \Delta_1$, $\frac{\partial}{\partial t_w}G(t_1^r,t_w^r) = \Delta_2$ and $\frac{\partial}{\partial t_w}F(t_1^r,t_w^r) = \Delta_3$.
The $2nd$ condition for the existence of maximum value of $AEP(t_1,t_w)$ at the point (t_1^r,t_w^r) is $\frac{\partial^2}{\partial t_1^2}(AEP(t_1^r,t_w^r)) < 0$. i.e., $\frac{\partial}{\partial t_1}(F(t_1^r,t_w^r)) < 0$, since $\frac{\partial}{\partial t_1}(AEP(t_1,t_w)) = F(t_1,t_w)$.
i.e., $\Delta_1 < 0$, since $\frac{\partial}{\partial t_1}F(t_1^r,t_w^r) = \Delta_1$.
The $3rd$ condition for the existence of maximum value of $AEP(t_1,t_w)$ at the point (t_1^r,t_w^r) is $\frac{\partial^2}{\partial t_w^2}(AEP(t_1^r,t_w^r)) < 0$. i.e., $\frac{\partial}{\partial t_w}(G(t_1^r,t_w^r)) < 0$, since $\frac{\partial}{\partial t_w}(AEP(t_1,t_w)) = G(t_1,t_w)$.
i.e., $\Delta_2 < 0$, since $\frac{\partial}{\partial t_w}G(t_1^r,t_w^r) = \Delta_2$. Now, the proof is complete. □

Lemma 5.5. *There does not exist the maximum average profit $AEP(t_1^r,t_w^r)$ if $\Delta_1 > 0$ and $\Delta_2 > 0$.*

Lemma 5.6. *There does not exist the maximum average profit $AEP(t_1^r,t_w^r)$ if $\Delta_1\Delta_2 - \Delta_3^2 < 0$.*

5.4 Solution Procedure

From equation (5.8), it is seen that in the proposed model, the objective function $AEP(t_1,t_w)$ is highly nonlinear. Here, t_1 and t_w are two decision variables. Also T is a function of t_1 and t_w obtained according to Lemma 5.3. Since the objective function is highly nonlinear, hence to get the optimal solution of the proposed model the following algorithms have been developed.

Algorithm 5.1. *For a fixed x, suppose $x = x_0$, the value of y from $\psi(x,y) = 0$ can be obtained as follows:*

step 1: For $x = x_0$, compute $\psi(x_0,y) = 0$.

step 2: Select (y_1,y_2) such that $\psi(x_0,y_1)\psi(x_0,y_2) < 0$. Then by Rolle's theorem there exist a root of $\psi(x_0,y) = 0$, between y_1 and y_2.

step 3: Calculate $m = \frac{(y_1+y_2)}{2}$, be the midpoint of the interval (y_1,y_2).

step 4: Compute the signs of $\psi(x_0,y_1)$, $\psi(x_0,m)$, and $\psi(x_0,y_2)$.

step 5: If $\psi(x_0,y_1)\psi(x_0,m) < 0$, then a root of $\psi(x_0,y) = 0$ lies between y_1 and m. In this case, replace y_2 by m. Otherwise, a root of $\psi(x_0,y) = 0$ lies between m and y_2, then replace y_1 by m.

step 6*: Repeat steps* 3 *through* 5 *until* $|y_1 - y_2| < 10^{-\varepsilon}$ *where* ε *is a tolerance limit.*

step 7*: Then the root of* $\psi(x_0, y) = 0$ *is* m *such that* $m = \frac{(y_1+y_2)}{2}$.

Algorithm 5.2. *Since there is no possibility to get the general explicit solution due to absence of linearity of the profit function, to get the maximum profit the following procedure has been devised according to Lemma* 5.4 *and Algorithm* 5.1*. Here the optimal values of* T, t_1, t_w *and* $AEP(t_1, t_w)$ *are denoted by* T^*, t_1^*, t_w^* *and* AEP^* *respectively.*

step 1*: Initialize all parameters associated with the objective function* $AEP(t_1, t_w)$.

step 2*: Set an interval* (t_{10}, t_{11}) *where* $t_{10} \in (0, T_0)$ *and* $t_{11} \in (0, T_0)$*. Here* $t_w \leq T_0$ *where* T_0 *also is initialized.*

step 3*: Compute* t_{w0F}, t_{w1F}, t_{w0G} *and* t_{w1G} *for* t_w *from* $F(t_{10}, t_w) = 0$, $F(t_{11}, t_w) = 0$, $G(t_{10}, t_w) = 0$ *and* $G(t_{11}, t_w) = 0$ *respectively by Algorithm* 5.1.

step 4*: Compute* $\triangle_{t_{10}} = t_{w0F} - t_{w0G}$ *and* $\triangle_{t_{11}} = t_{w1F} - t_{w1G}$.

step 5*: If* $\triangle_{t_{10}} \triangle_{t_{11}} < 0$*, i.e., the signs of* $\triangle_{t_{10}}$ *and* $\triangle_{t_{11}}$ *are opposite, then compute* $t_{1m} = \frac{(t_{10}+t_{11})}{2}$.

step 6*: Compute* t_{w1mF} *and* t_{w1mG} *for* t_w *from* $F(t_{1m}, t_w) = 0$ *and* $G(t_{1m}, t_w) = 0$ *respectively by Algorithm* 5.1.

step 7*: Calculate* $\triangle_{t_{1m}} = \triangle_{t_{w1mF}} - \triangle_{t_{w1mG}}$.

step 8*: Compare* $\triangle_{t_{1m}}$ *with* $\triangle_{t_{10}}$*. If* $\triangle_{t_{10}} \triangle_{t_{1m}} < 0$*, i.e., the signs of* $\triangle_{t_{10}}$ *and* $\triangle_{t_{1m}}$ *are opposite, then replace* t_{11} *by* t_{1m}*. Otherwise replace* t_{10} *by* t_{1m}.

step 9*: Repeat steps* 5 *through* 8 *until the absolute values of* $(t_{10} - t_{1m})$ *or* $(\triangle_{t_{10}} - \triangle_{t_{1m}})$ *or* $(\triangle_{t_{10}} - \triangle_{t_{1m}})$ *are within the tolerance limits.*

step 10*: The root of* $F(t_1, t_w) = 0$ *and* $G(t_1, t_w) = 0$ *is* $({t_1}^r, {t_w}^r)$ *where* ${t_1}^r = t_{1m}$ *and* $t_{wr} = \frac{t_{w0F}+t_{w1F}}{2}$ *or* $\frac{t_{w0G}+t_{w1G}}{2}$.

step 11*: Compute* $\triangle_1$*,* $\triangle_2$ *and* $\triangle_3$ *at the point* (t_1^r, t_w^r) *where* $\frac{\partial F}{\partial t_1} = \Delta_1$*,* $\frac{\partial G}{\partial t_w} = \Delta_2$ *and* $\frac{\partial F}{\partial t_w} = \Delta_3$*.*

step 12*: If* $\Delta_1 < 0$*,* $\Delta_2 < 0$ *and* $\Delta_1\Delta_2 > \Delta_3^2$*, then according Lemma* 5.4 *then* $({t_1}^r, {t_w}^r)$ *be the optimal solution. So* $t_1^* = {t_1}^r$*,* $t_w^* = {t_w}^r$ *and calculate* T^* *by Lemma* 5.3*. Also calculate* $AEP^* = AEP(t_1^*, t_w^*)$*.*

step 13*: If* $\Delta_1 > 0$*,* $\Delta_2 > 0$ *by Lemma* 5.5*, or* $\Delta_1\Delta_2 - \Delta_3^2 < 0$ *by Lemma* 5.6*, then* $({t_1}^r, {t_w}^r)$ *is not optimal solution. In this case, goto step* 1 *and change some parametric values.*

step 14*: Print the optimal values* t_1^**,* t_w^**,* T^* *and* AEP^**.*

5.5 Numerical Illustrations

It is considered that a production inventory system that produces perfect and defective items, continuously fills up the customer demand. The inspection process that screens out the defective items is also imperfect. Since in a production system the time at which a process goes from in-control to out-of-control in a cycle has been considered random which is exponentially distributed with mean $\frac{1}{\lambda}$. Similarly, the parameters for inspection errors and rework rate have been considered as uniform distribution. The probability density functions of the inspection errors and rework rate are mostly taken from the history of a machine and workers. Using the above mentioned solution procedure (Section 5.4), the optimum values of t_1, t_w, T and the average expected total profit, $AEP(t_1, t_w)$ have been calculated for the following values of the parameters of the illustrated model:
$P = 100$ unit per unit time, $D_0 = 45$ unit per unit time, $\theta_1 = 0.05$, $\theta_2 = 0.12$, $s = \$85$ per unit, $\lambda = 0.01$, $c_p = \$25$ per unit, $c_{sr} = \$3$ per unit, $c_r = \$15$ per unit, $(c_t + c_l) = \$5.5$ per unit, $h_c = \$1.5$ per unit per unit time, $h_c = \$1.5$ per unit per unit time, $a = \$15$, $v_{max} = 10$, $v_{min} = 3$, $v = 8$, $B_1 = \$58$, $B_0 = \$45$ per unit time, $A_0 = \$257$, $k = 1$, $K = 25$.
The probability density functions of the inspection errors (m_1 and m_2), fraction of rejecting non-defective items (γ) due to type I error and rework rate (δ) are considered as follows:

$$\phi(m_1) = \begin{cases} \frac{1}{\alpha}, & 0 \leq m_1 \leq \alpha \\ 0, & \text{otherwise} \end{cases} \qquad \phi(m_2) = \begin{cases} \frac{1}{\beta}, & 0 \leq m_2 \leq \beta \\ 0, & \text{otherwise} \end{cases}$$

$$\phi(\gamma) = \begin{cases} \frac{1}{\xi}, & 0 \leq \gamma \leq \xi \\ 0, & \text{otherwise} \end{cases} \qquad \phi(\delta) = \begin{cases} \frac{1}{\mu}, & 0 \leq \delta \leq \mu \\ 0, & \text{otherwise} \end{cases}$$

Now we calculate $E[m_2]$, $E[\delta]$, $E[(1-\gamma)m_1]$, $E[\delta(1-m_2)]$, and $E[(1-\delta)(1-m_2)]$ and the values are given by

$$E[m_2] = \int_0^\beta m_2\phi(m_2)\,d\delta = \frac{\beta}{2}, \quad E[\delta] = \int_0^\mu \delta\phi(\delta)\,d\delta = \frac{\mu}{2}$$
$$E[(1-\gamma)m_1] = \int_0^\alpha m_1\phi(m_1)\,dm_1 \int_0^\xi (1-\gamma)\phi(\gamma)\,d\gamma = \frac{\alpha}{2}(1-\frac{\xi}{2})$$
$$E[\delta(1-m_2)] = \int_0^\mu \delta\phi(\delta)\,d\delta \int_0^\alpha (1-m_2)\phi(m_2)\,dm_1 = \frac{\mu}{2}(1-\frac{\beta}{2})$$
$$E[(1-\delta)(1-m_2)] = \int_0^\alpha (1-\delta)\phi(\delta)\,d\delta \int_0^\beta (1-m_2)\phi(m_2)\,dm_1 = (1-\frac{\mu}{2})(1-\frac{\beta}{2})$$

Substituting the above expressions in the profit function in equation (5.8), we obtain the optimal values of the expected average profit when $\alpha = 0.04$, $\beta = 0.06$, $\xi = 0.0004$ and $\mu = 0.60$, $\eta = 0.10$, $\rho = 0.50$, $b_0 = 0.40$, $k = 0.80$.

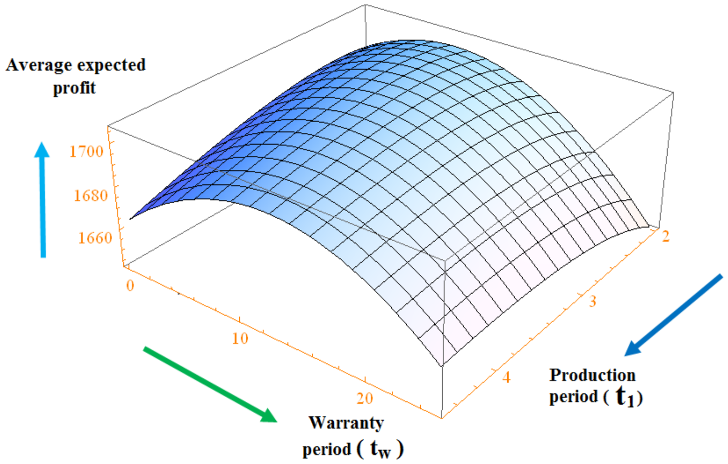

Figure 5.4: Expected average profit vs production period (t_1) and warranty period (t_w).

For this data set, Figure 5.4 shows the average expected profit as a function of t_1 and t_w. From this figure it is guarantied that the average expected profit is concave. So there exist unique solution of (t_1, t_w) that maximize the average expected profit $AEP(t_1, t_w)$. The optimal solutions for the given parametric set with different type of demand rate are represented by following Table 5.2, 5.3 and 5.4:

Table 5.2: Optimal result of the illustrated model

	Production period (t_1^* umit)	Warranty period (t_w^* unit)	Business cycle period (T^* unit)	Expected average profit ($AEP^*(t_1^*, t_w^*)$ \$)
Optimal value	2.59	7.79	4.93	1706.15

Table 5.3: Optimal result of the model with the effect of only warranty period on demand

	Production period (t_1^* umit)	Warranty period (t_w^* unit)	Business cycle period (T^* unit)	Expected average profit ($AEP^*(t_1^*, t_w^*)$ \$)
Optimal value	2.26	5.17	4.61	1624.77

Table 5.4: Optimal result of the model with the effect of only discount on demand

	Production period (t_1^* umit)	Warranty period (t_w^* unit)	Business cycle period (T^* unit)	Expected average profit ($AEP^*(t_1^*, t_w^*)$ \$)
Optimal value	2.28	–	4.73	1694.55

5.5.1 Sensitivity Analysis

The change in the values of the system parameters can take place an important role in decision-making about the system due to uncertainties and dynamic market conditions. In order to examine the implications of these changes in the values of parameters, the sensitivity analysis will be of great help in a decision-making process. Here, the sensitivity analysis with respect to the parameters such as α, β, μ, η, k, ρ, λ, and b have been carried out. The results of the sensitivity analysis are shown in Table 5.5, 5.6, 5.7, 5.8, 5.9, 5.10 and 5.11.

Table 5.5: Sensitivity analysis w.r.t. the probability of Type I error (α)

Parameter (α)	Production period (t_1^* unit)	Warranty period (t_w^* unit)	Business cycle period (T^* unit)	Expected average profit ($AEP^*(t_1^*, t_w^*)$ \$)
0.04	2.59	7.78	4.93	1706.15
0.08	2.66	6.89	5.01	1669.81
0.12	2.72	5.95	5.09	1632.42
0.16	2.80	4.98	5.18	1593.94
0.20	2.87	3.97	5.27	1554.34

• From Table 5.5, we see that the warranty period (t_w^*), the business cycle period (T^*) and the expected average profit $(AEP^*(t_1^*, t_w^*))$ decrease with the increase of α i.e., probability of a Type I error. But, the production period (t_1^*) increases when α increases.

Table 5.6: Sensitivity analysis w.r.t. the probability of Type II error (β)

Parameter (β)	Production period (t_1^* unit)	Warranty period (t_w^* unit)	Business cycle period (T^* unit)	Expected average profit $(AEP^*(t_1^*, t_w^*)\ \$)$
0.06	2.59	7.78	4.93	1706.15
0.12	2.56	7.39	4.89	1694.23
0.18	2.53	7.01	4.86	1682.38
0.24	2.50	6.63	4.83	1670.60
0.30	2.48	6.25	4.80	1658.89

• From Table 5.6, we see that the production period (t_1^*), warranty period (t_w^*), the business cycle period (T^*) and the expected average profit $(AEP^*(t_1^*, t_w^*))$ decrease with the increase of β i.e., probability of a Type II error.

Table 5.7: Sensitivity analysis w.r.t. rework rate (μ) and reworked cost (c_r) simultaneously

Parameter (μ)	Reworked cost (c_r \$)	Production period (t_1^* unit)	Warranty period (t_w^* unit)	Business cycle time (T^* unit)	Expected average profit $(AEP^*(t_1^*, t_w^*)\ \$)$
0.45	11	2.56	7.81	4.90	1692.84
0.50	12	2.57	7.82	4.91	1697.85
0.55	13	2.58	7.83	4.92	1702.78
0.60	15	2.59	7.79	4.93	1706.15
0.65	20	2.58	7.61	4.92	1704.75
0.70	25	2.58	7.42	4.92	1702.96
0.75	28	2.58	7.22	4.92	1700.77

• From Table 5.7 it is observed that when μ **and** c_r **increases simultaneously**, the production period (t_1^*), warranty period (t_w^*) and the expected average profit $(AEP^*(t_1^*, t_w^*))$ initially increase, then decrease due to the rapidly increase of average rework cost for defective item.

Table 5.8: Sensitivity analysis of t_1, t_w, T and EAP w.r.t. η for a fixed k

Parameter (η)	Production period (t_1^* unit)	Warranty period (t_w^* unit)	Business cycle period (T^* unit)	Expected average profit ($AEP^*(t_1^*, t_w^*)$ \$)
0.06	2.44	6.67	4.78	1671.61
0.08	2.51	7.22	4.85	1688.52
0.10	2.59	7.78	4.92	1706.15
0.12	2.68	8.38	5.01	1724.55
0.14	2.76	9.01	5.11	1743.78

• Table 5.8 signifies that when k **is fixed**, the production period (t_1^*), warranty period (t_w^*), the business cycle period (T^*) and the expected average profit ($AEP^*(t_1^*, t_w^*)$) increase together **due to the increase of discount rate** η.

Table 5.9: Sensitivity analysis of t_1, t_w, T and EAP w.r.t. η and k simultaneously

Parameter (η and k)	Production period (t_1^* unit)	Warranty period (t_w^* unit)	Business cycle period (T^* unit)	Expected average profit ($AEP^*(t_1^*, t_w^*)$ \$)
$\eta = 0.05$, $k = 1.2$	2.50	7.08	4.83	1684.23
$\eta = 0.07$, $k = 1.0$	2.54	7.43	4.88	1695.04
$\eta = 0.10$, $k = 0.8$	2.59	7.78	4.92	1706.15
$\eta = 0.12$, $k = 0.6$	2.55	7.50	4.88	1697.24
$\eta = 0.14$, $k = 0.4$	2.47	6.94	4.81	1679.98

• Table 5.9 shows that when η **increases and** k **decreases simultaneously**, the production period (t_1^*),warranty period (t_w^*), the business cycle period (T^*) and the average expected profit ($AEP^*(t_1^*, t_w^*)$) initially increase, after that decrease.

Table 5.10: Sensitivity analysis of t_1, t_w, T and EAP w.r.t. λ

Parameter (λ)	Production period (t_1^* unit)	Warranty period (t_w^* unit)	Business cycle period (T^* unit)	Expected average profit ($AEP^*(t_1^*, t_w^*)$ \$)
0.005	2.26	7.01	4.32	1690.68
0.010	2.59	7.78	4.92	1706.15
0.015	3.10	8.87	5.84	1724.09
0.020	3.99	10.54	7.41	1745.99
0.025	5.73	13.48	10.40	1774.95

• Table 5.10 explores that when the value of λ increases, both the production period (t_1^*) and warranty period (t_w^*) increase as well as the corresponding business cycle period T^* and expected average profit $(AEP^*(t_1^*, t_w^*))$ also increase due to the increase of the in-control state.

Table 5.11: Sensitivity analysis of t_1, t_w, T and EAP w.r.t. ρ

Parameter (ρ)	Production period (t_1^* unit)	Warranty period (t_w^* unit)	Business cycle period (T^* unit)	Expected average profit ($AEP^*(t_1^*, t_w^*)$ \$)
0.45	2.36	2.38	4.77	1695.54
0.50	2.59	7.78	4.92	1706.15
0.55	2.87	12.34	5.14	1726.22
0.60	3.22	16.33	5.44	1753.88
0.65	3.71	19.96	5.91	1788.05

• Table 5.11 shows that when the value of ρ increases, all of the production period (t_1^*), warranty period (t_w^*) and the business cycle period (T^*) increases together. In this case it is also observed that the expected average profit $(AEP^*(t_1^*, t_w^*))$ increases **due to increase of the demand rate**.

5.5.2 Practical Implication

There are many practical implications of this proposed model. As for example, it is very practicable in the manufacturing system for mobile phones. At the time of production few defective units (like, scratching, disorder shape, etc) are produced and then some of them be repaired to sell at the market. Sometimes the company gives a discount on selling price and increases the warranty period to increase selling rate. The decision manager of the company decides the maximization of the profit function, considering the warranty period of each product and the length of the production cycle. For such a real life problem, the present model can be implemented. From this study some managerial insights have been drawn which are very useful for the decision maker of any newly established mobile company.

5.6 Conclusion

In this chapter, we have studied the combined effects of deterioration of production process, inspection errors, warranty policy and discount on selling price. In practice, generally both production and inspection processes in a manufacturing system are not perfect. In most of the existing literature, there is a consideration of imperfect production except inspection errors. Here, a profit maximization model has been developed with two types of inspection errors with a demand depending on warranty period and discount on selling price. To solve such complex objective function, a computational algorithm have been devised to determine the optimal warranty period and optimal production period considering the randomness of the time from in-control to out-control state, inspection errors and defective rate. Finally, numerical examples has been presented to explore the feasibility of the proposed model from which the following insights have been drawn.

(i) From Table 5.2, 5.3 and 5.4 it is concluded that the average expected profit is maximum when manufacturer gives both effects such as (a) selling price discount, and (b) warranty period policy on the sale because of attraction of customer.

(ii) Again, from Table 5.5 and 5.6 it is inferred that the expected average profit decreases as the probability of a Type I and Type II error increase. This is because of (a) addition to the loss of incorrectly rejection of a non-defective item, and (b) return and penalty cost for a defective item to be sold as a non-defective item which is returned from market.

(iii) Table 5.7 shows that when the probability of reworking rate and average reworked cost simultaneously increase, initially average expected profit increases, after that average expected profit decrease due to some portion of defective item to be reworked with a minimum reworked cost and rest portion of defective item to be reworked with a large amount reworked cost. So any manufacturer company can find the optimal reworked rate from this study.

(iv) Table 5.9 shows that when selling price discount(η) increases and the corresponding effective parameter (k) simultaneously decreases, initially the average expected profit increases, after that the average expected profit decreases. Because at first selling price discount attracts more customers. As a result, the demand rate increases. But latter, though discount rate increases, the rate of demand does not increase as much as previous due to market saturation. So from this study, any manufacturer company can find the optimal selling price discount rate.

Chapter 6

Two layers supply chain in an imperfect production inventory model with two storage facilities under reliability consideration

6.1 Introduction

Supply chain model is the new era in the imperfect production inventory management. It is the oversight of materials, information and finances as they move in a process from supplier to manufacturer to wholesaler to retailer to consumer. It involves coordinating and integrating these flows both within and among companies. The investigation of supply chain management for defective and non-defective items has been published by several researchers such as Weng [223], Munson and Rosenblatt [157], Khouja [117], Yao et al. [234], Wang et al. [216].

Inventory management is generally attracted for large stock for several reasons such as an attractive price discount for bulk purchase, the replenishment cost, transportation cost, the demand of an item and so on. Therefore, due to space limitation of showroom, one (or sometimes more than one) warehouse(s) is hired on rental basis to store the excess items. The secondary warehouse (SW) may be located away from the Showroom or nearer to the showroom. The actual service to the customer is done at the showroom only. Hartely [90] first introduced the basic two warehouses problem in his book "Operations Research - A Managerial Emphasis". After Hartely [90], a number of research papers have been published by the different authors. Among them, the work done by Sarma [189], Dave [53], Goswami and Chaudhuri [76], Pakkala and Achary [161], Bhunia and Maiti [11], Benkherouf [10], Zhou [243], Kar et al. [111] and Chung & Huang [45] are worth mentioning. Dey et al. [57] considered a finite time horizon inventory problem for a deteriorating item having two separate warehouses with interval-valued lead time under inflation and a time value of money. Liang and Zhou [134] investigated a two-warehouse inventory model for deteriorating items under conditionally permissible delay in payments. Hariga [88] proposed an EOQ model with multiple storage facilities where both owned and rented warehouses had limited stock capacity. They assumed the rented warehouse having higher unit holding costs than the own warehouse but offered better preservation resulting in a lower rate of deterioration for the goods than in the own warehouse.

The production before the due time to meet the unexpected demand may cause to spoilage as it cannot be stored till the particular time arises or it cannot be perfectly estimated the demand. As a result, firm faces the problem of modeling the production of seasonal items, estimating the reliability of the factors of production to undertake the orders from the customers, budgeting the costs to be incurred to set up the EPQ model. Hazari et. al [98], Panda and Maiti [163] and others discussed several dimensional EPQ model under the consideration of reliability. But, none has considered the reliability conception in a supply chain model with two warehouse facilities. The production of complete perfect items is not possible in reality due to unrealistic assumption of 100% reliable system. Moreover, the imperfect production also may arise to switch over "in-control" state to "out-of-control" state in a long run production system.

In this chapter, a supply chain model consisting of manufacturer, retailer and customer has been considered. The production system may undergo an "out-of-control" state from an "in-control" state, after a certain time that follows a probability density function. The density function varies with reliability of the machinery system that may be controlled by new technologies, investing more capital. A mixture of perfect and imperfect quality items are produced by the manufacturer. After some rework, some repairable portion of imperfect quality items is transformed into perfect quality items and some of non repairable portion of imperfect items are sold with reduced price to the retailer. Retailer purchases both perfect and imperfect quality items and sales both items to the customers through his/her respective showrooms of finite capacities at a market place. In this chapter, it has been considered that the holding cost of perfect quality items in the secondary warehouse is less than the holding cost of the showroom as the nature of the items are non-deteriorating and so having no preservation cost. The perfect quality items are transported to the showroom via secondary warehouse and the less perfect quality items are directly transported to the related showroom. For this purpose, transportation cost is incurred to transport both quality items at the respective showrooms from the production center. In order to optimize the production rate and reliability parameter(the decision variables), the integrated average profit function of the chain is maximized. Finally, a numerical example has been provided to illustrate the feasibility of the model.

6.2 Notations and Assumptions

To develop the proposed model, following notations and assumptions have been used.

6.2.1 Notations

The following notations have been used to develop the model.

$q_{1m}(t)$	:	Inventory level of perfect quality items for the manufacturer at any time t.
$q_{2m}(t)$	:	Inventory level of less perfect quality items for the manufacturer at any time t.
$q_{1r}(t)$	:	Inventory level of perfect quality items for the retailer at any time t.
$q_{2r}(t)$	:	Inventory level of less perfect quality items for the retailer at any time t.
P	:	Production rate per unit time.
t_1	:	Production run-time in one period.
β	:	Percentage of imperfect quality items per unit time.
δ	:	Percentage of rework of defective units per unit time.
γ	:	Percentage of imperfect items per unit time which are not reworked and suitable for sale through reduction.
D_r	:	Demand rate of perfect quality items for the retailer.
D'_r	:	Demand rate of imperfect quality items for the retailer.
D_c	:	Demand rate of perfect quality items for the customer.
D'_c	:	Demand rate of imperfect quality items for the customer.
τ	:	An exponential random variable with mean $\frac{1}{\lambda}$ and denotes the time at which the production system shifts from 'in-control' state to the 'out-of-control' state.
$f(\tau)$	:	Probability density function of τ.
λ	:	Reliability parameter.
$V(\lambda)$	:	Development Cost per unit time.
T	:	Total time in one period.
c_p	:	Production cost per unit.
s_c	:	Screening cost per unit item.
r_{cm}	:	Reworking cost per unit for manufacturer.
A_m	:	Set up cost of manufacturer, $A_m = A_{m0} + A_{m1}P^{\xi}$, $\xi > 0$.
h_m	:	Holding cost per unit for per unit time for perfect item in manufacturer.
h'_m	:	Holding cost per unit for per unit time for imperfect item in manufacturer.
c_d	:	Disposal cost per unit.
s_m	:	Selling price per unit of perfect quality items for manufacturer.
s'_m	:	Selling price per unit less perfect quality items for manufacturer.
A_r	:	Set up cost of retailer.
h_r	:	Holding cost per unit item per unit time of perfect quality items in PW_1 of retailer.
h_{rs}	:	Holding cost per unit item per unit time of perfect quality items of retailer in secondary ware house.

h'_r	:	Holding cost per unit item per unit time of less perfect quality items in PW_2 for retailer.
s_r	:	Selling price per unit of perfect quality items for retailer at PW_1.
s'_r	:	Selling price per unit of less perfect quality items for retailer at PW_2.
S	:	Capacity of secondary warehouse.
W_j	:	Capacity of PW_j(j=1,2).

6.2.2 Assumptions

The following assumptions have been used to develop the model.

(i) Manufacturer produces a mixture of perfect and imperfect quality items. Some portion of imperfect items are reworked at a cost.

(ii) Production rate (P) is a decision variable satisfying $P_{min} \leq P \leq P_{max}$.

(iii) Lead time is negligible and manufacturer has ignored the machine breakdown.

(iv) In the 'in-control' state (the system is in better condition) of the production system it produces perfect items and after shifting the system in the out-control state (i.e., after occurrence of τ), the condition of production system decreases with time and produces defective items. If the production rate P increases the occupance of τ is very quickly, i.e., mean time of failure decreases with P (i.e., the production system shifts from 'in-control' state to 'out-control' state very quickly) and produce more non conforming items then earlier. So, a development cost is required to control the occurrence of τ and smaller value of λ. We take the development cost $V(\lambda)$ as (cf. Mettas [149])

$$V(\lambda) = A + Be^{(1-k)\frac{\lambda_{max}-\lambda}{\lambda-\lambda_{min}}} \tag{6.1}$$

where, $\lambda \in [\lambda_{min}, \lambda_{max}]$, A is the fixed cost like labor and energy costs which is independent of reliability factor (λ), B is the cost of technology and design complexity for production when $\lambda = \lambda_{max}$. The constant k represents the feasibility of increasing reliability of the production system lies between 0 and 1.

(v) Reliability parameter of a production system is defined as $\lambda = \frac{\text{number of failures}}{\text{total unit of operating hours}}$. Generally, in an imperfect production system, number of failures (i.e., number of imperfect items) increases with the increases of production rate P. i.e., the value of reliability parameter λ is dependent upon P and it is increasing with P. Here we take $\lambda = \lambda_1 \phi(P)$, where $\phi(P)$ is an increasing function of P and mean time failure $\frac{1}{\lambda_1 \phi(P)}$ is a decreasing function of P (i.e., the production system will shifts from 'in-control' state to 'out-control' state rapidly if P increases) where λ_1 is a parameter satisfying $0 \leq \lambda_1 < \frac{1}{\phi(P)}$.
Since $f(P)$ is increasing function of P so P_{min} and P_{max} respectively gives minimum

and maximum value of λ (i.e., λ_{min} and λ_{max}). Therefore, $\lambda_{min} = \lambda_1\phi(P)\Big]_{P=P_{min}}$ and $\lambda_{max} = \lambda_1\phi(P)\Big]_{P=P_{max}}$.

(vi) In this chapter, the unit production cost has been considered as a function of development cost and production rate in the following form

$$C(P) = C_r + \frac{V(\lambda)}{P^{\delta_1}} + \alpha_1 P^{\delta_2} + \alpha_2 P^{\delta_3},\ \ \delta_1, \delta_2, \delta_3 > 0 \tag{6.2}$$

where (a) C_r is the material cost per unit which is fixed.
(b) The second term is the development cost.
(iii) The third term $\alpha_1 P^{\delta_2}$ is wear and tear cost.
(c) The fourth term $\alpha_2 P^{\delta_3}$ is environment protection cost assuming that the cost due to the measures taken for the environment protection. For example, in a thermal electricity plant, the inject of 'ash' in the atmosphere depends upon the rate of production. If the production is more, the amount of required raw material i.e., impure coal is more and hence the amount of 'fly ash' is more. Now-a-days, some measures are taken to reduce the 'fly ash' amount. Thus the cost due to this measure varies with the production rate.

(vii) It is assumed that, demand rates of retailers to the manufacturer is constant. But the customers' demand is stock dependent and selling price dependent for the perfect quality items and less perfect items respectively. Since the storage space of the showroom for perfect quality items is limited due to space problem and the demand of the corresponding items is stock dependent, hence a secondary warehouse is hired by the retailer on rental basis to store the excess amount of perfect quality items and these items are continuously transferred to the showroom concerned. The literature suggests that the holding cost of secondary warehouse per unit item per unit time is more than the holding cost of showroom due to the preservation cost for maintaining the quality of the product and other costs related to holding large quantity of the product in the secondary warehouse.

$(viii)$ Set up cost of manufacturer has been considered as production rate dependent. Also assume that the set up cost of retailer depends on the demand rate of retailer to the manufacturer.

(ix) It is well known that the quality and imperfectness of a product depends on raw material, labour experience, machine component, production rate, production-run time etc. Here we assume that the defective rate (β) depends upon the production rate (P) of the production center and simultaneously it depends upon the time length $(t - \tau)$ in the out-control state and is given by

$$\beta = \begin{cases} 0\,, & 0 \le t \le \tau \\ (\theta_0 - \frac{\theta_1}{P^{\delta_5}})(t-\tau)^{\delta_4}\,, & \tau \le t \le t_1 \end{cases} \tag{6.3}$$

where θ_0 and θ_1 be the positive constants.
The above construction of β have the following properties:

Proposition 6.1. *If P increases then $(\theta_0 - \frac{\theta_1}{P^{\delta_5}})$ (for fixed value of δ_5) increases and when $P \to \infty$ then $\beta \to \theta_0(t-\tau)^{\delta_4}$, where θ_0 is the maximum value of $(\theta_0 - \frac{\theta_1}{P^{\delta_5}})$.*

Proposition 6.2. *For fixed value of P if we increase the value of parameter δ_5 then $(\theta_0 - \frac{\theta_1}{P^{\delta_5}})$ increases and it become θ_0 for large value of δ_5.*

Proposition 6.3. *If $t \to \tau$ then $\beta \to 0$ and if $t \to t_1$ then $\beta \to \beta_{max}$, where β_{max} is the maximum value of β. i.e., in out control state, defective rate increases with the δ_4 power of time length of the out control state, whatever may be the production rate.*

Proposition 6.4. *If $\delta_4 \neq 0$, $\delta_5 = 0$ then $(\theta_0 - \frac{\theta_1}{P^{\delta_5}}) = \theta_0 - \theta_1$ i.e., β does not dependon production rate P.*

Proposition 6.5. *If $\delta_4 = 0$, $\delta_5 \neq 0$ then β does not depend upon the time length inthe outcontrol state.*

Proposition 6.6. *If $\delta_4 = 0$, $\delta_5 = 0$ then production system has constant defective rate $(\theta_0 - \theta_1)$.*

(x) After reworking, the defective items are restored to its original quality.

(xi) Here two adjacent showrooms PW_1 and PW_2 have been considered by the retailer to store the perfect items and imperfect items respectively.

6.3 Mathematical Formulation of the Proposed Model

In this article, the production rate and demand rate of the inventory system is constant. During the regular production-run-time, the production system may undergo 'out-of-control' state after a random time τ that follows exponential probability distribution function with mean $\frac{1}{\lambda_1 \phi(P)}$ and during that time rate of producing imperfect items is β. When the system is in "in-control", then the system may produce perfect items and the probability of producing perfect items at the rate P. The defective items which are produced during regular production-run-time are reworked with rate δ. Inventory holding cost per unit of perfect items is much more than the holding cost of imperfect items.

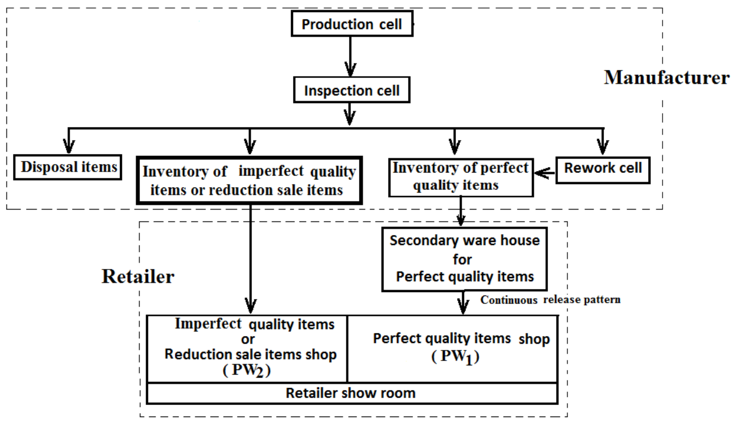

Figure 6.1: Schematic representation of the production inventory model

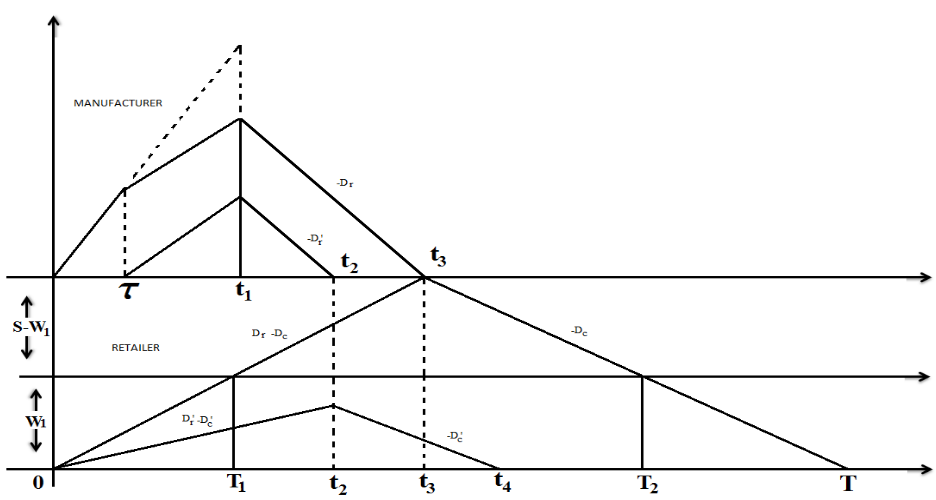

Figure 6.2: Pictorial representation of inventory situation of the integrated model

6.3.1 Formulation of the Manufacturer

At the beginning of the production, the process is assumed to be in "in-control" state. After a random time τ with mean $\frac{1}{\lambda_1\phi(P)}$, the process may shift to an "out-of-control" state and may generate non-conforming quality items. Reliability of machines in a manufacturing system is generally assumed to be an exponential function of time t which is $R(t) = exp(-\lambda t)$. Since a unit either fails or survives, and one of these two mutually exclusive alternatives must occur, we have $R(t) = 1 - F(t)$. Here $F(t) = \int_0^t f(u)\,du$, where $f(u)$ denotes the failure probability density function such that $\int_0^\infty f(t)\,dt = 1$. Thus $f(t) = \frac{d}{dt}\{F(t)\} = \lambda_1\phi(P)e^{-\lambda_1\phi(P)t}$.
In our production system, two cases may arise:

Case I: when $0 \le \tau \le t_1$ the 'out-of-control' state occurs during production-run time

Here the production starts with production rate P. During production run time t_1, the inventory piles up, after adjusting the demand of retailer. As τ occurs in the time span $(0, t_1)$, the system produces good quality items during $[0, \tau]$ and it produces both good and defective items during $[\tau, t_1]$. The total good items produced during $[0, t_1]$ are used to meet the demand of perfect and less perfect item upto time t_2 and t_3 respectively.

For Perfect Quality Items of Manufacturer

The rate of change of inventory level of manufacturer for perfect quality items can be represented by the following differential equations:

$$\frac{dq_{1m}}{dt} = \begin{cases} P - D_r, & 0 \le t \le \tau \\ P - D_r - (1-\delta)\beta P, & \tau \le t \le t_1 \\ -D_r, & t_1 \le t \le t_3 \end{cases} \tag{6.4}$$

with boundary conditions $q_{1m}(0) = 0,\ q_{1m}(\tau) = (P - D_r)\tau$ and $q_{1m}(t_3) = 0$.
The solution of above differential equations are given by

$$q_{1m}(t) = \begin{cases} (P - D_r)t, & 0 \le t \le \tau \\ (P - D_r)t - (1-\delta)P(\theta_0 - \frac{\theta_1}{P^{\delta_5}})\frac{(t-\tau)^{\delta_4+1}}{\delta_4+1}, & \tau \le t \le t_1 \\ -D_r(t - t_3), & t_1 \le t \le t_3 \end{cases} \tag{6.5}$$

Lemma 6.1. *The manufacturer's production time length (t_1) and production rate (P) must satisfy the condition* $Pt_1 - (1-\delta)P(\theta_0 - \frac{\theta_1}{P^{\delta_5}})\frac{(t_1-\tau)^{\delta_4+1}}{\delta_4+1} = D_r t_3$.

Proof. From the continuity conditions of $q_{1m}(t)$ at $t = t_1$, the following is obtained,
$(P - D_r)t_1 - (1-\delta)P(\theta_0 - \frac{\theta_1}{P^{\delta_5}})\frac{(t_1-\tau)^{\delta_4+1}}{\delta_4+1} = -D_r(t_1 - t_3)$
i.e., $Pt_1 - (1-\delta)P(\theta_0 - \frac{\theta_1}{P^{\delta_5}})\frac{(t_1-\tau)^{\delta_4+1}}{\delta_4+1} = D_r t_3$. Hence the proof. □

Reworking Cost(RCM) for Manufacture

$$RCM = r_{cm}\int_{\tau}^{t_1} \delta\beta P\, dt = \frac{r_{cm}\delta}{\delta_4+1}P(\theta_0 - \frac{\theta_1}{P^{\delta_5}})(t_1 - \tau)^{\delta_4+1}.$$

Inventory holding cost for perfect items is:

$$\begin{aligned} HCM_1 &= h_m\Big[\int_0^{\tau} q_{1m}(t)dt + \int_{\tau}^{t_1} q_{1m}(t)dt + \int_{t_1}^{t_3} q_{1m}(t)dt\Big] \\ &= \frac{h_m}{2}\Big[(P - D_r)t_1^2 - \frac{2(1-\delta)}{(\delta_4+1)(\delta_4+2)}P(\theta_0 - \frac{\theta_1}{P^{\delta_5}})(t_1-\tau)^{\delta_4+2} + D_r(t_1 - t_3)^2\Big] \end{aligned}$$

For Imperfect Quality Items of Manufacturer

The rate of change of inventory level of manufacturer for imperfect quality items can be represented by the following differential equations:

$$\frac{dq_{2m}}{dt} = \begin{cases} \gamma(1-\delta)\beta P - D'_r, & \tau \leq t \leq t_1 \\ -D'_r, & t_1 \leq t \leq t_2 \end{cases} \tag{6.6}$$

with boundary conditions $q_{2m}(\tau) = 0,$ and $q_{2m}(t_2) = 0.$
The solution of above differential equations are given by

$$q_{2m}(t) = \begin{cases} \gamma(1-\delta)P(\theta_0 - \frac{\theta_1}{P^{\delta_5}})\frac{(t-\tau)^{\delta_4+1}}{\delta_4+1} - D'_r(t-\tau), & \tau \leq t \leq t_1 \\ -D'_r(t-t_2), & t_1 \leq t \leq t_2 \end{cases} \tag{6.7}$$

Lemma 6.2. *The manufacturer's production time length (t_1) and production rate (P) must satisfy the condition* $\gamma(1-\delta)P(\theta_0 - \frac{\theta_1}{P^{\delta_5}})\frac{(t_1-\tau)^{\delta_4+1}}{\delta_4+1} + D'_r\tau = D'_r t_2$.

Proof. From the continuity conditions of $q_{2m}(t)$ at $t = t_1$, the following is obtained,
$\gamma(1-\delta)P(\theta_0 - \frac{\theta_1}{P^{\delta_5}})\frac{(t_1-\tau)^{\delta_4+1}}{\delta_4+1} - D'_r(t_1-\tau) = -D'_r(t_1-t_2)$
i.e., $\gamma(1-\delta)P(\theta_0 - \frac{\theta_1}{P^{\delta_5}})\frac{(t_1-\tau)^{\delta_4+1}}{\delta_4+1} + D'_r\tau = D'_r t_2$. Hence the proof. □

Inventory holding cost for imperfect items is given by

$$HCM_2 = h'_m\Big[\int_{\tau}^{t_1} q_{2m}(t)dt + \int_{t_1}^{t_2} q_{1m}(t)dt\Big]$$
$$= \frac{h'_m}{2}\Big[\Big\{\frac{2\gamma(1-\delta)}{(\delta_4+1)(\delta_4+2)}P(\theta_0 - \frac{\theta_1}{P^{\delta_5}})(t_1-\tau)^{\delta_4+2} - D'_r\Big(t_1-\tau\Big)^2\Big\} + D'_r\Big(t_1-t_2\Big)^2\Big]$$

Production cost for the manufacturer= $c_p P t_1$.
Inspection cost= $s_c P t_1$.
Revenue of perfect quality items for the manufacturer= $s_m\int_0^{t_3} D_r dt = s_m D_r t_3$.
Revenue of imperfect quality items for the manufacturer= $s'_m\int_{\tau}^{t_2} D'_r dt, = s'_m D'_r(t_2-\tau)$.
Amount of Disposal Items
The rate of change of inventory level of the disposal amount during the period $(0, T)$ can be represented by the following differential equations:

$$\frac{dq_{3m}}{dt} = (1-\gamma)(1-\delta)\beta P, \quad \tau \leq t \leq t_1 \tag{6.8}$$

Total disposal amount during the period $(0, T)$ = $(1-\gamma)(1-\delta)P(\theta_0 - \frac{\theta_1}{P^{\delta_5}})\frac{(t_1-\tau)^{\delta_4+1}}{\delta_4+1}$

In this case, total expected profit of manufacturer $(TEPM)$ is given by

$$\begin{aligned} TEPM &= \Big[\int_0^{t_1} s_m D_r t_3 f(\tau)\,d\tau + s_m'\int_0^{t_1} D_r'(t_2-\tau)f(\tau)\,d\tau - c_p\int_0^{t_1} Pt_1 f(\tau)d\tau \\ &-s_c\int_0^{t_1} Pt_1 f(\tau)\,d\tau - r_{cm}\int_0^{t_1}\frac{\delta}{\delta_4+1}P(\theta_0-\frac{\theta_1}{P^{\delta_5}})(t_1-\tau)^{\delta_4+1}f(\tau)d\tau \\ &-\frac{h_m}{2}\int_0^{t_1}\Big[(P-D_r)t_1^2-\frac{2(1-\delta)}{(\delta_4+1)(\delta_4+2)}P(\theta_0-\frac{\theta_1}{P^{\delta_5}})(t_1-\tau)^{\delta_4+2} \\ &+D_r(t_1-t_3)^2\Big]f(\tau)d\tau - \frac{h_m'}{2}\int_0^{t_1}\Big[\Big\{\frac{2\gamma(1-\delta)}{(\delta_4+1)(\delta_4+2)}P(\theta_0-\frac{\theta_1}{P^{\delta_5}})(t_1-\tau)^{\delta_4+2} \\ &-D_r'\Big(t_1-\tau\Big)^2\Big\}+D_r'\Big(t_1-t_2\Big)^2\Big]f(\tau)d\tau - \int_0^{t_1} A_m f(\tau)\,d\tau \\ &-c_d\int_0^{t_1}(1-\gamma)(1-\delta)P(\theta_0-\frac{\theta_1}{P^{\delta_5}})\frac{(t_1-\tau)^{\delta_4+1}}{\delta_4+1}f(\tau)\,d\tau\Big] \end{aligned} \quad (6.9)$$

Case II: when $t_1 \le \tau \le \infty$, the "out-of-control" state does not occur within the production-run time

In this case, no "out-of-control" state occurs during production run-time. Then governing differential equations of the inventory system are given as follows:

For Perfect Quality Items of Manufacturer

$$\frac{dq_{1m}}{dt} = \begin{cases} P-D_r, & 0\le t\le t_1 \\ -D_r, & t_1\le t\le t_3 \end{cases} \quad (6.10)$$

with boundary conditions $q_{1m}(0)=0$, and $q_{1m}(t_3)=0$.
The solution of above differential equations are given by

$$q_{1m} = \begin{cases} (P-D_r)t, & 0\le t\le t_1 \\ -D_r(t-t_3), & t_1\le t\le t_3 \end{cases} \quad (6.11)$$

Inventory holding cost for perfect quality items is given by

$$HCM = h_m\Big[\int_0^{t_1} q_{1m}(t)\,dt + \int_{t_1}^{t_3} q_{1m}(t)\,dt\Big] = \frac{h_m}{2}\Big[(P-D_r)t_1^2 + D_r(t_1-t_3)^2\Big]$$

Production cost for the manufacturer= c_pPt_1.
Inspection cost= s_cPt_1.
Revenue of perfect quality items for the manufacturer= $s_m\int_0^{t_3} D_r dt = s_m D_r t_3$.

In this case, total expected profit of manufacturer $(TEPM)$ is given by

$$\begin{aligned} TEPM &= \Big[\int_{t_1}^{\infty} s_m D_r t_3 f(\tau)\,d\tau - \int_{t_1}^{\infty} c_p P t_1 f(\tau) d\tau - \int_{t_1}^{\infty} s_c P t_1 f(\tau)\,d\tau \\ &\quad -\frac{h_m}{2}\int_{t_1}^{\infty}\Big\{(P-D_r)t_1^2 + D_r(t_1-t_3)^2\Big\} f(\tau)d\tau - \int_{t_1}^{\infty} A_m f(\tau)\,d\tau\Big] \end{aligned} \tag{6.12}$$

Now combining Case I and Case II, by (6.9) and (6.12), the total expected profit is

$$\begin{aligned} TEPM &= s_m D_r t_3 + s'_m \int_0^{t_1} D'_r(t_2-\tau) f(\tau)\,d\tau - c_p P t_1 - s_c P t_1 \\ &\quad -\Big\{c_d(1-\gamma)(1-\delta) + r_{cm}\delta\Big\}\frac{P}{\delta_4+1}(\theta_0 - \frac{\theta_1}{P^{\delta_5}})\int_0^{t_1}(t_1-\tau)^{\delta_4+1} f(\tau) d\tau \\ &\quad -\frac{h_m}{2}\int_0^{t_1}\frac{2(1-\delta)}{(\delta_4+1)(\delta_4+2)} P(\theta_0 - \frac{\theta_1}{P^{\delta_5}})(t_1-\tau)^{\delta_4+2} f(\tau)\,d\tau \\ &\quad -\frac{h'_m}{2}\int_0^{t_1}\Big[\Big\{\frac{2\gamma(1-\delta)}{(\delta_4+1)(\delta_4+2)} P(\theta_0 - \frac{\theta_1}{P^{\delta_5}})(t_1-\tau)^{\delta_4+2} - D'_r\Big(t_1-\tau\Big)^2\Big\} \\ &\quad +D'_r\Big(t_1-t_2\Big)^2\Big] f(\tau)d\tau - A_m - \frac{h_m}{2}\Big\{(P-D_r)t_1^2 + D_r(t_1-t_3)^2\Big\} \end{aligned}$$

6.3.2 Formulation of the Retailer

For Perfect Quality Items of Retailer

In this case the demand rate (D_c) of customers at PW_1 has consider as stock dependent as the following form

$$D_c = \begin{cases} \alpha_1 + \beta_1 q_{1r}, & 0 \le t \le T_1 \\ \alpha_1 + \beta_1 W_1, & T_1 \le t \le T_2 \\ \alpha_1 + \beta_1 q_{1r}, & T_2 \le t \le T \end{cases}$$

Now the corresponding rate of change of inventory of perfect quality items are given by

$$\frac{dq_{1r}}{dt} = \begin{cases} (D_r - D_c), & 0 \le t \le t_1 \\ -D_c, & t_1 \le t \le T \end{cases} \tag{6.13}$$

with boundary conditions

$$q_{1r}(t) = \begin{cases} 0, & \text{at } t = 0 \\ W_1, & \text{at } t = T_1 \\ S, & \text{at } t = t_3 \\ W_1, & \text{at } t = T_2 \\ 0, & \text{at } t = T \end{cases}$$

Therefore the solutions of above differential equations are given by

$$q_{1r}(t) \;=\; \begin{cases} \frac{D_r-\alpha_1}{\beta_1}(1-e^{-\beta_1 t}), & 0 \le t \le T_1 \\ W_1 + \Big[D_r - (\alpha_1+\beta_1 W_1)\Big](t-T_1), & T_1 \le t \le t_3 \\ S - \Big[(\alpha_1+\beta_1 W_1)\Big](t-t_3), & t_3 \le t \le T_2 \\ -\dfrac{\alpha_1}{\beta_1}\Big[1-e^{-\beta_1(t-T)}\Big], & T_2 \le t \le T \end{cases} \tag{6.14}$$

Lemma 6.3. *The retailer demand rate (D_r) of perfect item and capacity of perfect quality items for retailer at PW_1 must satisfy the condition* $T_1 = \frac{1}{\beta_1} log \frac{(D_r-\alpha_1)}{D_r-\alpha_1-W_1\beta_1}$.

Proof. From the continuity conditions of $q_{1r}(t)$ at $t = T_1$, the following is obtained,
$\frac{D_r-\alpha_1}{\beta_1}(1-e^{-\beta_1 T_1}) = W_1$
i.e., $(1-\frac{W_1\beta_1}{D_r-\alpha_1}) = e^{-\beta_1 T_1}$ i.e., $T_1 = \frac{1}{\beta_1} log \frac{(D_r-\alpha_1)}{D_r-\alpha_1-W_1\beta_1}$. Hence the proof. □

Lemma 6.4. *The retailer demand rate (D_r), capacity of retailer in secondary warehouse (S) and capacity of retailer at PW_1 of perfect item must satisfy the condition* $t_3 = \frac{1}{\beta_1} log \frac{(D_r-\alpha_1)}{D_r-\alpha_1-W_1\beta_1} + \frac{S-W_1}{\{D_r-(\alpha_1+\beta_1 W_1)\}}$.

Proof. From the continuity conditions of $q_{1r}(t)$ at $t = t_3$, the following is obtained,
$W_1 + \Big\{D_r - (\alpha_1+\beta_1 W_1)\Big\}(t_3-T_1) = S$
i.e., $t_3 = \frac{1}{\beta_1} log \frac{(D_r-\alpha_1)}{D_r-\alpha_1-W_1\beta_1} + \frac{S-W_1}{\{D_r-(\alpha_1+\beta_1 W_1)\}}$, (using Lemma 6.3)
Hence the proof. □

Lemma 6.5. *The capacity of retailer in secondary warehouse (S) and total business time length T must satisfy the condition* $S - \Big\{(\alpha_1+\beta_1 W_1)\Big\}(T_2-t_3) = -\frac{\alpha_1}{\beta_1}\Big\{1-e^{-\beta_1(T_2-T)}\Big\}$.

Proof. Similarly proof to Lemma 6.4. □

Holding cost($HCRW$) of the secondary warehouse SW is given by

$$HCRW = h_{rs}\int_{T_1}^{t_3}\{q_{1r}(t)-W_1\}\,dt + h_{rs}\int_{t_3}^{T_2}\{q_{1r}(t)-W_1\}\,dt$$
$$= \frac{h_{rs}}{2}\Big[(D_r-(\alpha_1+\beta_1 W_1))(t_3-T_1)^2 - (\alpha_1+\beta_1 W_1)(T_2-t_3)^2 + 2(S-W_1)(T_2-t_3)\Big].$$

Holding cost($HCRS_1$) of the showroom PW_1 is given by

$$HCRS_1 = h_r\int_0^{T_1} q_{1r}(t)\,dt + h_r\int_{T_1}^{T_2} W_1\,dt + h_r\int_{T_2}^{T} q_{1r}(t)\,dt$$
$$= h_r\frac{D_r-\alpha_1}{\beta_1}\Big[T_1 + \frac{e^{-\beta_1 T_1}}{\beta_1} - \frac{1}{\beta_1}\Big] + W_1 h_r(T_2-T_1) - h_r\frac{\alpha_1}{\beta_1}\Big[(T-T_2) + \frac{1}{\beta_1} - \frac{e^{-\beta_1(T_2-T)}}{\beta_1}\Big].$$

Transportation cost($TCPR$) for perfect quality items of retailer is given by

$$\begin{aligned} TCPR &= c_{tr}\int_0^T D_C\,dt \\ &= c_{tr}\Big[(\alpha_1+\beta_1 W_1)(T_2-T_1)+D_rT_1-\frac{D_r}{\beta_1}+\frac{D_r-\alpha_1}{\beta_1}e^{-\beta_1 T_1}+\frac{\alpha_1}{\beta_1}e^{-\beta_1(T_2-T)}\Big]. \end{aligned}$$

Revenue from selling perfect quality items is given by

$$\begin{aligned} REPR &= s_r\int_0^T D_c\,dt \\ &= s_r\Big[(\alpha_1+\beta_1 W_1)(T_2-T_1)+D_rT_1-\frac{D_r}{\beta_1}+\frac{D_r-\alpha_1}{\beta_1}e^{-\beta_1 T_1}+\frac{\alpha_1}{\beta_1}e^{-\beta_1(T_2-T)}\Big]. \end{aligned}$$

Hence the total profit of the retailer from the perfect quality items is given by

$$\begin{aligned} &TPR_1 = (s_r-ctr)\Big[(\alpha_1+\beta_1 W_1)(T_2-T_1)+D_rT_1-\frac{D_r}{\beta_1}+\frac{D_r-\alpha_1}{\beta_1}e^{-\beta_1 T_1} \\ &+\frac{\alpha_1}{\beta_1}e^{-\beta_1(T_2-T)}\Big]-\Big[h_r\frac{D_r-\alpha_1}{\beta_1}\Big\{T_1+\frac{e^{-\beta_1 T_1}}{\beta_1}-\frac{1}{\beta_1}\Big\}+W_1h_r(T_2-T_1) \\ &-h_r\frac{\alpha_1}{\beta_1}\Big\{(T-T_2)+\frac{1}{\beta_1}-\frac{e^{-\beta_1(T_2-T)}}{\beta_1}\Big\}\Big]-A_r-s_mD_rt_3 \\ &-\frac{h_{rs}}{2}\Big[(D_r-(\alpha_1+\beta_1W_1))(t_3-T_1)^2-(\alpha_1+\beta_1W_1)(T_2-t_3)^2+2(S-W_1)(T_2-t_3)\Big] \end{aligned}$$

For Imperfect Quality Items of Retailer

In this case demand rate (D'_c) of customers at PW_2 is assumed as selling price dependent defined as $D'_c=(a-bs'_r)$ and corresponding change of on hand inventory of less perfect quality items are given by

$$\frac{dq_{2r}}{dt} = \begin{cases} D'_r-D'_c, & 0\le t\le t_2 \\ -D'_c, & t_2\le t\le t_4 \end{cases} \tag{6.15}$$

with boundary conditions $q_{2r}(0)=0$, and $q_{2r}(t_4)=0$.
The solution of above differential equations are given by

$$q_{2r} = \begin{cases} (D'_r-D'_c)t, & 0\le t\le t_2 \\ -D'_c(t-t_4), & t_2\le t\le t_4 \end{cases} \tag{6.16}$$

Lemma 6.6. *The retailer demand rate (D'_r) and customer demand rate (D'_c) must satisfy the condition* $t_2=\frac{D'_c}{D'_r}t_4$.

Proof. From the continuity conditions of $q_{2r}(t)$ at $t=t_2$, the following is obtained, $(D'_r-D'_c)t_2=-D'_c(t_2-t_4)$ i.e., $t_2=\frac{D'_c}{D'_r}t_4$. Hence the proof. □

Holding cost $(HCRS_2)$ of the showroom PW_2 is given by

$$HCRS_2 = h'_r \int_0^{t_2} q_{2r}(t)\,dt + h'_r \int_{t_2}^{t_4} q_{2r}(t)\,dt = \frac{h'_r}{2}\Big[(D'_r - D'_c){t_2}^2 + D'_c(t_2 - t_4)^2\Big].$$

Transportation cost$(TCIR)$ for less perfect quality items of retailer is given by

$$TCIR \;=\; c'_{tr} \int_0^{t_4} D'_c\,dt = c'_{tr} D'_c t_4.$$

Revenue$(REIR)$ from selling less perfect quality items of retailer is given by

$$REIR \;=\; s'_r \int_0^{t_4} D'_c\,dt = s'_r D'_c t_4.$$

Total Profit(TPR_2) of Retailer for less perfect quality items is given by

$$TPR_2 \;=\; (s'_r - c'_{tr})D'_c t_4 - \frac{h'_r}{2}\Big[(D'_r - D'_c){t_2}^2 + D'_c(t_2 - t_4)^2\Big] - A'_r - s'_m D'_r \lambda_1 \phi(P) t_1 (t_2 - t_1)$$

Total Profit of Retailer during $(0, T)$ is

$$\begin{aligned}
&TPR = TPR_1 + TPR_2\\
&= (s_r - c_{tr})\Big[(\alpha_1 + \beta_1 W_1)(T_2 - T_1) + D_r T_1 - \frac{D_r}{\beta_1} + \frac{D_r - \alpha_1}{\beta_1} e^{-\beta_1 T_1} + \frac{\alpha_1}{\beta_1} e^{-\beta_1 (T_2 - T)}\Big]\\
&-\frac{h_{rs}}{2}\Big[(D_r - (\alpha_1 + \beta_1 W_1))(t_3 - T_1)^2 - (\alpha_1 + \beta_1 W_1)(T_2 - t_3)^2 + 2(S - W_1)(T_2 - t_3)\Big]\\
&-\Big[h_r \frac{D_r - \alpha_1}{\beta_1}\Big\{T_1 + \frac{e^{-\beta_1 T_1}}{\beta_1} - \frac{1}{\beta_1}\Big\} + W_1 h_r (T_2 - T_1) - h_r \frac{\alpha_1}{\beta_1}\Big\{(T - T_2) + \frac{1}{\beta_1} - \frac{e^{-\beta_1 (T_2 - T)}}{\beta_1}\Big\}\Big]\\
&-\frac{h'_r}{2}\Big[(D'_r - D'_c){t_2}^2 + D'_c(t_2 - t_4)^2\Big] - s'_m D'_r \lambda_1 \phi(P) t_1 (t_2 - t_1)\\
&-s_m D_r t_3 + (s'_r - c'_{tr})D'_c t_4 - A_r - A'_r
\end{aligned}$$

6.3.3 Average Profit of the Proposed Model

Average expected profit of the model during $(0, T)$ is given by

$$AEP(P, T) = \frac{1}{T}[TEPM + TPR]$$

6.4 Scenarios of the Proposed Models

Depending upon the values of δ_4 and δ_5, different scenarios have been developed.

6.4.1 Scenario-1 (Same as above Model with Defective Rate is Dependent on both Production Rate and Production Run Time)

Taking $\delta_4 \neq 0$, $\delta_5 \neq 0$ i.e., β is dependent only production rate P and does depend production-run time. In this case the rate of defectiveness β has been considered as follows:

$$\beta = \begin{cases} 0\,, & 0 \leq t \leq \tau \\ (\theta_0 - \frac{\theta_1}{P^{\delta_5}})(t-\tau)^{\delta_4}\,, & \tau \leq t \leq t_1 \end{cases} \tag{6.17}$$

where θ_0 and θ_1 be the positive constants.

$$\begin{aligned} & TEPM_1(P,T) = s_m D_r t_3 + s'_m \int_0^{t_1} D'_r(t_2-\tau) f(\tau)\, d\tau - c_p P t_1 - s_c P t_1 - A_m \\ & -\Big\{c_d(1-\gamma)(1-\delta) + r_{cm}\delta\Big\} \frac{P}{\delta_4+1}(\theta_0 - \frac{\theta_1}{P^{\delta_5}}) \int_0^{t_1} (t_1-\tau)^{\delta_4+1} f(\tau) d\tau \\ & -\frac{h_m}{2}\int_0^{t_1} \frac{2(1-\delta)P}{(\delta_4+1)(\delta_4+2)}(\theta_0 - \frac{\theta_1}{P^{\delta_5}})(t_1-\tau)^{\delta_4+2} f(\tau)\, d\tau \\ & -\frac{h'_m}{2}\int_0^{t_1} \Big[\Big\{\frac{2\gamma(1-\delta)}{(\delta_4+1)(\delta_4+2)P}(\theta_0 - \frac{\theta_1}{P^{\delta_5}})(t_1-\tau)^{\delta_4+2} - D'_r\Big(t_1-\tau\Big)^2\Big\} \\ & +D'_r\Big(t_1-t_2\Big)^2\Big] f(\tau) d\tau - \frac{h_m}{2}\Big\{(P-D_r)t_1^2 + D_r(t_1-t_3)^2\Big\} \end{aligned}$$

Approximating the function $exp(-f(P)t_1)$ of its expansion as well as $\delta_4 = 1$, we have

$$(i) \int_0^{t_1} f(\tau)\, d\tau \approx \lambda_1\phi(P)t_1, \qquad (ii) \int_0^{t_1} \Big(t_2-\tau\Big) f(\tau)\, d\tau \approx \lambda_1\phi(P)t_1(t_2-t_1),$$

$$(iii) \int_0^{t_1} \Big(t_1-\tau\Big)^2 f(\tau)\, d\tau \approx \frac{\lambda_1\phi(P)t_1^3}{3}, \quad (iv) \int_0^{t_1} \Big(t_1-\tau\Big)^3 f(\tau)\, d\tau \approx \frac{\lambda_1\phi(P)t_1^4}{4}$$

(For details see appendix D)

$$\begin{aligned} TEPM_1(P,T) \;=\; & s_m D_r t_3 + s'_m D'_r \lambda_1\phi(P)t_1(t_2-t_1) - c_p P t_1 - s_c P t_1 \\ & -\frac{P}{6}\Big\{c_d(1-\gamma)(1-\delta) + r_{cm}\delta\Big\}(\theta_0 - \frac{\theta_1}{P^{\delta_5}})\lambda_1\phi(P)t_1^3 - A_m \\ & -\frac{h_m}{24}\Big[12\Big\{(P-D_r)t_1^2 + D_r(t_1-t_3)^2\Big\} - P(1-\delta)(\theta_0 - \frac{\theta_1}{P^{\delta_5}})\lambda_1\phi(P)t_1^4\Big] \\ & -\frac{h'_m}{24}\Big[\Big\{P\gamma(1-\delta)(\theta_0 - \frac{\theta_1}{P^{\delta_5}})t_1^3 - 4D'_r t_1^2\Big\} + 12D'_r\Big(t_1-t_2\Big)^2\Big]\lambda_1\phi(P)t_1 \end{aligned}$$

Average expected profit of the model during $(0,T)$ is given by

$$AEP(P,T) = \frac{1}{T}[TEPM_1 + TPR]$$

6.4.2 Scenario-2 (Same as above Model with Defective Rate is Production Rate Dependent and Independent of Production Time)

Taking $\delta_4 = 0, \delta_5 \neq 0$ i.e., β is dependent only production rate P and does depend production-run time. In this case the rate of defectiveness β has been considered as follows:

$$\beta = \begin{cases} 0\,, & 0 \leq t \leq \tau \\ (\theta_0 - \frac{\theta_1}{P^{\delta_5}})\,, & \tau \leq t \leq t_1 \end{cases} \tag{6.18}$$

where θ_0 and θ_1 be the positive constants.

$$\begin{aligned} &TEPM_2(P,T) = s_m D_r t_3 + s'_m \int_0^{t_1} D'_r(t_2-\tau)f(\tau)\,d\tau - c_p P t_1 - s_c P t_1 \\ &-\Big\{c_d(1-\gamma)(1-\delta) + r_{cm}\delta\Big\}\frac{P}{\delta_4+1}(\theta_0 - \frac{\theta_1}{P^{\delta_5}})\int_0^{t_1}(t_1-\tau)f(\tau)d\tau \\ &-\frac{h_m}{2}\Big\{(P-D_r)t_1^2 + D_r(t_1-t_3)^2\Big\} - \frac{h_m}{2}\int_0^{t_1}(1-\delta)P(\theta_0-\frac{\theta_1}{P^{\delta_5}})(t_1-\tau)^2 f(\tau)\,d\tau \\ &-\frac{h'_m}{2}\int_0^{t_1}\Big[\Big\{\gamma(1-\delta)P(\theta_0-\frac{\theta_1}{P^{\delta_5}}) - D'_r\Big\}\Big(t_1-\tau\Big)^2 + D'_r\Big(t_1-t_2\Big)^2\Big]f(\tau)d\tau - A_m \end{aligned}$$

Approximating the function $exp(-f(P)t_1)$ of its expansion, we have

$$(i)\int_0^{t_1} f(\tau)\,d\tau \approx \lambda_1\phi(P)t_1, \qquad (ii)\int_0^{t_1}\Big(t_2-\tau\Big)f(\tau)\,d\tau \approx \lambda_1\phi(P)t_1(t_2-t_1),$$
$$(iii)\int_0^{t_1}\Big(t_1-\tau\Big)f(\tau)\,d\tau \approx \frac{\lambda_1\phi(P)t_1^2}{2}, \quad (iv)\int_0^{t_1}\Big(t_1-\tau\Big)^2 f(\tau)\,d\tau \approx \frac{\lambda_1\phi(P)t_1^3}{3},$$

(For details see appendix D)
Total expected profit of manufacturer is given by

$$\begin{aligned} TEPM_2 \;=\; & s_m D_r t_3 + s'_m D'_r \lambda_1\phi(P)t_1(t_2-t_1) - c_p P t_1 - s_c P t_1 \\ & -\frac{1}{2}\Big\{c_d(1-\gamma)(1-\delta) + r_{cm}\delta\Big\}(\theta_0 - \frac{\theta_1}{P^{\delta_5}})P\lambda_1\phi(P)t_1^2 \\ & -\frac{h_m}{6}\Big[3\Big\{(P-D_r)t_1^2 + D_r(t_1-t_3)^2\Big\} - (1-\delta)(\theta_0-\frac{\theta_1}{P^{\delta_5}})\lambda_1\phi(P)t_1^3\Big] \\ & -\frac{h'_m}{6}\Big[\Big\{\gamma(1-\delta)(\theta_0-\frac{\theta_1}{P^{\delta_5}})P - D'_r\Big\}t_1^2 + 3D'_r\Big(t_1-t_2\Big)^2\Big]\lambda_1\phi(P)t_1 - A_m \end{aligned}$$

Average expected profit of the model during $(0,T)$ is given by

$$AEP(P,T) = \frac{1}{T}[TEPM_2 + TPR]$$

6.4.3 Scenario-3 (Same as above Model with Defective Rate is Production Time Dependent and Independent of Production Rate)

Taking $\delta_4 \neq 0$, $\delta_5 = 0$ i.e., β is not dependent on production rate P and depend only production-run time. In this case the rate of defectiveness β has been considered as follows:

$$\beta = \begin{cases} 0\,, & 0 \le t \le \tau \\ (\theta_0 - \theta_1)(t-\tau)^{\delta_4}\,, & \tau \le t \le t_1 \end{cases} \tag{6.19}$$

where θ_0 and θ_1 be the positive constants.

$$\begin{aligned} TEPM_3 &= s_m D_r t_3 + s'_m \int_0^{t_1} D'_r(t_2-\tau) f(\tau)\, d\tau - c_p P t_1 - s_c P t_1 \\ &-\Big\{c_d(1-\gamma)(1-\delta) + r_{cm}\delta\Big\}\frac{P}{\delta_4+1}(\theta_0-\theta_1)\int_0^{t_1}(t_1-\tau)^{\delta_4+1} f(\tau) d\tau \\ &-\frac{h_m}{2}\int_0^{t_1}\frac{2(1-\delta)}{(\delta_4+1)(\delta_4+2)}P(\theta_0-\theta_1)(t_1-\tau)^{\delta_4+2} f(\tau)\, d\tau \\ &-\frac{h'_m}{2}\int_0^{t_1}\Big[\Big\{\frac{2\gamma(1-\delta)}{(\delta_4+1)(\delta_4+2)}P(\theta_0-\theta_1)(t_1-\tau)^{\delta_4+2} - D'_r\Big(t_1-\tau\Big)^2\Big\} \\ &+D'_r\Big(t_1-t_2\Big)^2\Big] f(\tau) d\tau - A_m - \frac{h_m}{2}\Big\{(P-D_r)t_1^2 + D_r(t_1-t_3)^2\Big\} \end{aligned}$$

Approximating the function $exp(-f(P)t_1)$ of its expansion and $\delta_4 = 1$, we have

$(i)\ \int_0^{t_1} f(\tau)\, d\tau \approx \lambda_1\phi(P)t_1,$ $\quad(ii)\ \int_0^{t_1}\Big(t_2-\tau\Big) f(\tau)\, d\tau \approx \lambda_1\phi(P)t_1(t_2-t_1),$

$(iii)\ \int_0^{t_1}\Big(t_1-\tau\Big) f(\tau)\, d\tau \approx \frac{\lambda_1\phi(P)t_1^2}{2},$ $\quad(iv)\ \int_0^{t_1}\Big(t_1-\tau\Big)^2 f(\tau)\, d\tau \approx \frac{\lambda_1\phi(P)t_1^3}{3},$

$(v)\ \int_0^{t_1}\Big(t_1-\tau\Big)^3 f(\tau)\, d\tau \approx \frac{\lambda_1\phi(P)t_1^4}{4},$ (For details see appendix D)

Total expected profit of manufacturer is given by

$$\begin{aligned} TEPM_3 &= s_m D_r t_3 + s'_m D'_r \lambda_1\phi(P)t_1(t_2-t_1) - c_p P t_1 - s_c P t_1 \\ &-\frac{P}{6}\Big\{c_d(1-\gamma)(1-\delta) + r_{cm}\delta\Big\}(\theta_0-\theta_1)\lambda_1\phi(P)t_1^3 \\ &-\frac{h_m}{24}\Big[12\Big\{(P-D_r)t_1^2 + D_r(t_1-t_3)^2\Big\} - (1-\delta)P(\theta_0-\theta_1)\lambda_1\phi(P)t_1^4\Big] \\ &-\frac{h'_m}{24}\Big[\Big\{\gamma(1-\delta)P(\theta_0-\theta_1)t_1^3 - 4D'_r t_1^2\Big\} + 12D'_r\Big(t_1-t_2\Big)^2\Big]\lambda_1\phi(P)t_1 - A_m \end{aligned}$$

Average expected profit of the model during $(0, T)$ is given by

$$AEP(P,T) = \frac{1}{T}[TEPM_3 + TPR]$$

6.4.4 Scenario-4 (Same as above Model with Constant Defective Rate)

Taking $\delta_4 = 0$, $\delta_5 = 0$ i.e., β does not dependents on both production rate P and production-run time. In this case the rate of defectiveness β has been considered as follows:

$$\beta = \begin{cases} 0\,, & 0 \leq t \leq \tau \\ (\theta_0 - \theta_1)\,, & \tau \leq t \leq t_1 \end{cases} \tag{6.20}$$

where θ_0 and θ_1 be the positive constants.

$$\begin{aligned} TEPM_4 &= s_m D_r t_3 + s'_m \int_0^{t_1} D'_r (t_2 - \tau) f(\tau)\, d\tau - c_p P t_1 - s_c P t_1 \\ &-\Big\{c_d(1-\gamma)(1-\delta) + r_{cm}\delta\Big\} P(\theta_0 - \theta_1) \int_0^{t_1} (t_1 - \tau) f(\tau) d\tau \\ &-\frac{h_m}{2}\Big\{(P - D_r)t_1^2 + D_r(t_1 - t_3)^2\Big\} + \frac{h_m}{2}\int_0^{t_1} (1-\delta)\, P(\theta_0 - \theta_1)(t_1 - \tau)^2 f(\tau)\, d\tau \\ &-\frac{h'_m}{2}\int_0^{t_1} \Big[\Big\{\gamma(1-\delta)P(\theta_0 - \theta_1) - D'_r\Big\}(t_1 - \tau)^2 + D'_r\Big(t_1 - t_2\Big)^2\Big] f(\tau) d\tau - A_m \end{aligned}$$

Approximating the function $exp(-f(P)t_1)$ of its expansion, we have

$$(i) \int_0^{t_1} f(\tau)\, d\tau \approx \lambda_1 \phi(P) t_1, \qquad (ii) \int_0^{t_1} \Big(t_2 - \tau\Big) f(\tau)\, d\tau \approx \lambda_1 \phi(P) t_1 (t_2 - t_1),$$

$$(iii) \int_0^{t_1} \Big(t_1 - \tau\Big) f(\tau)\, d\tau \approx \frac{\lambda_1 \phi(P) t_1^2}{2}, \qquad (iv) \int_0^{t_1} \Big(t_1 - \tau\Big)^2 f(\tau)\, d\tau \approx \frac{\lambda_1 \phi(P) t_1^3}{3},$$

(For details see appendix D)
Total expected profit of manufacturer is given by

$$\begin{aligned} TEPM_4 &= s_m D_r t_3 + s'_m D'_r \lambda_1 \phi(P) t_1 (t_2 - t_1) - c_p P t_1 - s_c P t_1 \\ &-\frac{1}{2}\Big\{c_d(1-\gamma)(1-\delta) + r_{cm}\delta\Big\}(\theta_0 - \theta_1) P \lambda_1 \phi(P) t_1^2 \\ &-\frac{h_m}{6}\Big[3\Big\{(P - D_r)t_1^2 + D_r(t_1 - t_3)^2\Big\} - (1-\delta)(\theta_0 - \theta_1) P f(P) t_1^3\Big] \\ &-\frac{h'_m}{6}\Big[\Big\{\gamma(1-\delta)(\theta_0 - \theta_1)P - D'_r\Big\} t_1^2 + 3D'_r\Big(t_1 - t_2\Big)^2\Big] \lambda_1 \phi(P) t_1 - A_m \end{aligned}$$

Average expected profit of the model during $(0, T)$ is given by

$$AEP(P, T) = \frac{1}{T}[TEPM_4 + TPR]$$

6.5 Numerical Illustration

Consider a manufacturer of the plastic product produces water tank in India. This production system produces mixture of perfect and defective items. The inspection process that screens out the defective items. The fraction of defective items have been reworked. Owing to the regulations of environmental protection, the manufacturer considers the disposal cost with aim to reduce the negative effects to environment. The company delivers the water tank to a distributor/retailer (the buyer) continuously. The customer buy the product from retailer. The customer demand is affected by the displayed stock and selling price. From the historical data, the manufacture has taken following parametric values:
$\alpha_1 = 0.003$, $\alpha_2 = 0.005$, $\beta_1 = 0.40$, $\gamma = 0.80$, $\delta = 0.70$, $\delta_1 = 2$, $\delta_2 = 2$, $\delta_3 = 1$, $\lambda_1 = 0.004$, $\lambda_{max} = 0.80$, $\lambda_{min} = 0.20$, $\theta_0 = 0.12$, $\theta_1 = 0.01$, $D_r = 31$ units, D'_r=4 units, $A = \$4000$, $B = \$1000$, $a = \$28$, $b = 0.30$, $k = 0.5$, $A_{m0} = \$48$, $A_{m1} = \$0.05$, $A_r = \$85$, $A'_r = \$40$, c_r=\$24 per unit, $c_{sr} = \$0.80$ per unit, $r_{cm} = \$2$ per unit, $c_{tr} = \$0.20$ per unit, c'_{tr}=\$0.20 per unit, $c_d = \$0.6$ per unit, $h_m = \$0.90$ per unit per unit time, $h'_m = \$0.45$ per unit per unit time, $h_r = \$1.5$ per unit per unit time, $h'_r = \$0.55$, $h_{rs} = \$1.2$ per unit per unit time, $s_m = \$89$ per unit, $s'_m = \$59$ per unit, $s_r = \$122$ per unit, $s'_r = \$85$ per unit, $S = \$78$ pics, $W_1 = \$56$ pics.
Using the above parametric values, the optimum values of our proposed non-linear problem have been shown in following Table 6.1.

Table 6.1: Optimal results of the illustrated model when $\phi(P) = 1.29 + 0.59P$

Scenario	Production rate (P^* $unit$)	Production time (t_1^* $unit$)	Total business time (T^* $unit$)	Average Expected Profit ($AEP^*(P^*, T^*)$)
Scenario-1 $\delta_4 = 0, \delta_5 = 0$	65.07	2.15	6.25	\$ 1639.57
Scenario-2 $\delta_4 = 0, \delta_5 = 0$	64.24	2.20	6.23	\$ 1648.85
Scenario-3 $\delta_4 = 0, \delta_5 = 0$	62.18	2.25	6.19	\$ 1664.71
Scenario-4 $\delta_4 = 0, \delta_5 = 0$	63.02	2.22	6.21	\$ 1657.43

6.5.1 Sensitivity Analysis

In this section, we examine the effects of changes in the system parameters to study the sensitivity analysis of the proposed model with respect to some parameters based on Scenario-1. The results are presented in following Table 6.2, 6.3 and 6.4.

Table 6.2: Sensitivity analysis of $AEP^*(P^*, T^*)$ when λ_1 changes

λ_1	λ	$V(\lambda)$	Reworking Cost	$(AEP^*(P^*, T^*)$
0.001	0.039	\$4093.362	\$1.03	\$1639.65
0.003	0.119	\$4014.910	\$3.08	\$1639.63
0.004	0.158	\$4000.423	\$4.11	\$ 1639.57
0.005	0.198	\$4000.000	\$5.14	\$1639.37
0.007	0.283	\$26338.49	\$7.48	\$1488.06

Table 6.3: Sensitivity analysis of $AEP^*(P^*, T^*)$ when α_1 and α_2 changes

α_1, α_2	Unit Production Cost $C(P)$	Total Production Cost	Average Expected Profit $(AEP^*(P^*, T^*))$
$\alpha_1 = 0.000$, $\alpha_2 = 0.005$	\$ 25.27	\$ 3535.29	\$1935.74
$\alpha_1 = 0.001$, $\alpha_2 = 0.003$	\$29.37	\$ 4109.43	\$1840.055
$\alpha_1 = 0.003$, $\alpha_2 = 0.000$	\$37.65	\$ 5266.82	\$1647.15
$\alpha_1 = 0.003$, $\alpha_2 = 0.005$	\$37.97	\$ 5312.33	\$ 1639.57
$\alpha_1 = 0.005$, $\alpha_2 = 0.006$	\$46.50	\$ 6506.13	\$ 1440.605
$\alpha_1 = 0.006$, $\alpha_2 = 0.008$	\$50.87	\$ 7116.68	\$ 1338.84
$\alpha_1 = 0.007$, $\alpha_2 = 0.009$	\$55.16	\$ 7718.13	\$1238.60
$\alpha_1 = 0.001$, $\alpha_2 = 0.005$	\$29.50	\$ 4127.64	\$1837.02
$\alpha_1 = 0.002$, $\alpha_2 = 0.005$	\$33.73	\$ 4719.99	\$1738.29
$\alpha_1 = 0.003$, $\alpha_2 = 0.005$	\$37.97	\$ 5312.33	\$1639.57
$\alpha_1 = 0.004$, $\alpha_2 = 0.005$	\$42.20	\$ 5904.68	\$1540.84
$\alpha_1 = 0.005$, $\alpha_2 = 0.005$	\$46.44	\$ 6497.03	\$1442.12
$\alpha_1 = 0.003$, $\alpha_2 = 0.003$	\$37.84	\$ 5294.13	\$1642.60
$\alpha_1 = 0.003$, $\alpha_2 = 0.004$	\$37.90	\$ 5303.23	\$1641.08
$\alpha_1 = 0.003$, $\alpha_2 = 0.005$	\$37.97	\$ 5312.33	\$1639.57
$\alpha_1 = 0.003$, $\alpha_2 = 0.006$	\$38.03	\$ 5321.44	\$ 1638.05
$\alpha_1 = 0.003$, $\alpha_2 = 0.007$	38.10	\$ 5330.54	\$ 1636.53

Table 6.4: Sensitivity analysis of $AEP^*(P^*, T^*)$ when S and W_1 changes

S	W_1	Holding Cost of Retailer	Average Expected Profit $(AEP^*(P^*, T^*))$
$S = 68$	$W_1 = 56$	\$379.16	\$1641.87
$S = 73$	$W_1 = 56$	\$386.06	\$1640.72
$S = 78$	$W_1 = 56$	\$392.96	\$1639.57
$S = 83$	$W_1 = 56$	\$399.86	\$1638.42
$S = 88$	$W_1 = 56$	\$406.76	\$1637.27
$S = 78$	$W_1 = 46$	\$369.29	\$1306.53
$S = 78$	$W_1 = 51$	\$381.12	\$1473.05
$S = 78$	$W_1 = 56$	\$392.96	\$1639.57
$S = 78$	$W_1 = 58$	\$397.70	\$1706.17
$S = 78$	$W_1 = 61$	\$404.80	\$1806.08

The main conclusions from the sensitivity analysis are as follows:
(i) The sensitivity analyses of the development cost, reworking cost and average expected profit are shown in Table 6.2 when the parameter λ_1 increases or decreases. Initially increasing λ_1 decrease development cost $(V(\lambda))$, but reworking cost increasing and average expected profit (AEP) decreases.
(ii) From Table 6.3 explore that when the rate of wear-tear cost (α_1) and environment protection cost (α_2) increase, the average expected profit (AEP) decrease due to increase of unit production cost $(C(P))$.
(iii) Here the customer demand rate dependent on the stock of showroom (W_1). So When the the scale of stock (W_1) increase, the average expected profit (AEP) increase due to selling rate of retailer increase (D_r) and manufacturer holding cost (h_m) decreases (Table 6.4).

6.6 Conclusion

This chapter develops a two-layer supply chain production inventory model involving manufacturer, retailer and customers. In comparing with the existing literature on the supply chain, the followings are the main contributions in the proposed model:
After a random time, the process may shift to "out-of-control" state from "in-control" state during the production run and may generate non-conforming quality items. Inspection cost is incurred during the production run time and manufacturer continuously inspects as well as separates the perfect quality items, less perfect quality items, repairable items which are transformed into perfect quality items after some rework and reject items. Reworked cost is considered by the manufacturer to repair a certain percent of imperfect quality items. Demand rate of customers for perfect quality items and less perfect quality items are respectively assumed to be stock dependent and selling price dependent. Here retailer have two showrooms PW_1 and PW_2 of finite capacities at busy market place and the market demands of perfect and less perfect quality items are respectively met through the showrooms

PW_1 and PW_2. Retailer has a secondary warehouse SW of infinite capacity, away from busy market place, to store the excess amount perfect quality items from where the items are continuously transferred to the showroom. It is considered that the holding cost per unit per unit time at SW is less than the holding cost at PW_1 per unit per unit time. The repairing costs of corrective and preventive maintenance should also be considered, as these costs increase the unit production cost. Inventory and production decisions are made at the manufacturer and retailer levels. Actually, in this chapter the coordination between production and inventory decisions has been established across the supply chain so that the integrated average profit of the chain is maximum.

Part IV

Studies on Imperfect Production Inventory Systems in Fuzzy Environment

Chapter 7

Three layers supply chain in an imperfect production inventory model with two storage facilities under fuzzy rough environment[1]

7.1 Introduction

Business organizations all over the world are striving hard to evolve strategies to survive in the era of competition ushered in by globalization. Supply chain management (SCM) is one such strategy. It is an effective methodology and presents an integrated approach to resolve issues in sourcing customer service, demand flow and distribution. The focus is on the customer. The results are in the form of reduced operational costs, improved flow of supplies, reduction in delays of production and increased customer satisfaction. While the goal of supply chain management is to reduce cost of producing and reaching the finished products to the customers, inventory control is the means to achieve the goal. Researchers as well as practitioners in manufacturing industries have given importance to develop inventory control problems in supply chain management. All steps from supply of raw materials to finished products can be included into a supply chain, connecting raw materials supplier, manufacturer, retailer and finally customers. Recent reviews on supply chain management are provided by Weng [223], Munson and Rosenblatt [157], Yang and Wee [230], Khouja [117], Yao et al. [234], Chaharsooghi et al. [16], Wang et al. [216] and others.

[1]This model published in **Journal of Uncertainty Analysis and Applications, 2(1) (2014) 1-31**, SPRINGER.

Now a days, it is common to all industries that a certain percentage of produced or ordered items are mixed of perfect and imperfect quality. It is also important to a supply manager of any organization to control and maintain the inventories of perfect and imperfect quality items. Salameh and Jaber [184] developed an inventory model for imperfect quality items using the EPQ/EOQ formulae and assumed that inferior quality items are sold as a single batch at the end of the total screening process. Goyal and Cardenas-Barron [78] extended the idea of Salameh and Jaber's [184] model and proposed a practical approach to determine EPQ for items with imperfect quality. Yu et al. [238] generalized the models of Salameh and Jaber [184], incorpoting deterioration and partial back ordering. Liu and Yang [143] investigated a single stage production system with imperfect process delivering two types of defects: reworkable and non-reworkable items. The reworkable items are sent for reworking, whereas non-reworkable items are immediately discarded from the system. Panda and Maiti [163] represented a geometric programming approach for multi-item inventory models with price dependent demand under flexibility and reliability with imprecise space constraint. Ma et al. [146] considered the effects of imperfect production processes and the decision on whether and when to implement a screening process for defective items generated during a production run. Sana [187] develops two inventory models in an imperfect production system and showed that the inferior quality items could be reworked at a cost where overall production inventory costs could be reduced significantly. Sana [188] extended the idea of imperfect production process in three layer supply chain management system.

Warehousing is an integral part of every logistics system. We can define warehousing as that part of a firms logistics system that stores products (raw materials, parts, goods in process, finished goods). The two warehouse model for a finite and infinite time horizon is developed by several researchers (cf. Hartely [90], Pakkala and Achary [161], Bhunia and Maiti [11], Kar et al. [111] and others) already discussed in Section 6.1.

Dubois and Prade [63] first studied the fuzzification problem of rough sets. Furthermore, Morsi and Yakout [156] defined the upper and lower approximations of the fuzzy sets with respect to a fuzzy min-similarity relation. Additionally, Radzikowska and Kerre [167], Xu and Zhou [228], Liu and Sai [139], Chen [33] and others generalized the above definitions of the fuzzy rough set to a more general case. Different types of uncertainty such as randomness, fuzziness and roughness are common factors in any production inventory problem. But some problems in production inventory system occur both fuzziness and roughness simultaneously. In many cases, it is found that some inventory parameters involve both the fuzzy and rough uncertainties. For example, the inventory related costs holding cost, set-up cost, idle costs, etc. depend on several factors such as bank interest, inflation, etc. which are uncertain in fuzzy rough sense. To be more specific, inventory holding cost is sometimes represented by a fuzzy number and it depends on the storage amount which may be imprecise and range within an interval due to several factors such as scarcity of storage space, market fluctuation, human estimation thought process i.e. it may be represented by a rough set.
In this chapter, a supply chain model consisting of supplier, manufacturer and retailer has been considered. Here supplier receives the raw materials in a lot and then the superior

quality items of the raw materials are sold at a higher price to the manufacturer after the screening the imperfect raw materials as well as inferior quality items of the raw materials are also sold to another manufacturer at a reduced price in a single batch by the end of cent percent screening process. A mixture of perfect and imperfect quality items are produced by the manufacturer. After some rework, some repairable portion of imperfect quality items is transformed into perfect quality items and some of non repairable portion of imperfect items are sold with reduced price to the retailer. Retailer purchases both perfect and imperfect quality items and sales both items to the customers through his/her respective showrooms of finite capacities at a market place. Here customers' demand is stock dependent and selling price dependent for the perfect quality items and less perfect items respectively. Since the storage space of the showroom for perfect quality items is limited due to space problem and the demand of the corresponding items is stock dependent, hence a secondary warehouse is hired by the retailer on rental basis to store the excess amount of perfect quality items and these items are continuously transferred to the showroom concerned. The literature suggests that the holding cost of secondary warehouse per unit item per unit time is more than the holding cost of showroom due to the preservation cost for maintaining the quality of the product and other costs related to holding large quantity of the product in the secondary warehouse. But in this chapter it has been considered that the holding cost of perfect quality items in the secondary warehouse is less than the holding cost of the showroom as the nature of the items are non-deteriorating and so having no preservation cost. Here, transportation cost is also incurred to transport both quality items at the respective showrooms from the production center. Due to complexity of environment, inventory holding costs, idle costs, set-up costs and transportation costs are considered as fuzzy rough type and these are reduced to crisp ones using fuzzy rough expectation. In order to optimize the production rate and raw material order size (the decision variables), the average profit function of the manufacturer is maximized as the manufacturer acts as a leader (Stakelberg approach) and the supplier as well as retailer are the followers of that chain. The decision variables are also optimized by maximizing the integrated average profit function of the chain. Finally, a comparative study has been made between both approaches Stakelberg and integrated. A numerical example is provided to illustrate the feasibility of the model.

7.2 Notations and Assumptions

The following notations and assumptions have been used to develop the proposed model:

7.2.1 Notations

The following notations have been used to developed the model.

R	:	Replenishment lot size of the supplier.
P	:	Production rate for the manufacturer which is also the demand rate of supplier.
x	:	Screening rate of supplier.

θ	:	Percentage of inferior quality items in each lot received by the supplier.
t_1	:	Cycle length of supplier.
t'	:	Total Screening time of R units order per cycle.
A_s	:	Set up cost of supplier.
c_s	:	Purchase cost per unit item of supplier.
h_s	:	Holding cost per unit item for per unit time for supplier.
I_{cs}	:	Cost per unit idle time of supplier.
s_c	:	Screening cost per unit item.
w_s	:	Selling price per unit of superior quality item for supplier.
w'_s	:	Selling price per unit of inferior quality item for supplier.
β	:	Percentage of imperfect quality items suitable for rework to make perfect items.
γ	:	Percentage of imperfect items which are suitable for sale through reduction.
$q_{1m}(t)$	:	Inventory level of perfect quality items for the manufacturer at any time t.
$q_{2m}(t)$	:	Inventory level of less perfect quality items for the manufacturer at any time t.
D_r	:	Demand rate of the retailer for perfect quality items.
D'_r	:	Demand rate of the retailer for less perfect quality items.
A_m	:	Set up cost of manufacturer.
h_m	:	Holding cost per unit item for per unit time for perfect item in manufacturer.
h'_m	:	Holding cost per unit item for per unit time for imperfect item in manufacturer.
I_{cm}	:	Cost per unit idle time of manufacturer.
I_{sm}	:	Inspection cost per unit item for manufacturer.
r_{cm}	:	Reworking cost per unit item for manufacturer.
$C(P)$	:	Production cost per unit item.
s_m	:	Selling price per unit of perfect quality items for manufacturer.
s'_m	:	Selling price per unit less perfect quality items for manufacturer.
$q_{1r}(t)$	:	Inventory level of perfect quality items for the retailer at any time t.
$q_{2r}(t)$	:	Inventory level of less perfect quality items for the retailer at any time t.
A_r	:	Set up cost for perfect quality items of retailer.
A'_r	:	Set up cost for less perfect quality items of retailer.
h_r	:	Holding cost per unit item for per unit time of perfect quality items in PW_1 for retailer.
h_{rs}	:	Holding cost per unit item for per unit time of perfect quality items for the retailer in secondary ware house.
h'_r	:	Holding cost per unit item for per unit time of less perfect quality items in PW_2 for retailer.
s_r	:	Selling price per unit of perfect quality items for retailer PW_1.
s'_r	:	Selling price per unit of less perfect quality items for retailer PW_2 .
c_{tr}	:	Transportation cost perfect item for retailer.
c'_{tr}	:	Transportation cost of less perfect quality items for retailer.
S	:	Capacity of secondary warehouse.
W_j	:	Capacity of PW_j(j=1,2).
D_c	:	Demand rate of customer for perfect quality items PW_1.
D'_c	:	Demand rate of customer for less perfect quality itemsPW_2.
$\simeq$	:	Denotes the fuzzy rough parameters.

7.2.2 Assumptions

(i) Joint effect of supplier, manufacture and retailer is considered in a supply chain management.

(ii) Model is developed for single item products and lead time is negligible.

(iii) Production rate is a decision variable.

(iv) Demand for perfect quality items is deterministic and function of current stock level.

(v) Replenishment rate of manufacture is instantaneously infinite but its size is finite.

(vi) Unit production cost $C(P)$ per unit item is considered as $C(P) = L + \frac{G}{P} + HP$, where G be the total labor cost for manufacturing the items, L and H are respectively the material cost and tool/die cost per unit item.

(vii) The manufacturer has ignored the machine breakdown.

(viii) Cost of idle times of supplier and manufacturer are taken into account.

(ix) Showrooms PW_1 and PW_2 of retailer are adjacent.

7.3 Mathematical Formulation of the Proposed Model

Block diagram and pictorial representation of the proposed supply chain production inventory model are respectively depicted in Figure 7.1 and Figure 7.2. Formulation of the model for supplier, manufacturer and retailer are given in the subsections 7.3.1, 7.3.2 and 7.3.3 respectively.

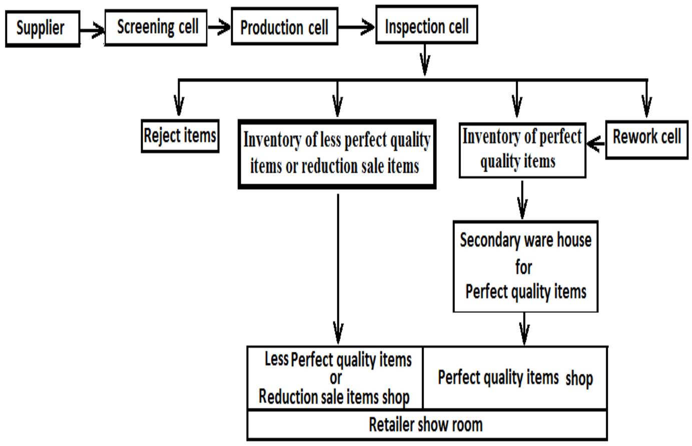

Figure 7.1: Block diagram representation of the proposed model

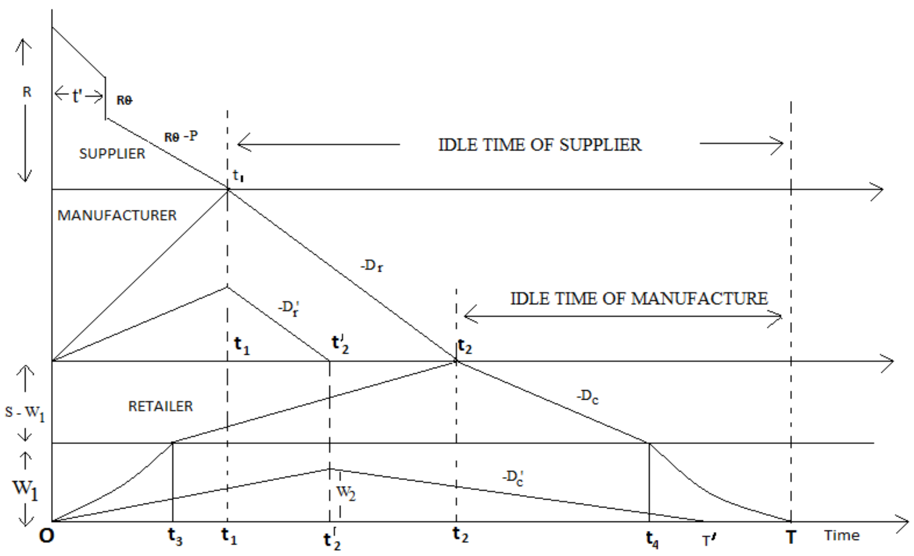

Figure 7.2: Pictorial representation of inventory situation of the integrated model

7.3.1 Formulation of the Supplier

Here R be the lot-size received by the supplier at $t = 0$. A screening process of the lot is conducted at a rate of x units per unit time and t' be the total screening time of R units. Defective items are kept in stock and sold prior to receiving the next shipment as a single batch at a discounted price of w_s' per unit. $R\theta$ is the number of inferior quality items withdrawn from inventory and t_1 is the cycle length of the supplier. The number of superior quality items in each lot, denoted by $N(R,\theta)$, is given by

$$N(R,\theta) = (R - R\theta). \tag{7.1}$$

Supplier supplies the superior quality items as a raw materials to the manufacturer at a rate P up to the time t_1 and to avoid shortages, it is assumed that the number of superior quality items $N(r,\theta)$ is at least equal to the demand during screening time t', i.e.,

$$N(R,\theta) \geq Pt' \tag{7.2}$$

Substituting equation(7.1) in equation(7.2) and replacing t' by $\frac{R}{x}$, the value of R is restricted to $R \leq 1 - \frac{P}{x}$.
Sales revenue from superior quality items per cycle $= w_s(R - R\theta)$.
Sales revenue from inferior quality items per cycle $= w_s' R\theta$.
Procurement cost for the supplier per cycle $=$ (Setup cost + Purchasing cost) $= A_s + c_s R$.
Screening cost per cycle $= s_c R$.
Holding cost during $(0, t_1) = h_s[\frac{(R-R\theta)t_1}{2} + \frac{R^2\theta}{x}]$.
Idle time cost per cycle $= I_{cs}(T - t_1)$.

Therefore, the average profit (APS) of the Supplier during $(0, T)$ is given by

$$\begin{aligned} APS &= \frac{1}{T}\Big[\{w_s(1-\theta) + w'_s\theta - c_s - s_c\}R - A_s - h_s R\Big\{\frac{(1-\theta)t_1}{2} + \frac{R\theta}{x}\Big\} - I_{cs}(T - t_1)\Big] \\ &= \frac{1}{T}\Big[- Z_{0s} + Z_{1s}R + Z_{2s}\frac{R}{P} - Z_{3s}\frac{R^2}{2P} - Z_{4s}R^2\Big]. \end{aligned} \quad (7.3)$$

where $t_1 = \frac{(R-R\theta)}{P}$ and $Z_{is},\ i = 0, 1, 2, 3, 4$ are independent of R and P.(See Appendix E)

7.3.2 Formulation of the Manufacturer

It is consider that a manufacturer produces some perfect and imperfect quality items at the rate of P units during the period $(0, t_1)$, receiving the raw material from the supplier at the same rate P during the period $(0, t_1)$. $Pe^{-\alpha t}$ and $P(1 - e^{-\alpha t})$ are respectively the expected quantity of perfect and imperfect quality items at any time t, where α be the reliability parameter given by $\alpha = \frac{\text{number of failures}}{\text{total units of operating hours}}$. Among the imperfect quality items, only $\beta P(1 - e^{-\alpha t}$ units per unit time become perfect quality after reworking and the portion $\gamma(1-\beta)P(1-e^{-\alpha t})$ are less perfect quality which are soled at a reduced price to the retailer. Here D_r and D'_r denote the demand rates of a retailer for perfect quality and less perfect quality items which are met by manufacturer during $(0, t_2)$ and $(0, t'_2)$ respectively.

For Perfect Quality Items of Manufacturer

The rate of change of inventory level of manufacturer for perfect quality items can be represented by the following differential equations:

$$\frac{dq_{1m}}{dt} = \begin{cases} Pe^{-\alpha t} + \beta P(1 - e^{-\alpha t}) - D_r, & 0 \le t \le t_1 \\ -D_r, & t_1 \le t \le t_2 \end{cases} \quad (7.4)$$

with boundary conditions $q_{1m}(t) = 0$ at $t = 0$ and $t = t_2$.
The solution of above differential equations are given by

$$q_{1m}(t) = \begin{cases} \frac{P}{\alpha}(1-\beta)(1 - e^{-\alpha t}) + (P\beta - D_r)t, & 0 \le t \le t_1 \\ D_r(t_2 - t), & t_1 \le t \le t_2 \end{cases} \quad (7.5)$$

From continuity at $t = t_1$, following condition is obtain

$$\frac{P}{\alpha}(1-\beta)(1 - e^{-\alpha t_1}) + (P\beta - D_r)t_1 = D_r(t_2 - t_1)$$

$$\text{which implies } t_2 = \frac{P}{D_r}\Big[\frac{1}{\alpha}(1-\beta)(1 - e^{-\alpha t_1}) + \beta t_1\Big]. \quad (7.6)$$

Now holding cost(HCM_1) for perfect quality items for manufacture is given by

$$\begin{aligned} HCM_1 &= h_m\int_0^{t_1} q_{1m}(t)\,dt + h_m\int_{t_1}^{t_2} q_{1m}(t)\,dt \\ &= \frac{h_m P}{\alpha}(1-\beta)t_1 - \frac{Ph_m}{\alpha^2}(1-\beta)(1 - e^{-\alpha t_1}) + h_m P\beta\frac{t_1^2}{2} - D_r h_m\frac{t_2^2}{2}. \end{aligned}$$

and reworking Cost(RCM) for Manufacture

$$RCM = r_{cm} \int_0^{t_1} P\beta(1-e^{-\alpha t})\,dt = r_{cm}P\beta\Big[t_1 - \frac{1}{\alpha}\{1-e^{-\alpha t_1}\}\Big].$$

For Less Perfect Quality Items of Manufacturer

The rate of change of inventory level of less perfect quality items for manufacturer can be represented by the following differential equations:

$$\frac{dq_{2m}}{dt} = \begin{cases} \gamma(1-\beta)P(1-e^{-\alpha t}) - D'_r, & 0 \le t \le t_1 \\ -D'_r, & t_1 \le t \le t'_2 \end{cases} \tag{7.7}$$

with boundary conditions $q_{2m}(t) = 0$ at $t = 0$ and $t = t'_2$.
The solution of above differential equations are given by

$$q_{2m}(t) = \begin{cases} -\frac{P}{\alpha}\gamma(1-\beta)(1-e^{-t\alpha}) + \Big[\gamma(1-\beta)P - D'_r\Big]t, & 0 \le t \le t_1 \\ D'_r(t'_2 - t), & t_1 \le t \le t'_2 \end{cases} \tag{7.8}$$

From continuity at $t = t_1$, following condition is obtain

$$t'_2 = \frac{1}{D'_r}\Big[-\frac{P}{\alpha}\gamma(1-\beta)(1-e^{-\alpha t_1}) + \gamma(1-\beta)Pt_1\Big]. \tag{7.9}$$

Now holding cost(HCM_2) for less perfect quality items for manufacture

$$\begin{aligned} HCM_2 &= h'_m \int_0^{t_1} q_{2m}(t)\,dt + h'_m \int_{t_1}^{t'_2} q_{2m}(t)\,dt \\ &= h'_m \frac{P\gamma}{\alpha^2}(1-\beta)\Big[(1-e^{-\alpha t_1}) - \alpha t_1\Big] + h'_m\Big[P\gamma(1-\beta) - D'_r\Big]\frac{{t_1}^2}{2} + h'_m\frac{D'_r}{2}(t'_2 - t_1)^2. \end{aligned}$$

Production cost for the manufacturer= $C(P)Pt_1$.
Inspection cost= $I_{sm}Pt_1$.
Holding cost for the manufacturer = $[HCM_1 + HCM_2]$
Set up cost of the manufacturer=A_m.
Idle time cost for the manufacturer= $I_{cm}(T - t_2)$.
Revenue of perfect quality items for the manufacturer $= s_m \int_0^{t_2} D_r dt = s_m D_r t_2$.
Revenue of less perfect quality items for the manufacturer $= s'_m \int_0^{t'_2} D'_r dt, = s'_m D'_r t'_2$.

Average Profit of Manufacturer

Average profit (APM) of manufacturer during the period $(0, T)$ is given by

$$\begin{aligned} APM &= \frac{1}{T}\Big[(s_m D_r t_2 + s'_m D'_r t'_2) - \Big\{w_s + C(P) + I_{sm}\Big\}Pt_1 - A_m - I_{cm}(T-t_2) \\ &-\frac{r_{cm}P\beta}{\alpha}\Big\{\alpha t_1 - (1-e^{-\alpha t_1})\Big\} - h_m\Big\{\frac{P}{\alpha^2}(1-\beta)\{\alpha t_1 - (1-e^{-\alpha t_1})\} + P\beta\frac{t_1^2}{2} - D_r\frac{t_2^2}{2}\Big\} \\ &-h'_m\Big\{\frac{P}{\alpha^2}\gamma(1-\beta)\{(1-e^{-\alpha t_1}) - \alpha t_1\} + \{\gamma(1-\beta)P - D'_r\}\frac{t_1^2}{2} + \frac{D'_r}{2}(t_1-t'_2)^2\Big\}\Big] \\ &= \frac{1}{T}\Big[-Z_{0m} - Z_{1m}\frac{R}{P} + Z_{2m}R + Z_{3m}\frac{R^2}{P^2} + Z_{4m}\frac{R^2}{P} + Z_{5m}R^2 - Z_{6m}PR^2\Big]. \end{aligned}$$

where $Z_{im},\ i = 0, 1, 2, ..., 6,$ are independent of R and P. (see Appendix E)

7.3.3 Formulation of the Retailer

Customers' demand for both perfect and less perfect quality items are met by the retailer through the adjacent showrooms PW_1 and PW_2 respectively. Retailer has a secondary warehouse SW to store the excess perfect quality items which are continuously transferred to the showroom PW_1. Less perfect quality items are directly transferred to the showroom PW_2. Transportation cost is taken into account to transfer each items from production center to the showrooms.

For Perfect Quality Items of Retailer

In this case the demand rate (D_c) of customers at PW_1 has consider as stock dependent as the following form

$$D_c = \begin{cases} \alpha_1 + \beta_1 q_{1r}, & 0 \le t \le t_3 \\ \alpha_1 + \beta_1 W_1, & t_3 \le t \le t_4 \\ \alpha_1 + \beta_1 q_{1r}, & t_4 \le t \le T \end{cases}$$

Now the corresponding rate of change of on hand inventory of perfect quality items are given by

$$\frac{dq_{1r}}{dt} = \begin{cases} D_r - D_c, & 0 \le t \le t_2 \\ -D_c, & t_2 \le t \le T \end{cases}$$

with boundary conditions

$$q_{1r}(t) = \begin{cases} 0, & \text{at } t = 0 \\ W_1, & \text{at } t = t_3 \\ S, & \text{at } t = t_2 \\ W_1, & \text{at } t = t_4 \\ 0, & \text{at } t = T \end{cases}$$

Therefore the solutions of above differential equations are given by

$$q_{1r}(t) = \begin{cases} \frac{D_r-\alpha_1}{\beta_1}(1-e^{-\beta_1 t}), & 0 \le t \le t_3 \\ W_1 + \Big[D_r - (\alpha_1+\beta_1 W_1)\Big](t-t_3), & t_3 \le t \le t_2 \\ S - \Big[(\alpha_1+\beta_1 W_1)\Big](t-t_2), & t_2 \le t \le t_4 \\ -\dfrac{\alpha_1}{\beta_1}\Big[1-e^{-\beta_1(t-T)}\Big], & t_4 \le t \le T \end{cases} \tag{7.10}$$

$$\text{Now, } q_{1r}(t_3) = W_1 \Rightarrow t_3 = -\frac{1}{\beta_1}\log(1-\frac{W_1\beta_1}{D_r-\alpha_1}) \tag{7.11}$$

$$q_{1r}(t_2) = S \text{ gives } t_4 = t_2 + \frac{S-W_1}{\alpha_1+\beta_1 W_1} \tag{7.12}$$

$$\text{and } W_1 + \Big[D_r - (\alpha_1+\beta_1 W_1)\Big](t_2-t_3) = S \tag{7.13}$$

$$q_{1r}(t_4) = W_1 \Rightarrow T = t_4 + \frac{1}{\beta_1}\log\Big[(1-\frac{D_r}{\alpha_1})(1-e^{-\beta_1 t_3})\Big] \tag{7.14}$$

Holding cost($HCRW$) of the secondary warehouse SW is given by

$$\begin{aligned} HCRW &= h_{rs}\int_{t_3}^{t_2}\{q_{1r}(t)-W_1\}\,dt + h_{rs}\int_{t_2}^{t_4}\{q_{1r}(t)-W_1\}\,dt \\ &= \frac{h_{rs}}{2}\Big[(D_r-(\alpha_1+\beta_1 W_1))(t_2-t_3)^2 - (\alpha_1+\beta_1 W_1)(t_2-t_4)^2\Big]. \end{aligned}$$

Holding cost($HCRS_1$) of the showroom PW_1 is given by

$$\begin{aligned} HCRS_1 &= h_r\int_0^{t_3} q_{1r}(t)\,dt + h_r\int_{t_3}^{t_4} W_1\,dt + \int_{t_4}^{T} q_{1r}(t)\,dt \\ &= h_r\frac{D_r-\alpha_1}{\beta_1}\Big[t_3 + \frac{e^{-\beta_1 t_3}}{\beta_1} - \frac{1}{\beta_1}\Big] + W_1 h_r(t_4-t_3) \\ &\quad -h_r\frac{\alpha_1}{\beta_1}\Big[(T-t_4) + \frac{1}{\beta_1} - \frac{e^{-\beta_1(t_4-T)}}{\beta_1}\Big]. \end{aligned}$$

Transportation cost($TCPR$) for perfect quality items of retailer is given by

$$\begin{aligned} TCPR &= c_{tr}\int_0^T D_C\,dt \\ &= c_{tr}\Big[(\alpha_1+\beta_1 W_1)(t_4-t_3) + D_r t_3 - \frac{D_r}{\beta_1} + \frac{D_r-\alpha_1}{\beta_1}e^{-\beta_1 t_3} + \frac{\alpha_1}{\beta_1}e^{-\beta_1(t_4-T)}\Big]. \end{aligned}$$

Revenue from selling perfect quality items is given by

$$\begin{aligned} REPR &= s_r\int_0^T D_c\,dt \\ &= s_r\Big[(\alpha_1+\beta_1W_1)(t_4-t_3)+D_rt_3-\frac{D_r}{\beta_1}+\frac{D_r-\alpha_1}{\beta_1}e^{-\beta_1t_3}+\frac{\alpha_1}{\beta_1}e^{-\beta_1(t_4-T)}\Big]. \end{aligned}$$

Hence the total profit of the retailer from the perfect quality items is given by

$$\begin{aligned} TPR_1 &= s_r\Big[(\alpha_1+\beta_1W_1)(t_4-t_3)+D_rt_3-\frac{D_r}{\beta_1}+\frac{D_r-\alpha_1}{\beta_1}e^{-\beta_1t_3}+\frac{\alpha_1}{\beta_1}e^{-\beta_1(t_4-T)}\Big] \\ &\quad -c_{tr}\Big[(\alpha_1+\beta_1W_1)(t_4-t_3)+D_rt_3-\frac{D_r}{\beta_1}+\frac{D_r-\alpha_1}{\beta_1}e^{-\beta_1t_3}+\frac{\alpha_1}{\beta_1}e^{-\beta_1(t_4-T)}\Big] \\ &\quad -\frac{h_{rs}}{2}\Big[(D_r-(\alpha_1+\beta_1W_1))(t_2-t_3)^2-(\alpha_1+\beta_1W_1)(t_2-t_4)^2\Big]-A_r \\ &\quad -h_r\Big[\frac{D_r-\alpha_1}{\beta_1}(t_3+\frac{e^{-\beta_1t_3}}{\beta_1}-\frac{1}{\beta_1})+W_1(t_4-t_3)-\frac{\alpha_1}{\beta_1}\Big\{(T-t_4) \\ &\quad +\frac{1}{\beta_1}-\frac{e^{-\beta_1(t_4-T)}}{\beta_1}\Big\}\Big]-s_m\Big[\frac{P}{\alpha^2}(1-\beta)\{\alpha t_1-(1-e^{-\alpha t_1})\}+P\beta\frac{t_1^2}{2}-D_r\frac{t_2^2}{2}\Big]. \end{aligned}$$

For Less Perfect Quality Items of Retailer

In this case demand rate (D_c') of customers at PW_2 is assumed as selling price dependent defined as $D_c'=(a-bs_r')$ and corresponding change of on hand inventory of less perfect quality items are given by

$$\frac{dq_{2r}}{dt} = \begin{cases} D_r'-D_c', & 0\le t\le t_2' \\ -D_c', & t_2'\le t\le T' \end{cases}$$

with boundary conditions $q_{2r}(t)=0$ at $t=0$ and $t=T'$.
The solution of above differential equations are given by

$$q_{2r}(t) = \begin{cases} (D_r'-D_c')t, & 0\le t\le t_2' \\ -D_c'(t-T'), & t_2'\le t\le T' \end{cases} \tag{7.15}$$

From continuity condition at $t=t_2'$, $T'=\frac{1}{D_c'}\Big[\gamma(1-\beta)Pt_1-\frac{P}{\alpha^2}\gamma(1-\beta)(1-e^{-\alpha t_1})\Big]$.
Also $q_{2r}(t_2')=W_2$ i.e., $W_2=(1-\frac{D_c'}{D_r'})\Big[-\frac{P}{\alpha}\gamma(1-\beta)(1-e^{-\alpha t_1})+\gamma(1-\beta)Pt_1\Big]$.
Holding cost$(HCRS_2)$ of the showroom PW_2 is given by

$$\begin{aligned} HCRS_2 &= h_r'\int_0^{t_2'}q_{2r}(t)\,dt+h_r'\int_{t_2'}^{T'}q_{2r}(t)\,dt \\ &= \frac{h_r'}{2}\Big[(D_r'-D_c'){t_2'}^2+D_c'(t_2'-T')^2\Big]. \end{aligned}$$

Transportation cost$(TCIR)$ for less perfect quality items of retailer is given by

$$TCIR = c'_{tr}\int_0^{T'} D'_c\,dt = c'_{tr}D'_cT'.$$

Revenue$(REIR)$ from selling less perfect quality items of retailer is given by

$$REIR = s'_r\int_0^{T'} D'_c\,dt = s'_rD'_cT'.$$

Total Profit(TPR_2) of Retailer for less perfect quality items is given by

$$\begin{aligned}TPR_2 &= \{s'_r - c'_{tr}\}D'_cT' - \frac{h'_r}{2}\Big[(D'_r - D'_c){t'_2}^2 + D'_c(t'_2 - T')^2\Big] - A'_r\\ &\quad -s'_m\Big[\frac{P\gamma}{\alpha^2}(1-\beta)\Big\{(1-e^{-t_1\alpha}) - \alpha t_1\Big\} + (P\gamma(1-\beta) - D'_r)\frac{{t_1}^2}{2} + \frac{D'_r}{2}(t_1 - t'_2)^2\Big].\end{aligned}$$

Average Profit of Retailer

Average Profit(APR) of Retailer during $(0,T)$ is

$$\begin{aligned}APR &= \frac{1}{T}[TPR_1 + TPR_2]\\ &= \frac{1}{T}\Big[s_r\Big\{(\alpha_1+\beta_1W_1)(t_4-t_3) + D_rt_3 - \frac{D_r}{\beta_1} + \frac{D_r-\alpha_1}{\beta_1}e^{-\beta_1t_3} + \frac{\alpha_1}{\beta_1}e^{-\beta_1(t_4-T)}\Big\}\\ &\quad -c_{tr}\Big\{(\alpha_1+\beta_1W_1)(t_4-t_3) + D_rt_3 - \frac{D_r}{\beta_1} + \frac{D_r-\alpha_1}{\beta_1}e^{-\beta_1t_3} + \frac{\alpha_1}{\beta_1}e^{-\beta_1(t_4-T)}\Big\}\\ &\quad -\frac{h_{rs}}{2}\Big\{(D_r - (\alpha_1+\beta_1W_1))(t_2-t_3)^2 - (\alpha_1+\beta_1W_1)(t_2-t_4)^2\Big\} + s'_rD'_cT'\\ &\quad -c'_{tr}D'_cT' - \frac{h'_r}{2}\Big\{(D'_r-D'_c){t'_2}^2 + D'_c(t'_2-T')^2\Big\} - h_r\Big\{\frac{D_r-\alpha_1}{\beta_1}(t_3 + \frac{e^{-\beta_1t_3}}{\beta_1}\\ &\quad -\frac{1}{\beta_1}) + W_1(t_4-t_3) - \frac{\alpha_1}{\beta_1}\{(T-t_4) + \frac{1}{\beta_1} - \frac{e^{-\beta_1(t_4-T)}}{\beta_1}\}\Big\} - s_m\Big\{\frac{P}{\alpha}(1-\beta)t_1\\ &\quad -\frac{P}{\alpha^2}(1-\beta)(1-e^{-\alpha t_1}) + P\beta\frac{t_1^2}{2} - D_r\frac{t_2^2}{2}\Big\} - s'_m\Big\{\frac{P\gamma}{\alpha^2}(1-\beta)(1-e^{-t_1\alpha})\\ &\quad -\frac{P\gamma}{\alpha}(1-\beta)t_1 + (P\gamma(1-\beta) - D'_r)\frac{{t_1}^2}{2} + \frac{D'_r}{2}(t_1-t'_2)^2\Big\}\Big] - A_r - A'_r\\ &= \frac{1}{T}\Big[Z_{0r} + Z_{1r}R + Z_{2r}\frac{R^2}{2P} + Z_{3r}R^2 + Z_{4r}\frac{R^3}{P} + Z_{5r}\frac{R^4}{P^2}\Big].\end{aligned}$$

where $Z_{ir},\ i = 0,1,...,5$ are independent of R and P (See Appendix E).

7.3.4 Integrated Average Profit

Average Profit(IAP) for the Integrated Model during $(0,T)$ is

$$\begin{aligned} IAP &= [APS + APM + APR] \\ &= \frac{1}{T}\Big[Z_0 + Z_1\frac{R}{P} + Z_2R - Z_3\frac{R^2}{2P} + Z_4\frac{R^2}{P^2} - Z_5PR^2 - Z_6R^2 + Z_7\frac{R^3}{P} + Z_8\frac{R^4}{P^2}\Big] \end{aligned}$$

where $Z_i,\ i = 0, 1, ..., 8$ are independent of R and P (See Appendix E).

7.3.5 Model in Fuzzy Rough Environment

In this environment, all holding cost, idle cost, setup cost and transportation cost have been considered fuzzy-rough parameters. Then the corresponding fuzzy-rough objective functions for supplier, manufacturer and retailer are given by

$$\begin{aligned} A\tilde{\bar{P}}S &= \frac{1}{T}\Big[-\tilde{\bar{Z}}_{0s} + \tilde{\bar{Z}}_{1s}R + \tilde{\bar{Z}}_{2s}\frac{R}{P} - \tilde{\bar{Z}}_{3s}\frac{R^2}{2P} - \tilde{\bar{Z}}_{4s}R^2\Big] \\ A\tilde{\bar{P}}M &= \frac{1}{T}\Big[-\tilde{\bar{Z}}_{0m} - \tilde{\bar{Z}}_{1m}\frac{R}{P} + \tilde{\bar{Z}}_{2m}R + \tilde{\bar{Z}}_{3m}\frac{R^2}{P^2} + \tilde{\bar{Z}}_{4m}\frac{R^2}{P} + \tilde{\bar{Z}}_{5m}R^2 - \tilde{\bar{Z}}_{6m}PR^2\Big] \\ \text{and } A\tilde{\bar{P}}R &= \frac{1}{T}\Big[\tilde{\bar{Z}}_{0r} + \tilde{\bar{Z}}_{1r}R + \tilde{\bar{Z}}_{2r}\frac{R^2}{2P} + \tilde{\bar{Z}}_{3r}R^2 + \tilde{\bar{Z}}_{4r}\frac{R^3}{P} + \tilde{\bar{Z}}_{5r}\frac{R^4}{P^2}\Big] \end{aligned}$$

Also the fuzzy-rough objective functions for integrated model is given by

$$\begin{aligned} I\tilde{\bar{A}}P &= [A\tilde{\bar{P}}S + A\tilde{\bar{P}}M + A\tilde{\bar{P}}R] \\ &= \frac{1}{T}\Big[\tilde{\bar{Z}}_0 + \tilde{\bar{Z}}_1\frac{R}{P} + \tilde{\bar{Z}}_2R - \tilde{\bar{Z}}_3\frac{R^2}{2P} + \tilde{\bar{Z}}_4\frac{R^2}{P^2} - \tilde{\bar{Z}}_5PR^2 - \tilde{\bar{Z}}_6R^2 + \tilde{\bar{Z}}_7\frac{R^3}{P} + \tilde{\bar{Z}}_8\frac{R^4}{P^2}\Big] \end{aligned}$$

where fuzzy rough parameters $\tilde{\bar{h}}_s, \tilde{\bar{h}}_m, \tilde{\bar{h}}'_m, \tilde{\bar{h}}_r, \tilde{\bar{h}}'_r, \tilde{\bar{h}}_{rs}, \tilde{\bar{A}}_s, \tilde{\bar{A}}_m, \tilde{\bar{A}}_r, \tilde{\bar{I}}_{cs}, \tilde{\bar{I}}_{cm}, \tilde{\bar{c}}_{tp}, \tilde{\bar{c}}'_{tp}$ are defined as follows,
$\tilde{\bar{h}}_s = (\bar{h}_{s1}, \bar{h}_{s2}, \bar{h}_{s3}, \bar{h}_{s4})$ with $\bar{h}_{st} \vdash ([h_{st2}, h_{st3}], [h_{st1}, h_{st4}])$, $0 \le h_{st1} \le h_{st2} < h_{st3} \le h_{st4}$,
$\tilde{\bar{h}}_m = (\bar{h}_{m1}, \bar{h}_{m2}, \bar{h}_{m3}, \bar{h}_{m4})$ with $\bar{h}_{mt} \vdash ([h_{mt2}, h_{mt3}], [h_{mt1}, h_{mt4}])$,
$0 \le h_{mt1} \le h_{mt2} < h_{mt3} \le h_{mt4}$.
$\tilde{\bar{h}}'_m = (\bar{h'}_{m1}, \bar{h'}_{m2}, \bar{h'}_{m3}, \bar{h'}_{m4})$ with $\bar{h'}_{mt} \vdash ([h'_{mt2}, h'_{mt3}], [h'_{mt1}, h'_{mt4}])$,
$0 \le h'_{mt1} \le h'_{mt2} < h'_{mt3} \le h'_{mt4}$.
$\tilde{\bar{h}}_r = (\bar{h}_{r1}, \bar{h}_{r2}, \bar{h}_{r3}, \bar{h}_{r4})$ with $\bar{h}_{rt} \vdash ([h_{rt2}, h_{rt3}], [h_{rt1}, h_{rt4}])$, $0 \le h_{rt1} \le h_{rt2} < h_{rt3} \le h_{rt4}$.
$\tilde{\bar{h}}'_r = (\bar{h'}_{r1}, \bar{h'}_{r2}, \bar{h'}_{r3}, \bar{h'}_{r4})$ with $\bar{h'}_{rt} \vdash ([h'_{rt2}, h'_{rt3}], [h'_{rt1}, h'_{rt4}])$, $0 \le h'_{rt1} \le h'_{rt2} < h'_{rt3} \le h'_{rt4}$.
$\tilde{\bar{h}}_{rs} = (\bar{h}_{rs1}, \bar{h}_{rs2}, \bar{h}_{rs3}, \bar{h}_{rs4})$ with $\bar{h}_{rst} \vdash ([h_{rst2}, h_{rst3}], [h_{rst1}, h_{rst4}])$,
$0 \le h_{rst1} \le h_{rst2} < h_{rst3} \le h_{rst4}$.
$\tilde{\bar{A}}_s = (\bar{A}_{s1}, \bar{A}_{s2}, \bar{A}_{s3}, \bar{A}_{s4})$ with $\bar{A}_{st} \vdash ([A_{st2}, A_{st3}], [A_{st1}, A_{st4}])$, $0 \le A_{st1} \le A_{st2} < A_{st3} \le A_{st4}$.
$\tilde{\bar{A}}_m = (\bar{A}_{m1}, \bar{A}_{m2}, \bar{A}_{m3}, \bar{A}_{m4})$ with $\bar{A}_{mt} \vdash ([A_{mt2}, A_{mt3}], [A_{mt1}, A_{mt4}])$,
$0 \le A_{mt1} \le A_{mt2} < A_{mt3} \le A_{mt4}$.

$\tilde{\bar{A}}_r = (\overline{A}_{r1}, \overline{A}_{r2}, \overline{A}_{r3}, \overline{A}_{r4})$ with $\overline{A}_{rt} \vdash ([A_{rt2}, A_{rt3}], [A_{rt1}, A_{rt4}])$, $0 \leq A_{rt1} \leq A_{rt2} < A_{rt3} \leq A_{rt4}$.
$\tilde{\bar{I}}_{cs} = (\overline{I}_{cs1}, \overline{I}_{cs2}, \overline{I}_{cs3}, \overline{I}_{cs4})$ with $\overline{I}_{cst} \vdash ([I_{cst2}, I_{cst3}], [I_{cst1}, I_{cst4}])$, $0 \leq I_{cst1} \leq I_{cst2} < I_{cst3} \leq I_{cst4}$.
$\tilde{\bar{I}}_{cm} = (\overline{I}_{cm1}, \overline{I}_{cm2}, \overline{I}_{cm3}, \overline{I}_{cm4})$ with $\overline{I}_{cmt} \vdash ([I_{cmt2}, I_{cst3}], [I_{cmt1}, I_{cmt4}])$,
$0 \leq I_{cmt1} \leq I_{cmt2} < I_{cmt3} \leq I_{cmt4}$.
$\tilde{\bar{c}}_{tp} = (\overline{c}_{tp1}, \overline{c}_{tp2}, \overline{c}_{tp3}, \overline{c}_{tp4})$ with $\overline{c}_{tpt} \vdash ([c_{tpt2}, c_{tpt3}], [c_{tpt1}, c_{tpt4}])$, $0 \leq c_{tpt1} \leq c_{tpt2} < c_{tpt3} \leq c_{tpt4}$.
$\tilde{\bar{c}}'_{tp} = (\overline{c'}_{tp1}, \overline{c'}_{tp2}, \overline{c'}_{tp3}, \overline{c'}_{tp4})$ with $\overline{c'}_{tpt} \vdash ([c'_{tpt2}, c'_{tpt3}], [c'_{tpt1}, c'_{tpt4}])$, $0 \leq c'_{tpt1} \leq c'_{tpt2} < c'_{tpt3} \leq c'_{tpt4}$.

7.3.6 Model in Equivalent Crisp Environment

In this environment, using Lemma 2.4 and Theorems 2.2 & 2.3, the fuzzy rough objective functions for supplier, manufacturer and retailer are given by

$$\begin{aligned}
EAPS &= E[A\tilde{\bar{P}}S] = \frac{1}{T}\Big[-E[\tilde{\bar{Z}}_{0s}] + E[\tilde{\bar{Z}}_{1s}]R + E[\tilde{\bar{Z}}_{2s}]\frac{R}{P} - E[\tilde{\bar{Z}}_{3s}]\frac{R^2}{2P} - E[\tilde{\bar{Z}}_{4s}]R^2\Big] \\
EAPM &= E[A\tilde{\bar{P}}M] = \frac{1}{T}\Big[-E[\tilde{\bar{Z}}_{0m}] - E[\tilde{\bar{Z}}_{1m}]\frac{R}{P} + E[\tilde{\bar{Z}}_{2m}]R + E[\tilde{\bar{Z}}_{3m}]\frac{R^2}{P^2} \\
&\quad + E[\tilde{\bar{Z}}_{4m}]\frac{R^2}{P} + E[\tilde{\bar{Z}}_{5m}]R^2 - E[\tilde{\bar{Z}}_{6m}]PR^2\Big] \\
\text{and } EAPR &= E[A\tilde{\bar{P}}R] \\
&= \frac{1}{T}\Big[E[\tilde{\bar{Z}}_{0r}] + E[\tilde{\bar{Z}}_{1r}]R + E[\tilde{\bar{Z}}_{2r}]\frac{R^2}{2P} + E[\tilde{\bar{Z}}_{3r}]R^2 + E[\tilde{\bar{Z}}_{4r}]\frac{R^3}{P} + E[\tilde{\bar{Z}}_{5r}]\frac{R^4}{P^2}\Big]
\end{aligned}$$

Also the objective functions for integrated model is given by

$$\begin{aligned}
EIAP &= E[I\tilde{\bar{A}}P] = \Big[E[A\tilde{\bar{P}}S] + E[A\tilde{\bar{P}}M] + E[A\tilde{\bar{P}}R]\Big] \\
&= \frac{1}{T}\Big[E[\tilde{\bar{Z}}_0] + E[\tilde{\bar{Z}}_1]\frac{R}{P} + E[\tilde{\bar{Z}}_2]R - E[\tilde{\bar{Z}}_3]\frac{R^2}{2P} + \tilde{\bar{Z}}_4\frac{R^2}{P^2} - E[\tilde{\bar{Z}}_5]PR^2 - \tilde{\bar{Z}}_6R^2 \\
&\quad + E[\tilde{\bar{Z}}_7]\frac{R^3}{P} + E[\tilde{\bar{Z}}_8]\frac{R^4}{P^2}\Big]
\end{aligned}$$

where $E[\tilde{\bar{h}}_s] = \frac{1}{16}\sum_{t=1}^{4}\sum_{k=1}^{4} h_{stk}$, $\quad E[\tilde{\bar{h}}_m] = \frac{1}{16}\sum_{t=1}^{4}\sum_{k=1}^{4} h_{mtk}$, $\quad E[\tilde{\bar{h}}'_m] = \frac{1}{16}\sum_{t=1}^{4}\sum_{k=1}^{4} h'_{mtk}$,

$$E[\tilde{\bar{h}}_r] = \frac{1}{16}\sum_{t=1}^{4}\sum_{k=1}^{4} h_{rtk}, \quad E[\tilde{\bar{h}}'_r] = \frac{1}{16}\sum_{t=1}^{4}\sum_{k=1}^{4} h'_{rtk}, \quad E[\tilde{\bar{h}}_{rs}] = \frac{1}{16}\sum_{t=1}^{4}\sum_{k=1}^{4} h_{rstk},$$

$$E[\tilde{\bar{A}}_s] = \frac{1}{16}\sum_{t=1}^{4}\sum_{k=1}^{4} A_{stk}, \quad E[\tilde{\bar{A}}_m] = \frac{1}{16}\sum_{t=1}^{4}\sum_{k=1}^{4} A_{mtk}, \quad E[\tilde{\bar{A}}_r] = \frac{1}{16}\sum_{t=1}^{4}\sum_{k=1}^{4} A_{rtk},$$

$$E[\tilde{\tilde{I}}_{cs}] = \frac{1}{16}\sum_{t=1}^{4}\sum_{k=1}^{4} I_{cstk}, \qquad E[\tilde{\tilde{I}}_{cm}] = \frac{1}{16}\sum_{t=1}^{4}\sum_{k=1}^{4} I_{cmtk}, \qquad E[\tilde{\tilde{c}}_{tp}] = \frac{1}{16}\sum_{t=1}^{4}\sum_{k=1}^{4} c_{tptk},$$

$E[\tilde{\tilde{c}}'_{tp}] = \frac{1}{16}\sum_{t=1}^{4}\sum_{k=1}^{4} c'_{tptk}$

7.3.7 Stakelberg Approach (Leader-follower Relationship)

In this case manufacturer is leader, supplier and retailer are followers. Also, optimum values of the average profit of supplier and retailer are obtained by putting the optimum value of the decision variables, which are obtained by optimizing the average profit of manufacturer. Using the equation (7.6), (7.11), (7.12), (7.13) and $(1-e^{-\alpha t_1}) \approx \alpha t_1$, in the equation (7.14), the relation $T = \frac{1-\theta}{D_r}R + t_0$ is obtained where t_0 is given by $t_0 = \frac{S-W_1}{\alpha_1+\beta_1 W_1} + \frac{1}{\beta_1}\log\Big[(1-\frac{D_r}{\alpha_1})(1-e^{-\beta_1 t_3})\Big]$ and it is independent of R and P and.

When both R and P are Decision Variables:

The expected average profit of manufacturer is given by

$$\begin{aligned}\text{EAPM(R,P)} &= \frac{1}{T}\Big[-E[\tilde{\tilde{Z}}_{0m}] - E[\tilde{\tilde{Z}}_{1m}]\frac{R}{P} + E[\tilde{\tilde{Z}}_{2m}]R + E[\tilde{\tilde{Z}}_{3m}]\frac{R^2}{P^2} + E[\tilde{\tilde{Z}}_{4m}]\frac{R^2}{P} \\ &\quad + E[\tilde{\tilde{Z}}_{5m}]R^2 - E[\tilde{\tilde{Z}}_{6m}]PR^2\Big]\end{aligned}$$

where $E[\tilde{\tilde{Z}}_{im}]$, i=0,1,2,...,6, are independent of R and P(see Appendix E).
The necessary conditions for maximum value of EAPM(R,P) are $\frac{\partial}{\partial R}(EAPM) = 0$ and $\frac{\partial}{\partial P}(EAPM) = 0$ which gives respectively

$$\begin{aligned}&\frac{(1-\theta)}{D_r T}E[\tilde{\tilde{Z}}_{0m}] - \Big\{1 - \frac{(1-\theta)R}{D_r T}\Big\}\Big\{\frac{1}{P}E[\tilde{\tilde{Z}}_{1m}] - E[\tilde{\tilde{Z}}_{2m}]\Big\} + \Big\{\frac{2R}{P} - \frac{(1-\theta)R^2}{D_r TP}\Big\} \\ &\times\Big\{\frac{1}{P}E[\tilde{\tilde{Z}}_{3m}] + E[\tilde{\tilde{Z}}_{4m}]\Big\} + \Big\{2R - \frac{(1-\theta)R^2}{D_r T}\Big\}\Big\{E[\tilde{\tilde{Z}}_{5m}] - PE[\tilde{\tilde{Z}}_{6m}]\Big\} = 0 \qquad (7.16)\end{aligned}$$

$$\text{and} \quad E[\tilde{\tilde{Z}}_{1m}]\frac{R}{P^2} - 2E[\tilde{\tilde{Z}}_{3m}]\frac{R^2}{P^3} - E[\tilde{\tilde{Z}}_{4m}]\frac{R^2}{P^2} - E[\tilde{\tilde{Z}}_{6m}]R^2 = 0 \qquad (7.17)$$

Solving (7.16) and (7.17), we can obtain the optimum value of R and P, say R^* and P^*.
If $\Big\{\frac{\partial^2}{\partial R^2}(EAPM)\Big\}\Big\{\frac{\partial^2}{\partial P^2}(EAPM)\Big\} - \Big\{\frac{\partial^2}{\partial P\partial R}(EAPM)\Big\}^2 > 0$, $\frac{\partial^2}{\partial R^2}(EAPM) < 0$, and $\frac{\partial^2}{\partial P^2}(EAPM) < 0$ holds for $R = R^*$ and $P = P^*$ then EAPM (R^*, P^*) is maximum.

Now $\frac{\partial^2}{\partial R^2}(EAPM)\Big]_{at\ (R^*,P^*)} < 0$

$$i.e.,\ -\frac{(1-\theta)^2}{{D_r}^2{T^*}^2}E[\tilde{\tilde{Z}}_{0m}] + \Big\{\frac{(1-\theta)}{D_rT^*P^*} - \frac{(1-\theta)^2R^*}{{D_r}^2{T^*}^2P^*}\Big\}E[\tilde{\tilde{Z}}_{1m}] + \Big\{\frac{(1-\theta)^2R^*}{{D_r}^2{T^*}^2} - \frac{(1-\theta)}{D_rT^*}\Big\}E[\tilde{\tilde{Z}}_{2m}] + \Big\{\frac{(1-\theta)^2{R^*}^2}{{D_r}^2{T^*}^2{P^*}^2} + \frac{1}{{P^*}^2} - \frac{2(1-\theta)R^*}{D_rT^*{P^*}^2}\Big\}E[\tilde{\tilde{Z}}_{3m}] + \Big\{\frac{(1-\theta)^2}{{D_r}^2{T^*}^2P^*} + \frac{1}{P^*} - \frac{2(1-\theta)R^*}{D_rT^*P^*}\Big\}E[\tilde{\tilde{Z}}_{4m}] + \Big\{1 - \frac{2(1-\theta)R^*}{D_rT^*P^*} + \frac{(1-\theta)^2{R^*}^2}{{D_r}^2{T^*}^2}\Big\}E[\tilde{\tilde{Z}}_{5m}] - \Big\{P^* - \frac{2(1-\theta)P^*R^*}{D_rT^*} + \frac{(1-\theta)^2{R^*}^2P^*}{{D_r}^2{T^*}^2}\Big\}E[\tilde{\tilde{Z}}_{6m}] < 0 \quad (7.18)$$

and $\frac{\partial^2}{\partial P^2}(EAPM)]_{at\ (R^*,P^*)} < 0$

$$i.e.,\ -E[\tilde{\tilde{Z}}_{1m}]\frac{R^*}{{P^*}^3} + 3E[\tilde{\tilde{Z}}_{3m}]\frac{{R^*}^2}{{P^*}^4} + E[\tilde{\tilde{Z}}_{4m}]\frac{{R^*}^2}{{P^*}^3} < 0 \quad (7.19)$$

$$\text{and } \Big[\Big\{\frac{\partial^2}{\partial R^2}(EAPM)\Big\}\Big\{\frac{\partial^2}{\partial P^2}(EAPM)\Big\} - \Big\{\frac{\partial^2}{\partial P\partial R}(EAPM)\Big\}^2\Big]_{at\ (R^*,P^*)} > 0 \quad (7.20)$$

Therefore, EAPM(R^*, P^*) is maximum if the relations (7.18), (7.19) and (7.20) holds and the corresponding optimum average profit of supplier and retailer are

$EAPS(R^*, P^*) = \frac{1}{T^*}\Big[- E[\tilde{\tilde{Z}}_{0s}] + E[\tilde{\tilde{Z}}_{1s}]R^* + E[\tilde{\tilde{Z}}_{2s}]\frac{R^*}{P^*} - E[\tilde{\tilde{Z}}_{3s}]\frac{{R^*}^2}{2P^*} - E[\tilde{\tilde{Z}}_{4s}]{R^*}^2\Big]$

$EAPR(R^*, P^*) = \frac{1}{T^*}\Big[E[\tilde{\tilde{Z}}_{0r}] + E[\tilde{\tilde{Z}}_{1r}[R^* + E[\tilde{\tilde{Z}}_{2r}]\frac{{R^*}^2}{2P^*} + E[\tilde{\tilde{Z}}_{3r}]{R^*}^2 + E[\tilde{\tilde{Z}}_{4r}]\frac{{R^*}^3}{P^*}$

$+E[\tilde{\tilde{Z}}_{5r}]\frac{{R^*}^4}{{P^*}^2}\Big]$ where $T^* = \frac{1-\theta}{D_r}R^* + t_0$

When P is only Decision Variable

The necessary conditions for maximum value of EAPM(P) is $\frac{d}{dP}(EAPM) = 0$

$$\text{i.e., } E[\tilde{\tilde{Z}}_{1m}]\frac{R}{P^2} - 2E[\tilde{\tilde{Z}}_{3m}]\frac{R^2}{P^3} - E[\tilde{\tilde{Z}}_{4m}]\frac{R^2}{P^2} - E[\tilde{\tilde{Z}}_{6m}]R^2 = 0 \quad (7.21)$$

which gives the optimum value of P, say P^{**}.
If $\frac{d^2}{dP^2}(EAPM) < 0$ hold for $P = P^{**}$ then EAPM(P^{**}) is maximum.

$$\text{Now } \frac{d^2}{dP^2}(EAPM)]_{at\ P=P^{**}} < 0 \text{ gives } E[\tilde{\tilde{Z}}_{1m}] - 3E[\tilde{\tilde{Z}}_{3m}]\frac{R}{P^{**}} - E[\tilde{\tilde{Z}}_{4m}]R < 0 \quad (7.22)$$

Therefore, $EAPM(P^{**})$ is maximum if the relation (7.22) hold and corresponding optimum average profit of supplier and retailer are respectively

$EAPS(P^{**}) = \frac{1}{T}\Big[- E[\tilde{\tilde{Z}}_{0s}] + E[\tilde{\tilde{Z}}_{1s}]R + E[\tilde{\tilde{Z}}_{2s}]\frac{R}{P^{**}} - E[\tilde{\tilde{Z}}_{3s}]\frac{R^2}{2P^{**}} - E[\tilde{\tilde{Z}}_{4s}]R^2\Big]$

$EAPR(P^{**}) = \frac{1}{T}\Big[E[\tilde{\tilde{Z}}_{0r}] + E[\tilde{\tilde{Z}}_{1r}]R + E[\tilde{\tilde{Z}}_{2r}]\frac{R^2}{2P^{**}} + E[\tilde{\tilde{Z}}_{3r}]R^2 + E[\tilde{\tilde{Z}}_{4r}]\frac{R^3}{P^{**}}$

$+E[\tilde{\tilde{Z}}_{5r}]\frac{R^4}{{P^{**}}^2}\Big]$ where $T = \frac{1-\theta}{D_r}R + t_0$

7.3.8 Integrated Approach

When both R and P are Decision Variables:
The necessary conditions for maximum value of $EIAP(R,P)$ are $\frac{\partial}{\partial R}(EIAP) = 0$ and $\frac{\partial}{\partial P}(EIAP) = 0$ which gives respectively

$$\Big\{1 - \frac{(1-\theta)R}{D_rT}\Big\}\Big\{\frac{1}{P}E[\tilde{\tilde{Z}}_1] + E[\tilde{\tilde{Z}}_2]\Big\} - \Big\{1 + \frac{(1-\theta)R}{2D_rT}\Big\}\Big\{\frac{R}{P}E[\tilde{\tilde{Z}}_3]$$
$$+2PRE[\tilde{\tilde{Z}}_5] + 2RE[\tilde{\tilde{Z}}_6]\Big\} - \frac{(1-\theta)}{D_rT}E[\tilde{\tilde{Z}}_0] + \Big\{\frac{2R}{P^2} - \frac{(1-\theta)R^2}{D_rTP^2}\Big\}E[\tilde{\tilde{Z}}_4]$$
$$+\Big\{\frac{3R^2}{P} - \frac{(1-\theta)R^3}{D_rTP}\Big\}E[\tilde{\tilde{Z}}_7] + \Big\{\frac{4R^3}{P^2} - \frac{(1-\theta)R^4}{D_rTP^2}\Big\}E[\tilde{\tilde{Z}}_8] = 0 \tag{7.23}$$

$$\text{and } \Big[E[\tilde{\tilde{Z}}_1] + E[\tilde{\tilde{Z}}_3]\frac{R}{2} + 2E[\tilde{\tilde{Z}}_4]\frac{R}{P} + E[\tilde{\tilde{Z}}_5]RP^2 + E[\tilde{\tilde{Z}}_7]R^2 + 2E[\tilde{\tilde{Z}}_8]\frac{R^2}{P}\Big] = 0 \tag{7.24}$$

Solving (7.23) and (7.24), we can obtain the optimum value of R and P, say R^* and P^*.
If $\Big\{\frac{\partial^2}{\partial R^2}(EIAP)\Big\}\Big\{\frac{\partial^2}{\partial P^2}(EIAP)\Big\} - \Big\{\frac{\partial^2}{\partial P \partial R}(EIAP)\Big\}^2 > 0$, $\frac{\partial^2}{\partial R^2}(EIAP) < 0$ and $\frac{\partial^2}{\partial P^2}(EIAP) < 0$ holds for $R = R^*$ and $P = P^*$ then EIAP(R^*, P^*) is maximum.
Now $\frac{\partial^2}{\partial R^2}(EIAP)\Big]_{at(R^*,P^*)} < 0$

$$i.e., \frac{2(1-\theta)^2}{{D_r}^2{T^*}^2}E[\tilde{\tilde{Z}}_0] + \frac{2(1-\theta)}{D_rT^*}\Big\{\frac{(1-\theta)R^*}{D_rT^*} - 1\Big\}\Big\{\frac{1}{P}E[\tilde{\tilde{Z}}_1] + E[\tilde{\tilde{Z}}_2]\Big\}$$
$$-\Big\{1 + \frac{2(1-\theta)^2{R^*}^2}{{D_r}^2{T^*}^2P^*} - \frac{2(1-\theta)R^*}{D_rT^*P^*}\Big\}E[\tilde{\tilde{Z}}_3] + \Big\{\frac{2}{{P^*}^2} + \frac{2(1-\theta)^2{{R^*}^2}^2}{{D_r}^2T^*}{P^*}^2$$
$$-\frac{4(1-\theta)R^*}{D_rT^*P^*}\Big\}E[\tilde{\tilde{Z}}_4] - \Big\{2P^* + \frac{2(1-\theta)^2P^*{R^*}^2}{{D_r}^2{T^*}^2} - \frac{4(1-\theta)P^*R^*}{D_rT^*}\Big\}E[\tilde{\tilde{Z}}_5]$$
$$-\Big\{2 + \frac{2(1-\theta)^2{R^*}^2}{{D_r}^2{T^*}^2} - \frac{4(1-\theta)R^*}{D_rT}\Big\}E[\tilde{\tilde{Z}}_6] + \Big\{\frac{6R^*}{P^*} + \frac{2(1-\theta)^2{R^*}^3}{{D_r}^2{T^*}^2P^*}$$
$$-\frac{6(1-\theta){R^*}^2}{D_rT^*P^*}\Big\}E[\tilde{\tilde{Z}}_7] + \Big\{\frac{12{R^*}^2}{{P^*}^2} + \frac{2(1-\theta)^2{R^*}^4}{{D_r}^2{T^*}^2{P^*}^2} - \frac{8(1-\theta){R^*}^3}{D_rT^*{P^*}^2}\Big\}E[\tilde{\tilde{Z}}_8] < 0 \tag{7.25}$$

and $\frac{\partial^2}{\partial P^2}(EIAP)\Big]_{at(R^*,P^*)} < 0$

$$i.e., \Big[2E[\tilde{\tilde{Z}}_1]\frac{R^*}{{P^*}^3} + E[\tilde{\tilde{Z}}_3]\frac{{R^*}^2}{{P^*}^3} + 6E[\tilde{\tilde{Z}}_4]\frac{{R^*}^2}{{P^*}^4} + 2E[\tilde{\tilde{Z}}_7]\frac{{R^*}^3}{{P^*}^3} + 6E[\tilde{\tilde{Z}}_8]\frac{{R^*}^4}{{P^*}^4}\Big] < 0 \tag{7.26}$$

$$\text{and } \Big[\Big\{\frac{\partial^2}{\partial R^2}(EIAP)\Big\}\Big\{\frac{\partial^2}{\partial P^2}(EIAP)\Big\} - \Big\{\frac{\partial^2}{\partial P\partial R}(EIAP)\Big\}^2\Big]_{at(R^*,P^*)} > 0 \tag{7.27}$$

Therefore, IAP(R^*, P^*) is maximum if the relations (7.25), (7.26) and (7.27) holds and corresponding optimum integrated average profit of the supply chain is

$$\begin{aligned} EIAP(R^*,P^*) &= \frac{1}{T^*}\Big[E[\tilde{\tilde{Z}}_0]+E[\tilde{\tilde{Z}}_1]\frac{R^*}{P^*}+E[\tilde{\tilde{Z}}_2]R^*-Z_3\frac{R^{*2}}{2P^*}+E[\tilde{\tilde{Z}}_4]\frac{R^{*2}}{P^{*2}} \\ &\quad -E[\tilde{\tilde{Z}}_5]P^*R^{*2}-E[\tilde{\tilde{Z}}_6]R^{*2}+E[\tilde{\tilde{Z}}_7]\frac{R^*3}{P^*}+E[\tilde{\tilde{Z}}_8]\frac{R^{*4}}{P^{*2}}\Big]. \\ &\quad \text{where } T^*=\frac{1-\theta}{D_r}R^*+t_0 \end{aligned}$$

When P is only Decision Variable:
The necessary conditions for maximum value of EIAP(P) is $\frac{d}{dP}(EIAP)=0$
$i.e.,\ E[\tilde{\tilde{Z}}_1]\frac{R}{P^2}-E[\tilde{\tilde{Z}}_3]\frac{R^2}{2P^2}+2E[\tilde{\tilde{Z}}_4]\frac{R^2}{P^3}+E[\tilde{\tilde{Z}}_5]R^2-E[\tilde{\tilde{Z}}_7]\frac{R^3}{P^2}+2E[\tilde{\tilde{Z}}_8]\frac{R^4}{P^3}=0$
which gives the optimum value of P, say P^{**}.
If $\frac{d^2}{dP^2}(EIAP)<0$ hold for $P=P^{**}$ then $EIAP(P^{**})$ is maximum.
Now $\frac{d^2}{dP^2}(EIAP)]_{at\ P=P^{**}}<0$ gives

$$-2E[\tilde{\tilde{Z}}_1]\frac{R}{P^{**3}}+E[\tilde{\tilde{Z}}_3]\frac{R^2}{P^{**3}}-6E[\tilde{\tilde{Z}}_4]\frac{R^2}{P^{**4}}-2E[\tilde{\tilde{Z}}_7]\frac{R^3}{P^{**3}}-6E[\tilde{\tilde{Z}}_8]\frac{R^4}{P^{**4}}<0 \quad (7.28)$$

Therefore, $EIAP(P^{**})$ is maximum if the relation (7.28) hold and corresponding maximum integrated average profit of the supply chain is

$$\begin{aligned} EIAP(P^{**}) &= \frac{1}{T}\Big[E[\tilde{\tilde{Z}}_0]+E[\tilde{\tilde{Z}}_1]\frac{R}{P^{**}}+Z_2R-Z_3\frac{R^2}{2P^{**}}+E[\tilde{\tilde{Z}}_4]\frac{R^2}{P^{**2}} \\ &\quad -Z_5P_2R^2-E[\tilde{\tilde{Z}}_6]R^2+E[\tilde{\tilde{Z}}_7]\frac{R^3}{P^{**}}+E[\tilde{\tilde{Z}}_8]\frac{R^4}{P^{**2}}\Big]. \end{aligned}$$

7.4 Numerical Illustration

To illustrate the proposed production inventory model, we consider the following numerical data in Table 7.1 and 7.2. The optimal values of the decision variables and corresponding profits are given in Table 7.3 and 7.4. Also sensitivity analysis has been performed of the profits, production rate (P) and inventory level (R) of supplier with respect to different parameters are shown in Figure 7.3 to Figure 7.12.

Table 7.1: Values of different crisp parameters

Parameters	Value	Parameters	Value	Parameter	Value	Parameter	Value
θ	0.11	r_{cm}	2	C_s	40	S	335
x	20	s_m	158	w_s	90	W_1	45
α	0.10	s'_m	108	w'_s	55	W_2	16
β	0.44	s_r	176	R_1	16	a	51
γ	0.78	s'_r	135	H	0.02	b	0.17
α_1	20	S_c	0.35	G	10	D'_r	35
β_1	0.2	D_r	36				

Table 7.2: Values of different fuzzy-rough parameters

Fu-Ro parameters	Fu-Ro value	Input values	Expected value	value
$\tilde{\tilde{h}}_s$	near roughly(0.5)	$(\overline{.49}, \overline{.50}, \overline{.51}, \overline{.52})$ with ([-.04,.04],[-.08,.08])	E[$\tilde{\tilde{h}}_s$]	.505
$\tilde{\tilde{h}}_m$	near roughly(1.7)	$(\overline{1.5}, \overline{1.6}, \overline{1.7}, \overline{1.8})$ with ([-0.2,0.2],[-0.4,0.4])	E[$\tilde{\tilde{h}}_m$]	1.65
$\tilde{\tilde{h}}'_m$	near roughly(1.5)	$(\overline{1.35}, \overline{1.40}, \overline{1.50}, \overline{1.55})$ with ([-.18,.18],[-.36,.36])	E[$\tilde{\tilde{h}}'_m$]	1.45
$\tilde{\tilde{h}}_r$	near roughly(1.6)	$(\overline{1.54}, \overline{1.6}, \overline{1.63}, \overline{1.67})$ with ([-.12,.12],[-.24,.24])	E[$\tilde{\tilde{h}}_r$]	1.61
$\tilde{\tilde{h}}'_r$	near roughly(1.3)	$(\overline{1.15}, \overline{1.22}, \overline{1.3}, \overline{1.37})$ with ([-.11,.11],[-.22,.22])	E[$\tilde{\tilde{h}}'_r$]	1.26
$\tilde{\tilde{h}}_{rs}$	near roughly(1.8)	$(\overline{1.67}, \overline{1.72}, \overline{1.80}, \overline{1.85})$ with ([-.13,.13],[-.26,.26])	E[$\tilde{\tilde{h}}_{rs}$]	1.78
$\tilde{\tilde{A}}_s$	near roughly(420)	$(\overline{416}, \overline{420}, \overline{425}, \overline{428})$ with ([-.30,.30],[-.60,.60])	E[$\tilde{\tilde{A}}_s$]	422.25
$\tilde{\tilde{A}}_m$	near roughly(500)	$(\overline{486}, \overline{495}, \overline{500}, \overline{510})$ with ([-.29,.29],[-.58,.58])	E[$\tilde{\tilde{A}}_m$]	491.50
$\tilde{\tilde{A}}_r$	near roughly(400)	$(\overline{395}, \overline{400}, \overline{408}, \overline{410})$ with ([-.35,.35],[-.70,.70])	E[$\tilde{\tilde{A}}_r$]	403.25
$\tilde{\tilde{I}}_{cs}$	near roughly(3)	$(\overline{2.5}, \overline{2.8}, \overline{3.0}, \overline{3.15})$ with ([-.3,.3],[-.6,.6])	E[$\tilde{\tilde{I}}_{cs}$]	2.86
$\tilde{\tilde{I}}_{cm}$	near roughly(2)	$(\overline{1.9}, \overline{1.96}, \overline{2.0}, \overline{2.04})$ with ([-.21,.21],[-.42,.42])	E[$\tilde{\tilde{I}}_{cm}$]	1.97
$\tilde{\tilde{c}}_{tp}$	near roughly(1.4)	$(\overline{1.36}, \overline{1.40}, \overline{1.43}, \overline{1.47})$ with ([-.14,.14],[-.28,.28])	E[$\tilde{\tilde{c}}_{tp}$]	1.42
$\tilde{\tilde{c}}'_{tp}$	near roughly(1.0)	$(\overline{.90}, \overline{.97}, \overline{1.0}, \overline{1.06})$ with ([-.01,.01],[-.02,.02])	E[$\tilde{\tilde{c}}'_{tp}$]	0.98

where value of $\tilde{\tilde{h}}_s$ is near roughly (0.5) $= (\overline{.49}, \overline{.50}, \overline{.51}, \overline{.52})$ with oscillation $([-.04, .04], [-.08, .08])$ means that $\overline{.49} \vdash ([.45, .53], [.41, .57])$, $\overline{.50} \vdash ([.46, .54], [.42, .58])$, $\overline{.51} \vdash ([.47, .55], [.43, .59])$ and $\overline{.52} \vdash ([.48, .56], [.44, .60])$.

Table 7.3: Optimal result when P and R are decision variables

$Approach$	$Total\,profit$	$EAPS$	$EAPM$	$EAPR$	R	P
$Stakelberg$	5126.616	1543.341	1814.635	1656.160	3217.589	64.9821
$Integrated$	5176.606	1596.751	1872.695	1707.160	3189.629	69.8521

Table 7.4: Optimal result when P is decision variable and $R = 3225$

$Approach$	$Total\,profit$	$EAPS$	$EAPM$	$EAPR$	P
$Stakelberg$	5012.614	1512.752	1714.346	1751.426	63.64570
$Integrated$	5049.634	1582.912	1759.146	1707.576	67.37870

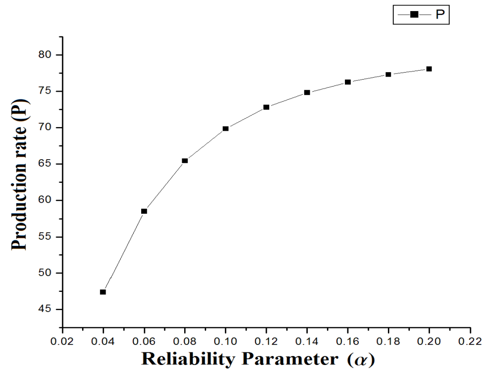

Figure 7.3: Reliability parameter (α) vs P

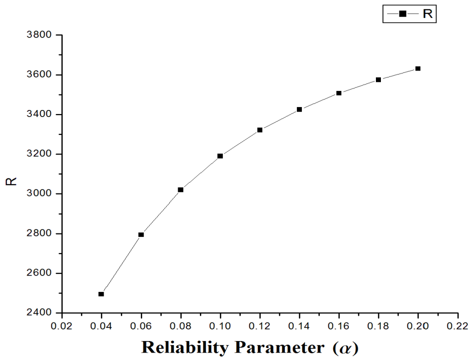

Figure 7.4: Reliability parameter (α) vs R

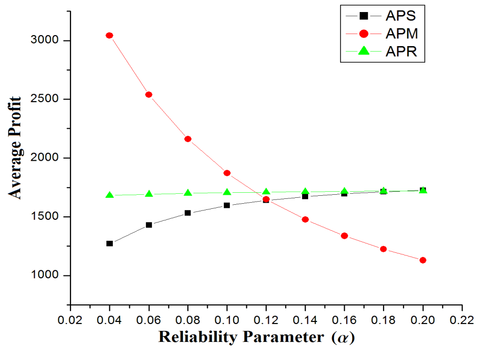

Figure 7.5: Reliability parameter vs profit

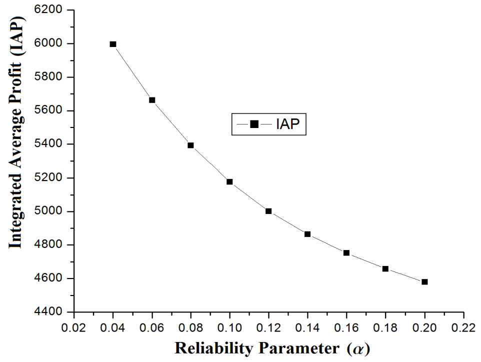

Figure 7.6: Reliability parameter (α) vs IAP

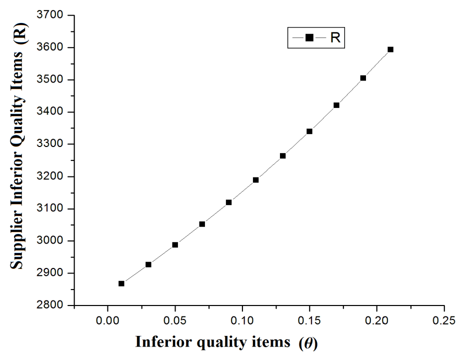

Figure 7.7: Defective rate (θ) vs R

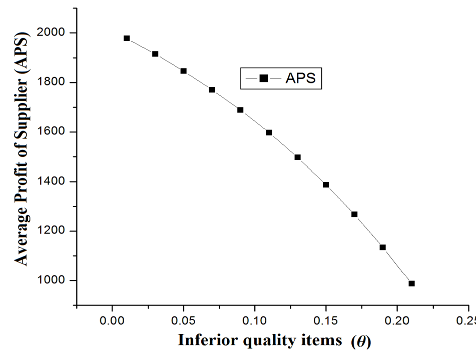

Figure 7.8: Defective rate (θ) vs APS

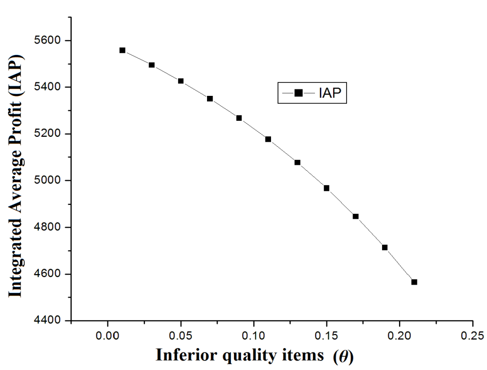

Figure 7.9: Defective rate (θ) vs IAP

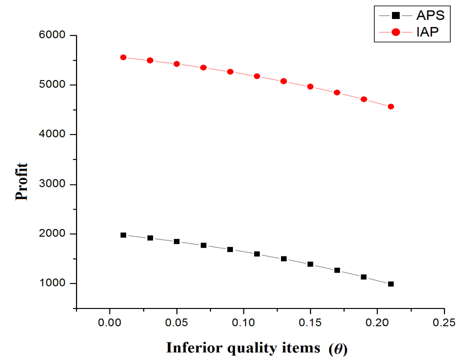

Figure 7.10: Defective rate (θ) vs APS, IAP

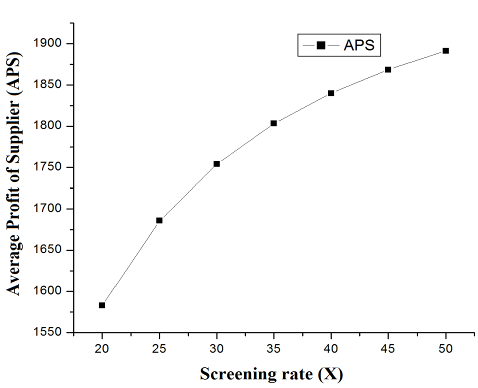

Figure 7.11: Screening rate (x) vs APS

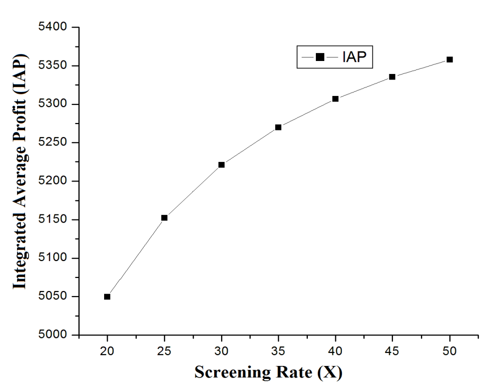

Figure 7.12: Screening rate (x) vs IAP

7.4.1 Discussion

From Table 7.3 and 7.4, it is observed that profits under the integrated approach is greater than the Stakelberg approach and hence the former approach is better than the later approach. The sensitivity analysis in section 7.4 shows that with increase of the reliability parameter α, (i) the profits of the supplier and retailer are slightly increasing (Figure 7.5), (ii) the values of P and R are gradually increasing (Figure 7.3 & 7.4) but (iii) both profits of the manufacturer (APM) and the integrated profit (IAP) are gradually decreasing (Figure 7.5 & 7.6). From Figure 7.7 to 7.10 it is also seen that the values of APS and IAP are decreasing but initial amount of inventory level of supplier is increasing with increase of θ. It is also noted that the values of APS and IAP increase with the screening rate (x) of the supplier.

7.5 Conclusion

This chapter develops a three layer supply chain production inventory model involving supplier, manufacturer and retailer as the members of the chain who are responsible for performing the raw materials into finished product and make them available to satisfy customers' demand in time. In comparing with the existing literature on the supply chain, the followings are the main contributions in the proposed model:
Inspection cost is incurred during the production run time and manufacturer continuously inspects as well as separates the perfect quality items, less perfect quality items, repairable items which are transformed into perfect quality items after some rework, and rejected items. Here, reworked cost is considered by the manufacturer to repair a certain percent of imperfect quality items. Demand rate of customers for perfect quality items and less perfect quality items are respectively assumed to be stock dependent and selling price dependent. Here retailer have two showrooms PW_1 and PW_2 of finite capacities at busy market place and the market demands of perfect and less perfect quality items are respectively met through the showrooms PW_1 and PW_2. Retailer has a secondary warehouse SW of infinite capacity, away from busy market place, to store the excess amount perfect quality items from where the items are continuously transferred to the showroom. It is considered that the holding cost per unit per unit time at SW is less than the holding cost at PW_1 per unit per unit time. The repairing costs of corrective and preventive maintenance should also be considered, as these costs increase the unit production cost. Inventory and production decisions are made at the supplier, manufacturer and retailer levels. Actually in this chapter, the coordination between production and inventory decisions has been established across the supply chain so that integrated average profit of the chain is maximum.

Chapter 8

A fuzzy imperfect production inventory model based on fuzzy differential and fuzzy integral method

8.1 Introduction

The fuzzy set concept has been used to treat the uncertainty in the classical inventory model. This set theory, originally introduced by Zadeh [239], provides a feasible approach to deal with the fuzzy uncertainty problem. The considerable attention has been attracted to fuzzy circumstance in the literature. For example, Park [164] used fuzzy inventory cost in economic order quantity model. Chang [24] discussed how to obtain the economic production quantity, when the quantity of demand is uncertain. Chen and Hsieh [29] established a fuzzy economic production model to treat the inventory problem with all the parameters and variables being fuzzy numbers. Hsieh [93], Lee and Yao [128], and Lin and Yao [137] also wrote some papers about the fuzzy production model. Except these there exist many other papers such as Das et al. [51] in which uncertainties have been solved using fuzzy set theory.

Traditional EPQ models assume that all items made are of perfect quality. However, in real world manufacturing systems, due to manufacturing operator, machine-component and/or other factors, generation of defective items is inevitable. It is more realistic to assume that all industries produce a certain percent of imperfect quality items. Such a production process is called imperfect production (cf. Salameh and Jaber [184], Yoo et al. [237]). Among other researchers, Salameh and Jaber [184] developed an inventory model which accounted for imperfect quality items using the EPQ/EOQ formulae. They assumed defective items are sold as a single batch at the end of the total screening process. Moreover, Porteus [166] assumed that the probability of a shift from the 'in-control' state to the 'out-of-control' state has a given value for each production item. Rosenblatt and Lee [181] proposed an EPQ model that deals with imperfect quality. They assumed that at some random point in time the process might shift from an in-control to an out-of-control state, and a fixed percentage of defective items are produced. Cheng [32] examined an economic order quantity model with demand-dependent unit production cost and imperfect production processes. Several researchers have applied fuzzy or rough sets theory to solve production inventory problems. Recently, Chiu et al. [40], Jaber [100], Lin et al. [138], Chung and Hou [42], Lee [129], Lo et al. [145], Das et al. [51] have been carried out to address the issues of imperfect production and reduction of its corresponding quality costs. From the previous researchers, we can find that some papers discussed fuzzy costs, fuzzy demand etc. but till now no one has considered fuzzy defective rate, fuzzy time on which imperfect production to be started. Therefore, to study Fuzzy Economic Production Quantity (FEPQ) model considering the above lacunas with imperfect products which cannot be repaired is very important in vague environment.

Recently, it is noted that most of the consumers are increasingly demanding good-quality products. With a rise in consumer consciousness, providing defective products to customers not only increases service and replacement costs, but it may also cause inestimable damage to the credibility of the company. Regarding these imperfect systems, an understanding of the relationship among production, inventory, inspection, and maintenance can assist a manager to perform operation control and quality assurance in a more effective manner. In actual production processes, the process begins in a controlled state, but it may change to an out-of-control state as production proceeds, and some non-conforming items may appear. The conditions of a production system are tracked through inspection to ensure that consumers do not receive defective products. The purpose of the inspection is to determine the state of the production system and of product quality. The quality costs must be balanced with inspection costs when deciding on the frequency of the inspections to be performed. Therefore, the schedule of the inspections is essential. Wang and Sheu [210] generalized the model of Porteus [166] introducing a product inspection policy. Wang [214] extended Kim and Hong's [118] work considering a product inspection policy only at the end of the production run, instead of full inspections during a production run.

The classical inventory models developed for constant demand rate can be applied to both manufacturing and sales environment. In the case of certain consumer products, the consumption rate may be influenced by the stock levels. This phenomenon induces some researchers to consider stock dependent demand rate. The traditional literature dealing with inventory model usually assumed the market demand to be constant, stock dependent, time dependent and stochastic etc. in many real situations. However, it is very difficult to estimate the probability distribution of market demand due to the lack of historical data. Given the situations, they can only use linguistic terms, such as the market demand is about d_M, but not less than d_L and not larger than d_R, to describe the fuzzy market demand. In this case, the demand quantity is approximately specified based on the experience. Some papers have dealt with this case by applying fuzzy theory. Petrovic and Vujosevic [165] first proposed a newsboy-type problem with discrete fuzzy demand. Dutta et al. [67] presented a single-period inventory problem in an imprecise and uncertain mixed environment, and introduced demand as a fuzzy random variable. Zhen and Xiaoyu [242] considered the multi-product newsboy problem with fuzzy demands under budget constraint. Kao and Hsu [107] constructed a single-period inventory model with fuzzy demand. Li et al. [133] established two single-period inventory models in fuzzy environment, one of which assuming demand is stochastic while the holding and shortage costs are fuzzy, the other assuming the costs are deterministic but the demand is fuzzy. Lee and Yao [128] discussed the production inventory problems for fuzzy demand quantity. Taleizadeh et al. [201] extended an uncertain EOQ model for joint replenishment strategy with incremental discount policy and fuzzy rough demand.

Table 8.1: Summary of related literature for EPQ/EOQ models

Author(s)	Imperfect production	Fuzzy defective rate	Fuzzy time($\widetilde{\tau}$) which shifts in-control to out-control	Fuzzy demand rate	Formulation using FDE	Imprecise environment
Park [164]				√		√
Chang [24]	√					√
Chen and Hsieh [29]	√					√
Lee and Yao [128]	√			√		√
Lin and Yao [137]	√					√
Salameh and Jaber [184]	√					
Cheng [32]	√					
Wang [214]	√					√
Dutta et al. [67]	√			√		√
Zhen and Xiaoyu [242]	√					√
Kao and Hsu [107]	√			√		√
Taleizadeh et al. [201]				√		√
Present model	√	√	√	√	√	√

In this chapter, an imperfect production model has been considered with fuzzy defective rate. Here production starts with a constant production rate up to a variable time. At the beginning of a production process, the system is assumed to be in a controlled state and perfect items are produced. During production-run-time, the manufacturing process may shift to an 'out-of-control' state after certain time that follows a fuzzy number. In 'out-of-control' state, a percent of produced items is defective. The defective items are sold at a single lot after end of the production at a reduced cost. Then, two profit functions have been formulated and optimized through some numerical illustrations.

8.2 Notations and Assumptions

To formulate the proposed model, we have used the following notations and assumptions

8.2.1 Notations

The following notations are used throughout the chapter.

$q_1(t)$	:	On hand inventory at any time t in crisp environment for perfect quality item.
$\tilde{q}_1(t)$	:	On hand inventory at any time t in fuzzy environment for perfect quality item.
$q_2(t)$	:	On hand inventory at any time t in crisp environment for imperfect quality item.
$\tilde{q}_2(t)$	:	On hand inventory at any time t in fuzzy environment for imperfect quality item.
D	:	Demand rate of perfect quality items in crisp environment.
$\widetilde{D}$	:	Fuzzy demand rate of perfect quality items in fuzzy environment.
P	:	Production rate $(P > D)$.
$\widetilde{\beta}$	:	Fuzzy percentage of imperfect quality items per unit time with its α-cut $[\beta_\alpha^L, \beta_\alpha^R]$.
$\widetilde{\tau}$	:	Fuzzy time from which the production system shifts from 'in-control' state to the 'out-of-control' state with its α-cut $[\tau_\alpha^L, \tau_\alpha^R]$.
t_1	:	Duration of production run time.
c_p	:	Production cost per unit item.
c_{sr}	:	Screening cost per unit item.
h_c	:	Inventory holding cost per unit for perfect item in production center per unit time.
h'_c	:	Inventory holding cost per unit for imperfect item in production center per unit time.
s	:	Selling price of perfect item per unit.
s'	:	Selling price of imperfect item per unit.
T	:	Length of business period.
$\sim$	:	Symbol is used on the top of notations to represent fuzzy parameters.

8.2.2 Assumptions

The proposed model based on the following assumptions:

(i) The model is developed only for a single item manufacturer which produces the items at the rate of P.

(ii) Practically, it is seen that any production concern initially produces items to be perfect since all resources are fresh i.e., the production system initially is 'in-control-state'. After some times, the production system produces perfect items as well as imperfect items also since with the increase of time, the manufacturing system gradually breakdowns i.e., the system enters in out-of-control state. Therefore, the production system may be either in-control state or out-of-controled state. In this proposed model, it is assumed that the production system is initially being a controlled state upto time $\widetilde{\tau}$ which is consider as fuzzy after that the production system goes to 'out-of-control' state.

(iii) Normally the rate of defectiveness is not be a constant, because it may vary in a production system due to many factors such as production rate, machine component etc. So it should be taken a uncertain quantity. In this proposed model it is assumed that the defective rate ($\widetilde{\beta}$) has been considered as fuzzy.

(iv) It is a single item and single period (T) inventory model which is decision variable.

(v) Production rate (P) is constant.

(vi) In practical business world, sometimes it is seen that the demand of a retailer changes due to various factors according to his/her business policy. So in nature, it is vague and imprecise. For this reason, it has been considered a fuzzy demand of the retailer from the manufacturer. Now in fuzzy set theory [244], there are some standard fuzzy numbers such as triangular, trapezoidal, parabolic, general etc. which are considered for illustration.

(vii) All imperfect items which are produced in out-of-control state, are sold altogether at the end of production period at a reduced price.

8.3 Mathematical Formulation of the Proposed Model

In this paper, we consider an imperfect production inventory problem in which production starts at time $t = 0$ at the rate of P. Initially upto time $\widetilde{\tau}$ the system produces perfect item. Then it produces both good and defective items during $[\widetilde{\tau}, t_1]$. At time t_1 production stop. After that, from the stock demand of the customer is fulfilled upto time T. According to assumptions $\widetilde{\tau}$, $\widetilde{\beta}$ and $\widetilde{D}$ are taken as fuzzy numbers. Due to existence of fuzzy parameters, the inventory level at any time t is also fuzzy in nature. Since there exist productions of perfect items and imperfect items, hence here two separate inventories have been considered.

8.3.1 Formulation for Perfect Quality Items

In this case, the initially stock of the product of perfect items is zero then it starts production at the rate P. The system produces good quality items during $[0, \widetilde{\tau}]$ and it produces both good and defective items during $[\widetilde{\tau}, t_1]$, $\widetilde{\tau} \in (0, t_1)$. The total good items produced during $[0, t_1]$ are used to meet the demand of perfect item upto time T. Production of the cycle stops at time t_1.

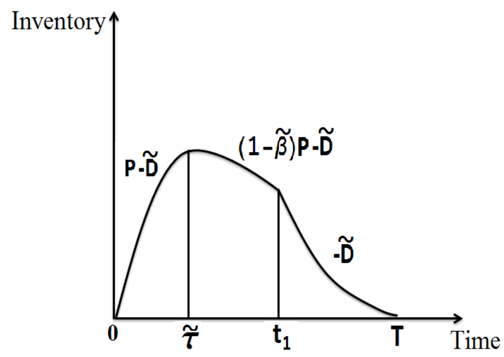

Figure 8.1: Pictorial representation of manufacturer's inventory of perfect quality item

Under such consideration the inventory level of perfect item $\widetilde{q}_1(t)$ at time t satisfies the following differential equations:

$$\frac{d\widetilde{q}_1(t)}{dt} = \begin{cases} P-\widetilde{D}, & 0 \leq t \leq \widetilde{\tau} \\ (P-\widetilde{D})-\widetilde{\beta}P, & \widetilde{\tau} \leq t \leq t_1 \\ -\widetilde{D}, & t_1 \leq t \leq T \end{cases} \tag{8.1}$$

subject to the conditions that, $\widetilde{q}_1[0]=0$, $\widetilde{q}_1[T]=0$.
To solve the fuzzy differential equations (8.1) first, we find out the solution $q_1(t)$ in crisp differential environment according to Chalco-Cano and Roman-Flores [19] is as follows

$$\frac{dq_1(t)}{dt} = \begin{cases} P-D, & 0 \leq t \leq \tau \\ (P-D)-\beta P, & \tau \leq t \leq t_1 \\ -D, & t_1 \leq t \leq T \end{cases} \tag{8.2}$$

subject to the conditions that, $q_1[0]=0$, $q_1[T]=0$.
The solution of the above differential equations are

$$q_1(t) = \begin{cases} (P-D)t, & 0 \leq t \leq \tau \\ (P-D)t+\beta P\tau-\beta Pt, & \tau \leq t \leq t_1 \\ D(T-t), & t_1 \leq t \leq T \end{cases} \tag{8.3}$$

Lemma 8.1. *Manufacturer's production rate (P), Demand rate (D), production time period (t_1) and business time period (T) must satisfy the condition*

$$(1-\beta)Pt_1+\beta P\tau = DT \tag{8.4}$$

Proof. From the continuity condition of $q_1(t)$ at $t = t_1$, the following is obtained

$$(P-D)t_1 + \beta P\tau - \beta P t_1 = -D(t_1 - T)$$

$$i.e., (1-\beta)Pt_1 + \beta P\tau = DT$$

Hence the proof.

□

As $q(t)$ is continuous for $t \geq 0$, the unique fuzzy solution of equation (8.1) according to Chalco-Cano and Roman-Flores [19] is given by

$$\widetilde{q}_1(t) = \begin{cases} (P-\widetilde{D})t, & 0 \leq t \leq \widetilde{\tau} \\ (P-\widetilde{D})t + \widetilde{\beta}P\widetilde{\tau} - \widetilde{\beta}Pt, & \widetilde{\tau} \leq t \leq t_1 \\ \widetilde{D}(T-t), & t_1 \leq t \leq T \end{cases} \tag{8.5}$$

with the condition $(1-\widetilde{\beta})Pt_1 + \widetilde{\beta}P\widetilde{\tau} = \widetilde{D}T$.

Theorem 8.1. *If $\widetilde{\tau}$ be a fuzzy number then (i) $t \leq \tau_\alpha^L$ for all α, provided that $t \leq \widetilde{\tau}$.*
ii) $t \geq \tau_\alpha^R$ for all α, provided that $t \geq \widetilde{\tau}$.

Proof. (i) Let $\widetilde{t}$ and $\widetilde{\tau}$ be two fuzzy numbers and $\widetilde{t} \leq \widetilde{\tau}$. Then the α-cuts of t and τ must satisfy $[t_\alpha^L, t_\alpha^R] \leq [\tau_\alpha^L, \tau_\alpha^R]$, which implies that $t_\alpha^L \leq \tau_\alpha^L$ and $t_\alpha^R \leq \tau_\alpha^R$.
But since t is precise so $t = t_\alpha^L = t_\alpha^R$. Then $t \leq \tau_\alpha^L$. Hence the proof.
(ii) Let $\widetilde{t}$ and $\widetilde{\tau}$ be two fuzzy numbers and $\widetilde{t} \geq \widetilde{\tau}$. Then $[t_\alpha^L, t_\alpha^R] \geq [\tau_\alpha^L, \tau_\alpha^R]$, which implies that $t_\alpha^L \geq \tau_\alpha^L$ and $t_\alpha^R \geq \tau_\alpha^R$.
But since t is precise so $t = t_\alpha^L = t_\alpha^R$. Then $t \geq \tau_\alpha^R$. Hence the proof. □

Hence, α-cut of above equation (8.5) is given by

$$\widetilde{q}_1(t)[\alpha] = [q_1^L(\alpha,t), q_1^R(\alpha,t)], \tag{8.6}$$

where

$$q_1^L(\alpha,t) = \begin{cases} (P-D_\alpha^R)t, & 0 \leq t \leq \tau_\alpha^L \\ (P-D_\alpha^R)t + P\beta_\alpha^L\tau_\alpha^L - P\beta_\alpha^R t, & \tau_\alpha^R \leq t \leq t_1 \\ D_\alpha^L(T-t), & t_1 \leq t \leq T \end{cases}$$

and

$$q_1^R(\alpha,t) = \begin{cases} (P-D_\alpha^L)t, & 0 \leq t \leq \tau_\alpha^R \\ (P-D_\alpha^L)t + P\beta_\alpha^R\tau_\alpha^R - P\beta_\alpha^L t, & \tau_\alpha^L \leq t \leq t_1 \\ D_\alpha^R(T-t), & t_1 \leq t \leq T \end{cases}$$

Lemma 8.2. *In fuzzy environment manufacturer's production time period (t_1) and business time period (T) must satisfy the condition either*
i) $\Big\{(1-\beta_\alpha^R)P + D_\alpha^L - D_\alpha^R\Big\}t_1 + P\beta_\alpha^L\tau_\alpha^L = D_\alpha^L T$
or ii) $\Big\{(1-\beta_\alpha^L)P + D_\alpha^R - D_\alpha^L\Big\}t_1 + P\beta_\alpha^R\tau_\alpha^R = D_\alpha^R T.$

Proof. From the continuity conditions of $q_1^L(\alpha,t)$ and $q_1^R(\alpha,t)$ at $t = t_1$, the followings are obtained respectively
i) $(P - D_\alpha^R)t_1 + P\beta_\alpha^L\tau_\alpha^L - P\beta_\alpha^R t_1 = D_\alpha^L(T - t_1)$
i.e., $\Big\{(1-\beta_\alpha^R)P + D_\alpha^L - D_\alpha^R\Big\}t_1 + P\beta_\alpha^L\tau_\alpha^L = D_\alpha^L T$
or, ii) $(P - D_\alpha^L)t_1 + P\beta_\alpha^R\tau_\alpha^R - P\beta_\alpha^L t_1 = D_\alpha^R(T - t_1)$
i.e., $\Big\{(1-\beta_\alpha^L)P + D_\alpha^R - D_\alpha^L\Big\}t_1 + P\beta_\alpha^R\tau_\alpha^R = D_\alpha^R T$
Now from Lemma 8.1, it is seen that there exist variabilities of t_1 and T for crisp value of β, τ and D. But in fuzzy environment, two relations are obtained. If t_1 and T satisfy both these two relations simultaneously, then there will be loss of variability of t_1 and T. Therefore to maintain variabilities of t_1 and T, they must satisfy either $\Big\{(1-\beta_\alpha^R)P + D_\alpha^L - D_\alpha^R\Big\}t_1 + P\beta_\alpha^L\tau_\alpha^L = D_\alpha^L T$ or $\Big\{(1-\beta_\alpha^L)P + D_\alpha^R - D_\alpha^L\Big\}t_1 + P\beta_\alpha^R\tau_\alpha^R = D_\alpha^R T$. Hence the proof. □

8.3.2 Formulation for Imperfect Quality Items

At the end of the screening process, the imperfect quality items are sold as a single lot. The inventory level $\widetilde{q}_2(t)$ at time t satisfies the following differential equation:

$$\frac{d\widetilde{q}_2(t)}{dt} = \widetilde{\beta}P, \ \ \widetilde{\tau} \le t \le t_1 \tag{8.7}$$

subject to the condition that, $\widetilde{q}_2[\tau] = 0$.
According to Chalco-Cano and Roman-Flores [19], first find out the solution $q_2(t)$ of the crisp differential equation

$$\frac{dq_2(t)}{dt} = \beta P, \ \ \tau \le t \le t_1 \tag{8.8}$$

subject to the condition that, $q_2[\widetilde{\tau}] = 0$.
The solution of the above differential equation is given by

$$q_2(t) = \beta P(t - \tau), \ \ \tau \le t \le t_1 \tag{8.9}$$

As $q(t)$ is continuous for each $t \geq 0$, the unique fuzzy solution (according to Chalco-Cano and Roman-Flores [19] of equation (8.7) is given by

$$\widetilde{q}_2(t) = \widetilde{\beta} P(t - \widetilde{\tau}), \;\; \widetilde{\tau} \leq t \leq t_1 \tag{8.10}$$

Hence, α-cut of above equation (8.10) is given by

$$\widetilde{q}_2(t)[\alpha] = [q_2^L(\alpha, t), q_2^R(\alpha, t)], \tag{8.11}$$

where

$$q_2^L(\alpha, t) = P\beta_\alpha^L(t - \tau_\alpha^R), \;\; \tau_\alpha^R \leq t \leq t_1, \tag{8.12}$$
$$q_2^R(\alpha, t) = P\beta_\alpha^R(t - \tau_\alpha^L), \;\; \tau_\alpha^L \leq t \leq t_1 \tag{8.13}$$

8.3.3 The Profit Function of the Proposed Model

The total production cost (PC) in the production system during the cycle $(0, T)$ is given by

$$PC = c_p \int_0^{t_1} P\,dt = c_p P t_1$$

Total screening cost (SC) in the production system during the cycle $(0, T)$ is given by

$$SC = c_{sr} \int_0^{t_1} P\,dt = c_{sr} P t_1$$

The total set up cost in the production system during the cycle $(0, T)$ =A_m

Theorem 8.2. *Let $\tilde{f}(x)$ be a bounded and closed-fuzzy-valued function defined on the closed fuzzy real number system $(\Re_{\boldsymbol{R}}/\sim)_{\boldsymbol{R}}$ and $\tilde{f}(x)$ be induced by $\tilde{f}(x)$. Suppose that $\tilde{b} \succeq \tilde{a}$ and there exists a fuzzy number $\tilde{\xi}$ such that $\tilde{a} \preceq \tilde{\xi} \preceq \tilde{b}$.*
(i) If $\tilde{f}(x)$ is non-negative as well as $f_\alpha^L(x)$ and $f_\alpha^R(x)$ be Riemann-integrable α-cut of $\tilde{f}(x)$ on $[a_\alpha^R, b_\alpha^L]$ and $[a_\alpha^L, b_\alpha^R]$ respectively for all α then

$$\Big(\int_{\tilde{a}}^{\tilde{b}} \tilde{f}(x)\,dx\Big)[\alpha] = \Big[\int_{a_\alpha^R}^{\xi_\alpha^L} f_\alpha^L(x)\,dx + \int_{\xi_\alpha^R}^{b_\alpha^L} f_\alpha^L(x)\,dx, \int_{a_\alpha^L}^{\xi_\alpha^R} f_\alpha^R(x)\,dx + \int_{\xi_\alpha^L}^{b_\alpha^R} f_\alpha^R(x)\,dx\Big]$$

Proof. If there exists a fuzzy number $\tilde{\xi}$ such that $\tilde{a} \preceq \tilde{\xi} \preceq \tilde{b}$ and $\widetilde{f}(x)$ is Riemann-integrable on $[\tilde{a}, \tilde{b}]$, for all α then

$$\Big(\int_{\tilde{a}}^{\tilde{b}} \tilde{f}(x)\,dx\Big)[\alpha] = \Big(\int_{\tilde{a}}^{\tilde{\xi}} \tilde{f}(x)\,dx + \int_{\tilde{\xi}}^{\tilde{b}} \tilde{f}(x)\,dx,\Big)[\alpha] \tag{8.14}$$

Now, $\Big(\int_{\tilde{a}}^{\tilde{\xi}}\tilde{f}(x)\,dx\Big)[\alpha]=\Big[\int_{a_\alpha^R}^{\xi_\alpha^L}f_\alpha^L(x)\,dx,\int_{a_\alpha^L}^{\xi_\alpha^R}f_\alpha^R(x)\,dx\Big]$
and $\Big(\int_{\tilde{\xi}}^{\tilde{b}}\tilde{f}(x)\,dx\Big)[\alpha]=\Big[\int_{\xi_\alpha^R}^{b_\alpha^L}f_\alpha^L(x)\,dx,\int_{\xi_\alpha^L}^{b_\alpha^R}f_\alpha^R(x)\,dx\Big]$
Therefore,

$$\begin{aligned}\Big(\int_{\tilde{a}}^{\tilde{\xi}}\tilde{f}(x)\,dx\Big)[\alpha]+\Big(\int_{\tilde{\xi}}^{\tilde{b}}\tilde{f}(x)\,dx\Big)[\alpha] &= \Big[\int_{a_\alpha^R}^{\xi_\alpha^L}f_\alpha^L(x)\,dx,\int_{a_\alpha^L}^{\xi_\alpha^R}f_\alpha^R(x)\,dx\Big]\\ &\quad+\Big[\int_{\xi_\alpha^R}^{b_\alpha^L}f_\alpha^L(x)\,dx,\int_{\xi_\alpha^L}^{b_\alpha^R}f_\alpha^R(x)\,dx\Big]\end{aligned}$$

$$\begin{aligned}\Rightarrow\ \Big(\int_{\tilde{a}}^{\tilde{\xi}}\tilde{f}(x)\,dx+\int_{\tilde{\xi}}^{\tilde{b}}\tilde{f}(x)\,dx\Big)[\alpha] &= \Big[\int_{a_\alpha^R}^{\xi_\alpha^L}f_\alpha^L(x)\,dx+\int_{\xi_\alpha^R}^{b_\alpha^L}f_\alpha^L(x)\,dx,\int_{a_\alpha^L}^{\xi_\alpha^R}f_\alpha^R(x)\,dx\\ &\quad+\int_{\xi_\alpha^L}^{b_\alpha^R}f_\alpha^R(x)\,dx\Big]\end{aligned}$$

Hence the proof. □

Theorem 8.3. *Let $\tilde{f}(\tilde{x})$ be a bounded and closed-fuzzy-valued function defined on the closed fuzzy real number system $(\Re_{\boldsymbol{R}}/\sim)_{\boldsymbol{R}}$ and $\tilde{f}(x)$ be induced by $\tilde{f}(\tilde{x})$. Suppose that $a\succeq b$ and there exists a fuzzy number $\tilde{\xi}$ such that $a\preceq\tilde{\xi}\preceq b$.*
(i) If $\tilde{f}(x)$ is non negative and $f_\alpha^L(x)$ and $f_\alpha^R(x)$ are Riemann-integrable α-cut of $\widetilde{f}(x)$ on $[a,\xi_\alpha^L]$ and $[\xi_\alpha^R,b]$, respectively, for all α then

$$\Big(\int_a^b\tilde{f}(x)\,dx\Big)[\alpha]=\Big[\int_a^{\xi_\alpha^L}f_\alpha^L(x)\,dx+\int_{\xi_\alpha^R}^{b}f_\alpha^L(x)\,dx,\int_a^{\xi_\alpha^R}f_\alpha^R(x)\,dx+\int_{\xi_\alpha^L}^{b}f_\alpha^R(x)\,dx\Big]\quad(8.15)$$

Proof. Similar proof of above theorem. □

Now α-cut of the total holding cost (HC) in the production system during the cycle $(0,T)$ is given by

$$\begin{aligned}HC[\alpha] &= \Big(h_m\int_0^T\widetilde{q}_1(t)\,dt+h_m'\int_{\widetilde{\tau}}^{t_1}\widetilde{q}_2(t)\,dt\Big)[\alpha]\\ &= \Big[h_m\int_0^T q_1^L(t,\alpha)\,dt+h_m'\int_{\tau^R}^{t_1}q_2^L(t,\alpha)\,dt,h_m\int_0^T q_1^L(t,\alpha)\,dt+h_m'\int_{\tau^L}^{t_1}q_2^R(t,\alpha)\,dt\Big]\\ &= \Big[HC_\alpha^L,HC_\alpha^R\Big](say),\end{aligned}$$

where

$$\begin{aligned}
HC^L_\alpha &= h_m\int_0^T q_1^L(t,\alpha)\,dt + h'_m\int_{\tau^R}^{t_1} q_2^L(t,\alpha)\,dt\\
&= h_m\Big[\int_0^{\tau^L} q_1^L(t,\alpha)\,dt + \int_{\tau^R}^{t_1} q_1^L(t,\alpha)\,dt + \int_{t_1}^{T} q_1^L(t,\alpha)\,dt\Big] + h'_m\int_{\tau^R}^{t_1} q_2^L(t,\alpha)\,dt\\
&= \frac{h_m}{2}\Big[(P-D^R_\alpha)\{t_1^2+(\tau^L_\alpha)^2-(\tau^R_\alpha)^2\} + 2P\beta^L_\alpha\tau^L_\alpha(t_1-\tau^R_\alpha) - P\beta^R_\alpha\{t_1^2-(\tau^R_\alpha)^2\}\\
&\quad +D^L_\alpha(T-t_1)^2\Big] + \frac{h'_m}{2}P(t_1-\tau^R_\alpha)^2
\end{aligned}$$

$$\begin{aligned}
HC^R_\alpha &= h_m\int_0^T q_1^R(t,\alpha)\,dt + h'_m\int_{\tau^L_\alpha}^{t_1} q_2^R(t,\alpha)\,dt\\
&= h_m\Big[\int_0^{\tau^R_\alpha} q_1^R(t,\alpha)\,dt + \int_{\tau^L_\alpha}^{t_1} q_1^R(t,\alpha)\,dt + \int_{t_1}^{T} q_1^R(t,\alpha)\,dt\Big] + h'_m\int_{\tau^L_\alpha}^{t_1} q_2^R(t,\alpha)\,dt\\
&= \frac{h_m}{2}\Big[(P-D^L_\alpha)\{t_1^2+(\tau^R_\alpha)^2-(\tau^L_\alpha)^2\} + 2P\beta^R_\alpha\tau^R_\alpha(t_1-\tau^L_\alpha) - P\beta^L_\alpha\{t_1^2-(\tau^L_\alpha)^2\}\\
&\quad +D^R_\alpha(T-t_1)^2\Big] + \frac{h'_m}{2}P(t_1-\tau^L_\alpha)^2
\end{aligned}$$

The α-cut of total revenue (SR) in the production system during the cycle $(0,T)$ is given by

$$\begin{aligned}
SR[\alpha] &= \Big(s\int_0^T \widetilde{D}\,dt + s'\widetilde{\beta}P(t_1-\widetilde{\tau})\Big)[\alpha]\\
&= \Big[s\int_0^T D^L_\alpha\,dt + s'\beta^L_\alpha P(t_1-\tau^R_\alpha),\, s\int_0^T D^R_\alpha\,dt + s'\beta^R_\alpha P(t_1-\tau^L_\alpha)\Big]\\
&= \Big[sD^L_\alpha T + s'\beta^L_\alpha P(t_1-\tau^R_\alpha),\, sD^R_\alpha T + s'\beta^R_\alpha P(t_1-\tau^L_\alpha)\Big]\\
&= \Big[SR^L_\alpha, SR^R_\alpha\Big](say),
\end{aligned}$$

where

$$SR^L_\alpha = s\int_0^T D^L_\alpha\,dt + s'P\beta^L_\alpha(t_1-\tau^R_\alpha) = sD^L_\alpha T + s'P\beta^L_\alpha(t_1-\tau^R_\alpha),$$

$$SR^R_\alpha = s\int_0^T D^R_\alpha\,dt + s'P\beta^R_\alpha(t_1-\tau^L_\alpha) = sD^R_\alpha T + s'P\beta^R_\alpha(t_1-\tau^L_\alpha)$$

The α-cut of the total Profit (TP) in the production system during the cycle $(0,T)$ is given by

$$\widetilde{TP}[\alpha] = [TP^L_\alpha(t_1,T), TP^R_\alpha(t_1,T)](say),$$

where

$$TP^L_\alpha(t_1,T) = \frac{1}{T}\Big[SR^L_\alpha - HC^R_\alpha - PC - SC - A_m\Big],$$

$$TP^R_\alpha(t_1,T) = \frac{1}{T}\Big[SR^R_\alpha - HC^L_\alpha - PC - SC - A_m\Big]$$

Finally, the model becomes:

$$\begin{aligned}
&\text{Max } TP_\alpha^L(t_1,T)=\frac{1}{T}\Big[SR_\alpha^L-HC_\alpha^R-PC-SC-A_m\Big],\\
&\text{Max } TP_\alpha^R(t_1,T)=\frac{1}{T}\Big[SR_\alpha^R-HC_\alpha^L-PC-SC-A_m\Big]\\
&\text{such that: } 0<\tau_\alpha^R<t_1,\ \ 0<t_1<T,\\
&\text{and}\Big\{(1-\beta_\alpha^R)P+D_\alpha^L-D_\alpha^R\Big\}t_1+P\beta_\alpha^L\tau_\alpha^L=D_\alpha^LT\\
&\text{or}\Big\{(1-\beta_\alpha^L)P+D_\alpha^R-D_\alpha^L\Big\}t_1+P\beta_\alpha^R\tau_\alpha^R=D_\alpha^RT.
\end{aligned}$$

8.4 Solution Procedure

To get the optimum value of the production run time (t_1), business period (T) and average profit in the proposed model, the following steps are necessary.

Step-1: Input the suitable values of crisp and fuzzy parameters of $TP_\alpha^L(t_1,T)$ and $TP_\alpha^R(t_1,T)$.

Step-2: Compute the left α-cut (τ_α^L) and right α-cut (τ_α^R) of fuzzy parameter $\widetilde{\tau}$ as follows:
(i) If $\widetilde{\tau}$ be TFN such as $\widetilde{\tau}=(\tau_0-\Delta_1,\tau_0,\tau_0+\Delta_2)$ then $\tau_L^\alpha=(\tau_0-\Delta_1)+\alpha\Delta_1$ and $\tau_R^\alpha=(\tau_0+\Delta_2)-\alpha\Delta_2$.
(ii) If $\widetilde{\tau}$ be TrFN such as $\widetilde{\tau}=(\tau_0-\Delta_1,\tau_0-\Delta_2,\tau_0+\Delta_3,\tau_0+\Delta_4)$ then $\tau_L^\alpha=(\tau_0-\Delta_1)+\alpha(\Delta_1-\Delta_2)$ and $\tau_R^\alpha=(\tau_0+\Delta_4)-\alpha(\Delta_4-\Delta_3)$.
(iii) If $\widetilde{\tau}$ be PFN such as $\widetilde{\tau}=(\tau_0-\Delta_1,\tau_0,\tau_0+\Delta_2)$ respectively then $\tau_L^\alpha=\tau_0-\sqrt{\alpha}\Delta_1$ and $\tau_R^\alpha=\tau_0+\sqrt{\alpha}\Delta_2$.
(iv) If $\widetilde{\tau}$ be GFN such that $\widetilde{\tau}=(\tau_0-\Delta_1,\tau_0-\Delta_2,\tau_0+\Delta_3,\tau_0+\Delta_4)$ then $\tau_L^\alpha=(\tau_0-\Delta_1)+\sqrt{\alpha}(\Delta_1-\Delta_2)$ and $\tau_R^\alpha=(\tau_0+\Delta_4)-\sqrt{\alpha}(\Delta_4-\Delta_3)$.
Similarly, compute left and right α-cuts of other two fuzzy parameters $\widetilde{\beta}$ and $\widetilde{D}$ for TFN, TrFN, PFN and GFN.

Step-3: Maximize the profit $TP_\alpha^L(t_1,T)$ and obtain the optimal values of $t_1,T,TP_\alpha^R(t_1,T)$ and $TP_\alpha^{L*}(t_1,T)$ for different values of α using the standard LINGO software.

Step-4: Maximize the profit $TP_\alpha^R(t_1,T)$ and obtain the optimal values of $t_1,T,TP_\alpha^L(t_1,T)$ and $TP_\alpha^{R*}(t_1,T)$ for different values of α using the standard LINGO software.

Step-5: Maximize the both $TP_\alpha^L(t_1,T)$ and TP_α^R by Fuzzy Programming Technique (FPT) and obtain the optimal values of t_1, T and average profit for different values of α using the standard LINGO software as follows:

Step-6: From the results of step-3 and step-4, the following pay off matrix can be constructed:

$$\begin{pmatrix} TP_\alpha^{L*}(t_1,T) & TP_\alpha^R(t_1,T)\\ TP_\alpha^L(t_1,T) & TP_\alpha^{R*}(t_1,T)\end{pmatrix}$$

Step-7: From this pay off matrix, two values U_j and L_j are defined such that they are the upper and lower bounds of the j-th objective for each j=1, 2 respectively. Here, L_j = higher acceptable level of achievement, U_j = aspired level of achievement for maximization, which are computed as follows:

$$U_1 = \max\{TP_\alpha^{L*}(t_1, T)\},\ U_2 = \max\{TP_\alpha^{R}(t_1, T)\}$$
$$L_1 = \min\{TP_\alpha^{L}(t_1, T)\},\ L_2 = \min\{TP_\alpha^{R*}(t_1, T)\}$$

Step-8: Then the membership functions $\mu_1(TP_\alpha^L(t_1, T))$ and $\mu_2(TP_\alpha^R(t_1, T))$ corresponding to the objective functions of $TP_\alpha^L(t_1, T)$ and $TP_\alpha^R(t_1, T)$ are constructed linearly as follows:

$$\mu_1(TP_\alpha^L(t_1, T)) = \begin{cases} 0, & \text{if } TP_\alpha^L(t_1, T) \leq L_1 \\ \frac{TP_\alpha^L(t_1,T)-L_1}{U_1-L_1}, & \text{if } L_1 \leq TP_\alpha^L(t_1, T) \leq U_1 \\ 1, & \text{if } TP_\alpha^L(t_1, T) \geq U_1 \end{cases}$$

$$\mu_2(TP_\alpha^R(t_1, T)) = \begin{cases} 0, & \text{if } TP_\alpha^R(t_1, T) \leq L_2 \\ \frac{TP_\alpha^R(t_1,T)-L_2}{U_2-L_2}, & \text{if } L_2 \leq TP_\alpha^R(t_1, T) \leq U_2 \\ 1, & \text{if } TP_\alpha^R(t_1, T) \geq U_2 \end{cases}$$

Step-9: Finally, according to the Zimmermann [244] method, the multi-objective programming problem is reduced to the following single objective programming problem:

$$\begin{aligned}
&\text{Max } \lambda \\
&\text{such that} \\
&\mu_1(TP_\alpha^L(t_1, T)) \geq \lambda, \\
&\mu_2(TP_\alpha^R(t_1, T)) \geq \lambda, \\
&0 < \tau_\alpha^R < t_1,\ \ 0 < t_1 < T,\ \lambda \in [0, 1], \\
&\text{and} \left\{(1-\beta_\alpha^R)P + D_\alpha^L - D_\alpha^R\right\}t_1 + P\beta_\alpha^L\tau_\alpha^L = D_\alpha^L T \\
&\text{or} \left\{(1-\beta_\alpha^L)P + D_\alpha^R - D_\alpha^L\right\}t_1 + P\beta_\alpha^R\tau_\alpha^R = D_\alpha^R T.
\end{aligned}$$

8.5 Numerical Illustrations

To illustrate numerically the proposed model, according to Lemma 8.2 we have two relations

$$\left\{(1-\beta_\alpha^R)P + D_\alpha^L - D_\alpha^R\right\}t_1 + P\beta_\alpha^L\tau_\alpha^L = D_\alpha^L T$$

and

$$\left\{(1-\beta_\alpha^L)P + D_\alpha^R - D_\alpha^L\right\}t_1 + P\beta_\alpha^R\tau_\alpha^R = D_\alpha^R T$$

, one of which is taken to compute t_1 and T optimizing objective functions. At first considering the relation

$$\Big\{(1-\beta_\alpha^R)P + D_\alpha^L - D_\alpha^R\Big\}t_1 + P\beta_\alpha^L\tau_\alpha^L = D_\alpha^L T$$

the proposed model is optimized for the following examples and then the other relation

$$\Big\{(1-\beta_\alpha^L)P + D_\alpha^R - D_\alpha^L\Big\}t_1 + P\beta_\alpha^R\tau_\alpha^R = D_\alpha^R T$$

has been considered to get the optimum solution but in this case latter it has been shown that this relation is not acceptable due to some in-feasibility of the solution.

Example 8.1. *The following parametric values are used to illustrate the model:*

$C_p = \$30$, $C_{sr} = \$2$, $A_m = \$5200$, $s = \$59$, $s' = \$35$, $h_m = \$1.00$, $h'_m = \$0.50$, $P = 3380$. Here, fuzzy parameters are considered as triangular fuzzy number (TFN) and their different values are given below. $\widetilde{\tau} = (\tau_0 - \Delta_1, \tau_0, \tau_0 + \Delta_2) = (2, 3, 5)$, $\Delta_1 = 1$, $\Delta_2 = 0.5$, $\widetilde{\beta} = (\beta_0 - \sigma_1, \beta_0, \beta_0 + \sigma_2) = (0.07, 0.10, 0.15)$, $\sigma_1 = 0.03$, $\sigma_2 = 0.05$, $\widetilde{D} = (D_0 - \rho_1, D_0, D_0 + \rho_2) = (2500, 2559, 2631)$, $\rho_1 = 59$, $\rho_2 = 72$. This example is solved using LINGO-12.0. Table 8.2, 8.3 represent the optimum results when TP_α^L, TP_α^R are maximized separately and Table 8.4 represent the optimum results when TP_α^L and TP_α^R are maximized simultaneously.
For the above parametric values, the optimum business time period (T^*), optimum production time period (t_1^*), and profit interval ($[TP_\alpha^{L*}, TP_\alpha^{R*}]$) are obtained. Obtained results for different values of α are presented in Table 8.2, 8.3 and 8.4.

Table 8.2: Optimum results of Example 8.1 for maximizing TP_α^L

α	Production period (t_1^*)	Business period (T^*)	TP_α^{L*} (Max)	TP_α^R
0.00	8.445339	9.452128	47447.09	69007.36
0.25	7.884611	9.066641	52386.18	68234.01
0.50	7.100508	8.402024	57183.69	67540.85
0.75	5.982998	7.315060	61904.18	66988.16
0.99	4.345648	5.555799	66499.80	66701.16

Table 8.3: Optimum results of Example 8.1 for maximizing TP_α^R

α	Production period (t_1^*)	Business period (T^*)	TP_α^L	TP_α^{R*} (Max)
0.00	10.71312	11.93943	47248.61	69111.75
0.25	9.064714	10.38858	52327.64	68270.08
0.50	7.526072	8.888575	57175.25	67546.87
0.75	5.976583	7.307580	61904.18	66988.16
0.99	4.332328	5.539977	66499.78	66701.17

Table 8.4: Optimum results of Example 8.1 for maximizing both TP_α^L and TP_α^R

α	Production period (t_1^*)	Business period(T^*)	TP_α^{L*} (Max)	TP_α^{R*} (Max)	Average profit
0.00	9.513103	10.62325	47397.64	69085.74	58241.69
0.25	8.454594	9.705132	52371.56	68261.07	60316.32
0.50	7.310260	8.641835	57181.58	67545.36	62363.47
0.75	5.982997	7.315060	61904.18	66988.16	64446.17
0.99	4.339744	5.548787	66499.79	66701.17	66600.48

As expected, the left and right optimum profits increases and decreases respectively with the increase of α. At $\alpha = 0.99$, the above profit values are almost same.

Example 8.2. *Here, fuzzy parameters are considered as trapezoidal fuzzy number (TrFN) and their different values are taken as.*

$\widetilde{\tau} = (\tau_0 - \Delta_1, \tau_0 - \Delta_2, \tau_0 + \Delta_3, \tau_0 + \Delta_4) = (2, 2.5, 3.5, 5)$, $\Delta_1 = 1.25$, $\Delta_2 = 0.75$, $\Delta_3 = 0.25$, $\Delta_4 = 1.75$ $\widetilde{\beta} = (\beta_0 - \sigma_1, \beta_0 - \sigma_2, \beta_0 + \sigma_3, \beta_0 + \sigma_4) = (0.07, 0.10, 0.13, 0.15)$, $\sigma_1 = 0.05$, $\sigma_2 = 0.02$, $\sigma_3 = 0.01$, $\sigma_4 = 0.03$, $\widetilde{D} = (d_0 - \rho_1, d_0 - \rho_2, d_0 + \rho_3, d_0 + \rho_4) = (2500, 2549, 2571, 2631)$, $\rho_1 = 65$, $\rho_2 = 16$, $\rho_3 = 6$, $\rho_4 = 66$. Other parametric values are same as in Example 8.1. This example is solved using LINGO-12.0. Table 8.5, 8.6 represent the optimum results when TP_α^L, TP_α^R are maximized separately and Table 8.7 represent the optimum results when TP_α^L and TP_α^R are maximized simultaneously.

Table 8.5: Optimum results of Example 8.2 for maximizing TP_α^L

α	Production period (t_1^*)	Business period (T^*)	TP_α^{L*} (Max)	TP_α^{R*}
0.00	8.445339	9.452128	47447.09	69007.36
0.25	8.211077	9.327867	50801.53	68435.14
0.50	7.869975	9.079349	54132.44	67864.09
0.75	7.394903	8.672050	57450.46	67305.73
0.99	6.772804	8.081556	60639.97	66795.06

Table 8.6: Optimum results of Example 8.2 for maximizing TP_α^R

α	Production period (t_1^*)	Business period (T^*)	TP_α^L	TP_α^{R*} (Max)
0.00	10.71312	11.93943	47248.61	69111.75
0.25	9.666378	10.94183	50716.17	68484.49
0.50	8.696047	10.00548	54103.70	67882.37
0.75	7.772620	9.100049	57444.15	67310.16
0.99	6.907287	8.235475	60639.13	66795.72

Table 8.7: Optimum results of Example 8.2 for maximizing both TP_α^L and TP_α^R

α	Production period (t_1^*)	Business period(T^*)	TP_α^{L*} (Max)	TP_α^{R*} (Max)	Average profit
0.00	9.513103	10.62325	47397.64	69085.74	58241.69
0.25	8.909673	10.10263	50780.23	68472.17	59626.20
0.50	8.273025	9.531221	54125.26	67877.80	61001.53
0.75	7.581600	8.883601	57448.88	67309.05	62378.97
0.99	6.840087	8.158563	60639.76	66795.55	63717.66

For the above parametric values, the optimum business time period (T^*), optimum production time period (t_1^*), and profit interval ($[TP_\alpha^{L*}, TP_\alpha^{R*}]$) are obtained. Obtained results for different values of α are presented in Table 8.5, 8.6 and 8.7. As expected, the left and right optimum profits increases and decreases respectively with the increase of α.

Example 8.3. *In this example fuzzy parameters are considered as parabolic fuzzy number (PFN) type and their different values are:*

$\widetilde{\tau} = (\tau_0 - \Delta_1, \tau_0, \tau_0 + \Delta_2) = (2, 3, 5)$, $\Delta_1 = 1$, $\Delta_2 = 2$, $\widetilde{\beta} = (\beta_0 - \sigma_1, \beta_0, \beta_0 + \sigma_2) = (0.07, 0.10, 0.15)$, $\sigma_1 = 0.03$, $\sigma_2 = 0.05$, $\widetilde{D} = (D_0 - \rho_1, D_0, D_0 + \rho_2) = (2500, 2559, 2631)$, $\rho_1 = 59$, $\rho_2 = 72$. Other parametric values are same as in Example 8.1. This example is solved using LINGO-12.0. Table8.8 and 8.9 represent the optimum results when TP_α^L, TP_α^R are maximized separately and Table 8.10 represent the optimum results when TP_α^L, TP_α^R are maximized simultaneously.

Table 8.8: Optimum results of Example 8.3 for maximizing TP_α^L

α	Production period (t_1^*)	Business period (T^*)	TP_α^{L*} (Max)	TP_α^R
0.00	8.445339	9.452128	47447.09	69007.36
0.25	8.168350	9.274599	50114.31	68585.50
0.50	7.768098	8.975275	53218.01	68108.13
0.75	7.100508	8.402024	57183.69	67540.85
0.99	5.055331	6.336396	64752.49	66769.40

Table 8.9: Optimum results of Example 8.3 for maximizing TP_α^R

α	Production period (t_1^*)	Business period (T^*)	TP_α^L (Max)	TP_α^{R*}
0.00	10.71312	11.93943	47248.61	69111.75
0.25	9.808184	11.09378	50005.78	68647.74
0.50	8.796376	10.13124	53172.85	68136.67
0.75	7.526072	8.888575	57175.25	67546.87
0.99	4.980989	6.248691	64752.15	66769.71

Table 8.10: Optimum results of Example 8.3 for maximizing both TP_α^L and TP_α^R

α	Production period (t_1^*)	Business period(T^*)	TP_α^{L*} (Max)	TP_α^{R*} (Max)	Average profit
0.00	9.513103	10.62325	47397.64	69085.74	58241.69
0.25	8.951570	10.14348	50087.23	68632.21	59359.72
0.50	8.266637	9.535721	53206.74	68129.54	60668.14
0.75	7.310260	8.641835	57181.58	67545.36	62363.47
0.99	5.018012	6.292368	64752.40	66769.63	65761.02

For the above parametric values, the optimum business time period (T^*), optimum production time period (t_1^*), and profit interval ($[TP_\alpha^L, TP_\alpha^R]$) are obtained. Obtained results for different values of α are presented in Table 8.8, 8.9 and 8.10. As expected, the left and right optimum profits increases and decreases respectively with the increase of α. At $\alpha = 0.99$, the above profit values are almost same.

Example 8.4. *Here, fuzzy parameters are considered as general fuzzy number (GFN) and their different values are given.*

$\widetilde{\tau} = (\tau_0 - \Delta_1, \tau_0 - \Delta_2, \tau_0 + \Delta_3, \tau_0 + \Delta_4) = (2, 2.5, 3.5, 5)$, $\Delta_1 = 1$, $\Delta_2 = 0.25$, $\Delta_3 = 0.75$, $\Delta_4 = 2$, $\widetilde{\beta} = (\beta_0 - \sigma_1, \beta_0 - \sigma_2, \beta_0 + \sigma_3, \beta_0 + \sigma_4) = (0.07, 0.10, 0.13, 0.15)$, $\sigma_1 = 0.05$, $\sigma_2 = 0.02$, $\sigma_3 = 0.01$, $\sigma_4 = 0.03$, $\widetilde{D} = (d_0 - \rho_1, d_0 - \rho_2, d_0 + \rho_3, d_0 + \rho_4) = (2500, 2549, 2571, 2631)$, $\rho_1 = 65$, $\rho_2 = 16$, $\rho_3 = 6$, $\rho_4 = 66$. Other parametric values are same as in Example-1. This example is solved using LINGO-12.0. Table 8.11, 8.12 represent the optimum results when TP_α^L, TP_α^R are maximized separately and Table 8.13 represent the optimum results when TP_α^L, TP_α^R are maximized simultaneously.
For the above parametric values, the optimum business time period (T^*), optimum production time period (t_1^*), and profit interval ($[TP_\alpha^L, TP_\alpha^R]$) are obtained. Obtained results for different values of α are presented in Table 8.11, 8.12 and 8.13.

Table 8.11: Optimum results of Example 8.4 for maximizing TP_α^L

α	Production period (t_1^*)	Business period (T^*)	TP_α^{L*} (Max)	TP_α^R
0.00	8.445339	9.452128	47447.09	69007.36
0.25	7.869975	9.079349	54132.44	67864.09
0.50	7.487523	8.755217	56881.46	67400.04
0.75	7.117747	8.415065	58990.51	67054.82
0.99	6.757742	8.066704	60706.47	66784.82

Table 8.12: Optimum results of Example 8.4 for maximizing TP_α^R

α	Production period (t_1^*)	Business period (T^*)	TP_α^L	TP_α^{R*} (Max)
0.00	10.71312	11.93943	47248.61	69111.75
0.25	8.696047	10.00548	54103.70	67882.37
0.50	7.928822	9.254359	56872.92	67405.93
0.75	7.353021	8.682959	58988.00	67056.67
0.99	6.889372	8.217389	60705.66	66785.45

Table 8.13: Optimum results of Example 8.4 for maximizing both TP_α^L and TP_α^R

α	Production period (t_1^*)	Business period(T^*)	TP_α^{L*} (Max)	TP_α^{R*} (Max)	Average profit
0.00	9.513103	10.62325	47397.64	69085.74	58241.69
0.25	8.273025	9.531221	54125.26	67877.80	61001.53
0.50	7.704997	9.001196	56879.33	67404.46	62141.89
0.75	7.234143	8.547598	58989.88	67056.21	63023.05
0.99	6.823119	8.141546	60706.27	66785.29	63745.78

As expected, the left and right optimum profits increases and decreases respectively with the increase of α.

Example 8.5. *When* $\left\{(1-\beta^L_\alpha)P + D^R_\alpha - D^L_\alpha\right\}t_1 + P\beta^R_\alpha\tau^R_\alpha = D^R_\alpha T$, *the following parametric values are used to illustrate the model:*

$C_p = \$30$, $C_{sr} = \$2$, $A_m = \$5200$, $s = \$55$, $s' = \$35$, $h_m = \$1.0$, $h'_m = \$0.50$, $P = 3380$. Here, fuzzy parameters are considered as triangular fuzzy number (TFN) and their different values are given below. $\widetilde{\tau} = (\tau_0 - \Delta_1, \tau_0, \tau_0 + \Delta_2) = (2, 3, 5)$, $\Delta_1 = 1$, $\Delta_2 = 2$, $\widetilde{\beta} = (\beta_0 - \sigma_1, \beta_0, \beta_0 + \sigma_2) = (0.07, 0.10, 0.15)$, $\sigma_1 = 0.03$, $\sigma_2 = 0.05$, $\widetilde{D} = (D_0 - \rho_1, D_0, D_0 + \rho_2) = (2500, 2559, 2631)$, $\rho_1 = 59$, $\rho_2 = 72$. This example is solved using LINGO-12.0. Table 8.14,8.15 represent the optimum results when TP^L_α, TP^R_α are maximized separately.

Table 8.14: Optimum results of Example 8.5 for maximizing TP^L_α

α	Production period (t_1)	Business period(T)	TP^{L*}_α (Max)	TP^R_α
0.00	1.904205	3.333382	72465.08	91202.27
0.25	1.992095	3.252411	70387.39	84903.91
0.50	1.834233	2.883574	68885.25	78970.41
0.75	1.735770	2.604554	67219.03	72486.40
0.99	4.184291	5.377229	66696.28	66897.66

Table 8.15: Optimum results of Example 8.5 for maximizing TP^R_α

α	Production period (t_1)	Business period (T)	TP^L_α	TP^{R*}_α (Max)
0.00	1.904205	3.333382	72465.08	91202.27
0.25	1.992095	3.252411	70387.39	84903.91
0.50	1.834233	2.883574	68885.25	78970.41
0.75	1.468407	2.282920	67206.21	72582.51
0.99	4.169558	5.359706	66696.27	66897.67

Now from Table 8.14 and 8.15, it is observed that when $\alpha = 0.00$ then the optimum profit interval is $[TP^L_\alpha, TP^R_\alpha] = [72465.08, 91202.27]$. Again when $\alpha = 0.25$ then the optimum profit interval is $[TP_{\alpha}^{L}, TP_{\alpha}^{R}] = [70387.39, 84903.91]$. But here $[TP_{0.0}^{L}, TP_{0.0}^{R}] \supseteq [TP_{0.25}^{L}, TP_{0.25}^{R}]$ does not satisfied. This in-feasibility is also shown when α further is increasing. Therefore this relation has no roll to give the optimum solution of the model.

8.5.1 Comparison between the Optimum Average Profit by Fisher's t-test

Comparison between the optimum average profit due to TFN and TrFN by Fisher's t-test. In the fuzzy EPQ model, two optimum average profit have been obtained using TFN and TrFN. Now question is that does there exist any significance difference between these two values ? If exists, then how much ? To get this answer, it can be tested that the null hypothesis H_0: $\overline{AP}_{TFN}$ (mean of values of average profit for TFN) = $\overline{AP}_{TrFN}$ (mean of values of average profit for TrFN) against the alternative hypothesis H_1: $\overline{AP}_{TFN} \neq \overline{AP}_{TrFN}$ on the basis of the results presented in Tables8.4 and 8.7. This hypothesis can be tested using t-distribution.
The test statistic is

$$t = \frac{\overline{AP}_{TFN} - \overline{AP}_{TrFN}}{s\sqrt{(1/n_1) + (1/n_2)}}$$

which follows t-distribution with $(n_1 + n_2 - 2)$ degrees of freedom, where

$$s^2 = \frac{n_1 s_{TFN}^2 + n_2 s_{TrFN}^2}{n_1 + n_2 - 2}$$

Here $n_1 = 5$, $n_2 = 5$, $\overline{AP}_{TFN} = 62393.63$, $\overline{AP}_{TrFN} = 60993.21$, $s_{TFN}^2 = 10866309.90$, $s_{TrFN}^2 = 4695651.613$. Therefore, degrees of freedom (n_1+n_2-2)=8 and value of $t = 0.2840$. Since the evaluated value of $t <$ the tabulated value of $t_{0.05}$. we accept the null hypothesis H_0 with 95% confidence limit and conclude that there is no significant difference between the mean average profit (AP) for TFN and TrFN.

8.5.2 Sensitivity Analysis

To study the sensitivity analysis of the proposed model with respect to key parameters, the Example 8.1 has been considered. The optimum results of the model with the changes in the parameters Δ_1, Δ_2, σ_1, σ_2, ρ_1, ρ_2, P and s are given in Table 8.16 taking $\alpha = 0.5$.

Table 8.16: Sensitivity analysis on Example 8.1 w.r.t. Δ_1, Δ_2, σ_1, σ_2, ρ_1, ρ_2, P and S

Parameter	value	Production period (t_1)	business period (T)	TP_α^L	TP_α^R	Average profit
	0.75	7.250299	8.587478	57389.12	67422.17	62405.64
Δ_1	1.00	7.310260	8.641835	57181.58	67545.36	62363.47
	1.25	7.361048	8.685704	56976.58	67665.90	62321.24
	1.75	7.198704	8.514293	57409.15	67504.98	62457.07
Δ_2	2.00	7.310260	8.641835	57181.58	67545.36	62363.47
	2.25	7.415203	8.761817	56956.63	67584.60	62270.62
	0.02	7.229466	8.566165	57624.05	67683.21	62653.63
σ_1	0.03	7.310260	8.641835	57181.58	67545.36	62363.47
	0.04	7.375419	8.699629	56733.72	67410.27	62071.99
	0.04	7.083671	8.430101	57728.50	67646.06	62687.28
σ_2	0.05	7.310260	8.641835	57181.58	67545.36	62363.47
	0.06	7.526995	8.839341	56620.66	67454.61	62037.63
	55	7.317655	8.649237	57295.96	67535.26	62415.61
ρ_1	59	7.310260	8.641835	57181.58	67545.36	62363.47
	64	7.301025	8.632592	57038.64	67558.02	62298.33
	67	7.300695	8.638115	57256.03	67460.07	62358.05
ρ_2	72	7.310260	8.641835	57181.58	67545.36	62363.47
	77	7.319873	8.645592	57106.98	67630.55	62368.77
	3350	7.497850	8.775979	57266.24	67613.92	62440.08
P	3380	7.310260	8.641835	57181.58	67545.36	62363.47
	3420	7.079061	8.478815	57070.02	67458.47	62264.24
	54	7.310260	8.641835	44534.08	54570.36	49552.22
s	59	7.310260	8.641835	57181.58	67545.36	62363.47
	64	7.310260	8.641835	69829.08	80520.36	64188.79

Now, from Table 8.16 the following features of the proposed model have been observed:

• When Δ_1 increases, the production run time and business period also increase. But, the average profit decreases with increasing of Δ_1.

• When Δ_2 increases, the production run time and business period increase. But, the average profit decreases with increasing of Δ_2.

• When σ_1 increases, the production run time and business period also increase. But, the

average profit decreases with increasing of σ_1.
• When σ_2 increases, the production run time and business period also increase. But, the average profit decreases with increasing of σ_2.
• When ρ_1 increases, the production run time and business period also decrease. But, the average profit increases with increasing of ρ_1.
• When ρ_2 increases, the production run time and business period also increases. But, average profit increases with increasing of ρ_2.
• When the production rate (P) increases, the inventory increases as well as the production run time and business period reduce. But, the average profit increases with the production rate.
• When selling price of perfect item per unit (s) increases, the production run time and business period are not change. But, the average profit increases with the increasing of selling price (s).

8.5.3 Discussion

The optimum results of the proposed model are obtained from Table 8.4, 8.7,8.10 and 8.13 when the fuzzy numbers $\widetilde{\tau}$, $\widetilde{\beta}$, $\widetilde{D}$ have been considered as TFN, TrFN, PFN and GFN in Example 8.1, 8.2, 8.3 and 8.4 respectively. From these tables it is shown that when α increases then $(\frac{\Delta TP_{\alpha}^{L}}{\Delta\alpha})$ also increases but $(\frac{\Delta TP_{\alpha}^{R}}{\Delta\alpha})$ decreases. Again, from Figure 8.2, it is observed that the rate of increase left optimum profit $(\frac{\Delta TP_{\alpha}^{L}}{\Delta\alpha})$ is more than rate of decrease of right optimum profit $(\frac{\Delta TP_{\alpha}^{R}}{\Delta\alpha})$. Hence forth, the average profit(AP) i.e., $\frac{1}{2}(TP_{\alpha}^{L}+TP_{\alpha}^{R})$ is not same for all α. Ultimately, the average profit rises with the increase of α.

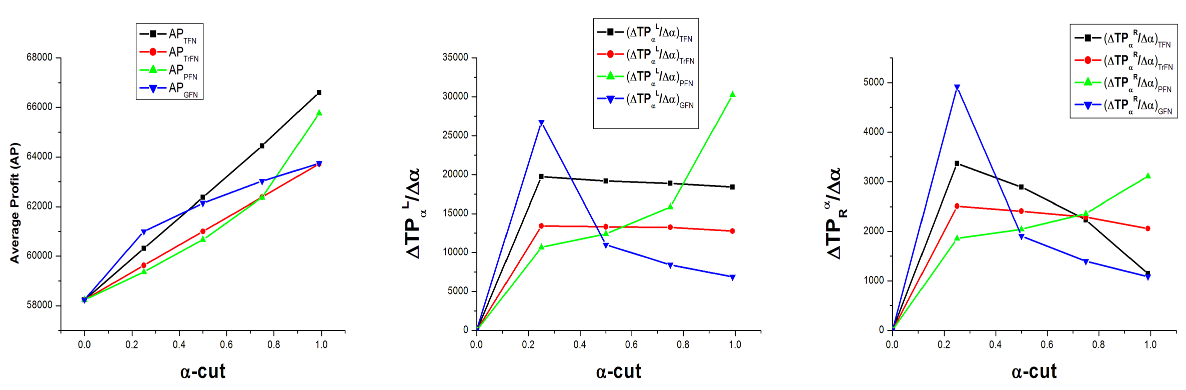

Figure 8.2: Graphical representation of average profit with α

From this Figure 8.2 it is also observed that the TFNs gives the maximum average profit among others, i.e., the ordering of optimum average profit is $AP_{TrFN} \leq AP_{GFN} \leq AP_{PFN} \leq AP_{TFN}$ where AP_{FN} indicates the optimum average profit for fuzzy number FN, though all fuzzy numbers have same spread.

8.6 Conclusion

The main contribution of this chapter is to develop a FEPQ model with fuzzy demand for perfect quality items with inspection of imperfect items. In the production run-time, the manufacturing system produces perfect items in the "in-control" state. Many research papers relating to the manufacturing process may shift to an 'out-of-control' state after certain time that follows constant/random. But, first time a fuzzy production inventory model has been developed, where the manufacturing process may shift to an 'out-of-control' state after certain time that follows a fuzzy number. During 'out-of-control' state, the process starts to produce defective items. The defective rate is considered as fuzzy number. Using fuzzy differential equation and fuzzy Riemann integration, an approach has been proposed, where α-cut of fuzzy profit is optimized to get optimal decision. It proposes a strategy for maximizing profit in an fuzzy imperfect production. The model is optimized for the production run time(t_1^*), business period(T^*) and profit interval. Here, four fuzzy numbers TFN, TrFN, PFN, and GNF have been used to illustrate the model for fuzzy parameters. Finally, Fuzzy Programming Technique(FPT) has been used to get the optimal solution. These models are applicable in the factory like steel, plastic etc. The present models can be extended to the rough, fuzzy-rough, random, fuzzy-random environment taking constant part of screening cost, holding cost, set-up cost, etc.

Chapter 9

GA approach for controlling GHG emission from industrial waste in two plant production and reproduction inventory model with interval valued fuzzy pollution parameters

9.1 Introduction

In an imperfect production process, 100% screening is required to identify perfect and defective units. Defective units may be reworked in the same cycle along with the normal production after some time from the initial commencement of production by engaging some additional labour forces and machinery. But all of the defective units cannot be considered for reworking. Some of the defective units should be avoided for rework as they may be of very poor quality and will be expensive to repair. Therefore, a certain percentage of the defective units may be considered for reworking and reworked items are assumed as good as new products. Non-reworkable defective units are treated as rejected items. So many researchers considered reworks in imperfect production sector and some of them are Hayek and Salameh [97], Chiu [35], Chiu et al. [39], Liao et al. [136], Wee et al. [219], Taleizadeh et al. [204], Wang [217] and others.

Generally, consumers dispose of the products after its use or at the end of the life-cycle of the product but recovery of the used products may be economically more attractive than disposal. In classical logistic system, material and related information flow (forward flow) is observed until the final products are delivered to the customer. But in reverse logistic system, used and reusable parts are returned from the customers to the producer (backward flow). Environmental consciousness forces companies to initiate such product recovery systems with their disposal (metal, glass, paper). In this way, natural resources can be saved for the future generations, so the firms can contribute to the sustainable development efforts. The importance of reverse logistics has increased significantly in the last two decades for a variety of legislative, environmental, and economic reasons (Fleischmann et al. [72]). Todays customers are more educated and demanding, and tend to be less inclined to purchase products that are not environmentally friendly. The growth of secondary markets emphasized the importance of reverse logistics (Tibben-Lembke [207]). In the last twenty four years, a lot of research has been investigated on reverse logistics. Authors such as Carter and Ellram [14], Jayaraman et al. [106], Rogers and Tibben-Lembke [176, 177], Dowlatshahi [60], Guide [81], Stock and Mulki [198], and Guide and Van Wassenhove [82] have described a broad range of reverse logistics systems and structures and analyzed a variety of attendant reverse logistics problems. Implementation of reverse logistics especially in product returns would allow not only for savings in inventory carrying cost, transportation cost, and waste disposal cost due to returned products, but also for the improvement of customer loyalty and future sales. In a broader sense, reverse logistics refers to the distribution activities involved in product returns, source reduction, conservation, recycling, substitution, reuse, disposal, refurnishment, repair and re-manufacturing. Recent reviews on reverse logistics are provided by Ferguson and Toktay [71], Mitra [151, 152], Ahmed and Jaber [4], Schulz [191], Xu [229], Hasanov et al. [91], Kim et al. [120, 121], Naeem et al. [159], Chen and Abrishami [34], Ahiska and Kurtul [3], Schulz [191], Rogers et al. [179], Ghosh and Dey [74] and others.

Rise in the temperature of earth, deterioration in Ozone layer, melting of glaciers, numerous natural calamities, phase-shift in environmental clock are the various rays converging at the point of alarm to nature or save environment. This alertness about saving the nature and its resources and realization about environment has given considerable attention to produce green Products in all fields worldwide. Along with producing green, reuse and recycle of used products also has been promoted to save the environment as much as possible.

Ever since the fuzzy set was proposed by Zadeh [239] fuzzy numbers have been widely studied, developed and applied to various fields. In fuzzy set, the degree of membership functions of the element in the universe is having a single value: either zero or one. Many times specialists are uncertain about the values of the membership of an element in a set. Hence, it is better to represent the values of the membership of an element in a set by intervals of possible real numbers instead of real numbers. An interval-valued fuzzy set on a universe X is a mapping from X to fall closed sub-intervals of the real interval [0, 1]. This type of fuzzy sets has been intensively investigated, not only its theoretical aspects, but also its numerous applications. As the parameters relating to the environmental pollution due to industrial waste are not fixed in nature so, in this model, we take them as interval valued fuzzy number.

This chapter deals with the combined effect of manufacturing and re-manufacturing for two types of quality items (item-I and item-II) produced in two different plants (plant-I and plant-II) in the same premises under single management system over a known-finite time horizon with consideration of environment pollution control through industrial waste management. Three types of inventories are involved in this network. The manufactured and re-manufactured items are stored in the first and second inventories. The used items returned from the market and rejected defective units are together collected in the third inventory for raw materials required for re-manufacturing process. The objective of this research is to propose a manufacturing/re-manufacturing policy that would minimize the uses of natural resource as raw materials and minimizes the environmental pollution from the used and non-reworkable defective units by industrial waste management considering pollution parameters as interval-valued fuzzy numbers.

9.2 Notations and Assumptions

The following notations and assumptions have been considered to develop the model:

9.2.1 Notations

The following notations are used throughout the chapter.

$q_{1i}(t)$	:	Inventory level of better quality item of ith cycle at time t in plant-I.
$q_{2i}(t)$	:	Inventory level of less better quality item of ith cycle at time t in plant-II.
$q_{3i}(t)$	:	Inventory level of the returned and non reworkable item of ith cycle at time t.
$q_{4i}(t)$	:	Inventory level of disposal item of ith cycle at time t.
β_1	:	Fraction of the better quality item in plant-I.
β_2	:	Fraction of the less better quality items in plant-II.
δ_1	:	Fraction of the re-workable item in plant-I.
δ_2	:	Fraction of the re-workable item in plant-II.
P_1	:	Production rate per unit time of 1st cycle in pant-I.
D_1	:	Demand rate in each cycle of plant-I.
t_1	:	Production time in each cycle which is considered as a decision variable.
T	:	Total time length of each cycle.
n	:	Total number of cycle.
k	:	Number of consecutive cycles in which the returned items from a cycle is considered.
c_p	:	Production cost per unit item per unit time in plant-I.
c'_p	:	Production cost per unit item per unit time in plant-II.
c_{sr}	:	Screening cost per unit item per unit time in plant-I.

c'_{sr}	:	Screening cost per unit item per unit time in plant-II.
r_c	:	Reworked cost per unit item per unit time in plant-I.
r'_c	:	Reworked cost per unit item per unit time in plant-II.
A_s	:	Set-up cost per cycle in plant-I.
A'_s	:	Set-up cost per cycle in plant-II.
h_c	:	Holding cost per unit item per unit time in plant-I.
h'_c	:	Holding cost per unit item per unit time in plant-II.
h''_c	:	Holding cost per unit item per unit time in the row material processing unit.
s	:	Selling price per unit item per unit time in plant-I.
s'	:	Selling price per unit item per unit time of plant-II.
c_w	:	Water pollution cost per unit .
c_g	:	Cost to control GHG emission per unit .
c_t	:	Transportation cost per unit to transport the disposal unit for landfills.
ER_g	:	Emission rate of GHG in landfills.
Q_4	:	Total disposal items.
P_{2i}	:	Production rate of ith cycle in plant-II, where $P_{2i} = \lambda P_{2(i-1)} = \lambda^{i-1}P_2$ with $P_{21} = P_2$.
D_{2i}	:	Demand rate of ith cycle in plant-II, where $D_{2i} = \mu D_{2(i-1)} = \mu^{i-1}D_2$ with $D_{21} = D_2$.
$\alpha_{ij}D_1$	:	Rate of return of item-I in plant-I collected from the market for j-th cycle when total time horizon consists of i-th cycles $(j = 1, 2, \ldots, i)$ and where we take $\alpha_{ii} = \alpha_1$ and $\alpha_{ij-1} = \alpha_{ij}h$ such that $\sum_{j=1}^{i} \alpha_{ij} = \alpha_1 \frac{h^i-1}{h-1}, \quad h > 1$.
$\alpha'_{ii}D_1$	:	Rate of return of item-I in plant-I collected from the market from the time period $[(i-1)T, (i-1)T + t_1]$ during $[(i-1)T + t_1, iT]$, where we take α'_{ii}=α'_1 ,$i = 1, 2, \ldots, n$.
$\beta_{ij}D_{2i}$	:	Rate of return of item-II in plant-II collected from the market for j-th cycle when total time horizon consists of i-th cycles $(j = 1, 2, \ldots, i)$ and where we take $\beta_{ii} = \alpha_2$ and$\beta_{ij-1} = \beta_{ij}h_1$ such that $\sum_{j=1}^{i} \beta_{ij} = \alpha_2 \frac{h_1^i-1}{h_1-1}, \quad h_1 > 1$.
$\beta'_{i,i-1}D_{2i}$	:	Rate of return of item-II items in plant-II collected from the market from the time period $[(i-2)T + t_1, (i-1)T]$ during $[(i-1)T, (i-1)T + t_1]$, where we take $\beta'_{i,i-1}$=α'_2, $i = 2, 3, \ldots, n$.
$\epsilon_w^{p_1}$	:	Percentage of water pollution per cycle per unit production per unit time produce from plant-I when production rate is P_1, where we take $\epsilon_w^{p_1}$=$\epsilon_w P_1^{\eta_{1w}-1}$, $\epsilon_w > 0, \eta_{1w} > 1$.
$\epsilon_g^{p_1}$	:	Percentage of GHG emission per cycle per unit production per unit time produce from plant-I when production rate is P_1, where we take $\epsilon_g^{p_1} = \epsilon_g P_1^{\eta_{1g}-1}$, $\epsilon_g > 0, \eta_{1g} > 1$.
$\epsilon_w^{p_2}$	:	Percentage of water pollution per cycle per unit production per unit time produce from plant-II when production rate is P_{2i}, where we take $\epsilon_w^{p_2} = \epsilon_w P_{2i}^{\eta_{2w}-1}, \quad \epsilon_w > 0, \ \eta_{2w} > 1$.
$\epsilon_g^{p_2}$	:	Percentage of GHG emission per cycle per unit production per unit time produce from plant-II when production rate is P_{2i}, where we take $\epsilon_g^{p_2} = \epsilon_g P_{2i}^{\eta_{2g}-1}, \ \ \epsilon_g > 0, \ \eta_{2g} > 1$.

9.2.2 Assumptions

The following assumptions have been made to developed the model.

(i) It is an imperfect production and reproduction inventory model in finite time horizon for two types of items (Item-I and Item-II).

(ii) Item-I is produced with raw materials from natural sources in plant-I and Item-II is produced in plant-II (re-manufacturing process) where the used and non-reworkable defective items are the raw materials. Also we assume that item-II is of less quality than the item-I.

(iii) The manufacturer produced mixture of perfect and defective (imperfect) quality items. Hence, the manufacturer decides to sale the perfect quality item after sorting the items in the inventory. So in this chapter, it is assumed that during the production period, the screening process has been occurred simultaneously which greater than demand rate.

(iv) Water pollution and GHG emission to the environment during production in plant-I and plant-II have been considered. Also we consider GHG emission from the disposal units (Industrial Solid waste) produced from both the plants and used items.

(v) n, k, λ and t_1 are decision variable.

(vi) The production rate and demand rate of 1st cycle of plant-I and plant-II are constant.

9.3 Mathematical Formulation of the Proposed Model

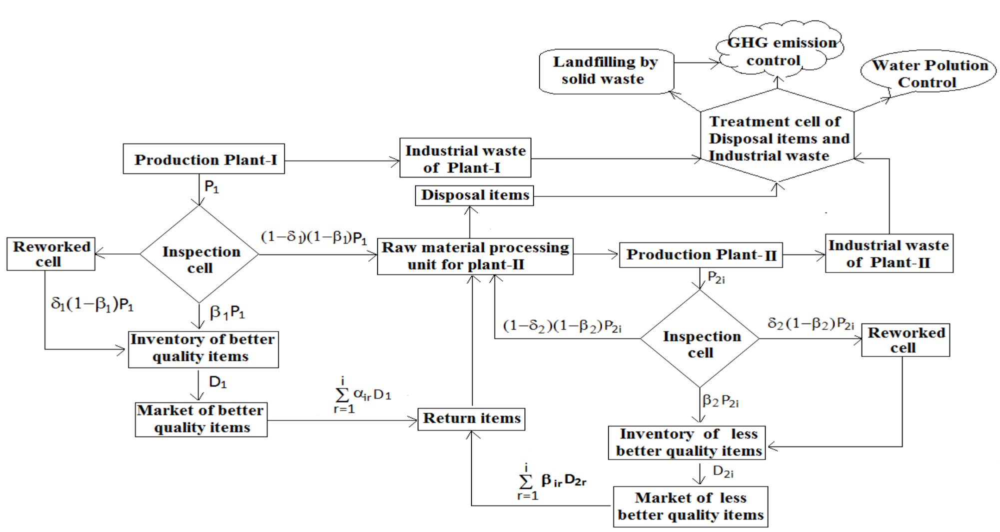

Figure 9.1: Schematic representation of the proposed model

We have considered an imperfect manufacturing system that produces perfect quality as well as imperfect quality items in plant-I and re-manufacturing of non-reworkable and returned items in plant-II with screening of defective units in both plants. Some imperfect quality products are reworked immediately at a cost to restore it to the original quality. For the development of this model, we assume that there are n cycles during the finite time horizon T. Here, non-reworkable items and returned items which are used items collected from the market are remanufactured. Before re-manufacturing, some collected non-reworkable items and returned items not to be in a position for re-manufacturing are disposed off at the rate of $(1-\gamma)\%$. This configuration is presented in Figure 9.1.

9.3.1 Formulation for Plant-I

For ith cycle: ($i = 1, 2, \ldots, n$)
In this case, the initial stock of the each cycle is zero and starts production with rate P_1. As production and reworking continues, inventory begins to pile up continuously after meeting demand with rate D_1. Production of ith cycle stops at time $(i-1)T + t_1$ and restarts at time iT for next cycle. Each cycle ends with zero inventory. It then repeats itself. Our problem may be precisely defined as follows. The differential equation of the item-I in the ith cycle during

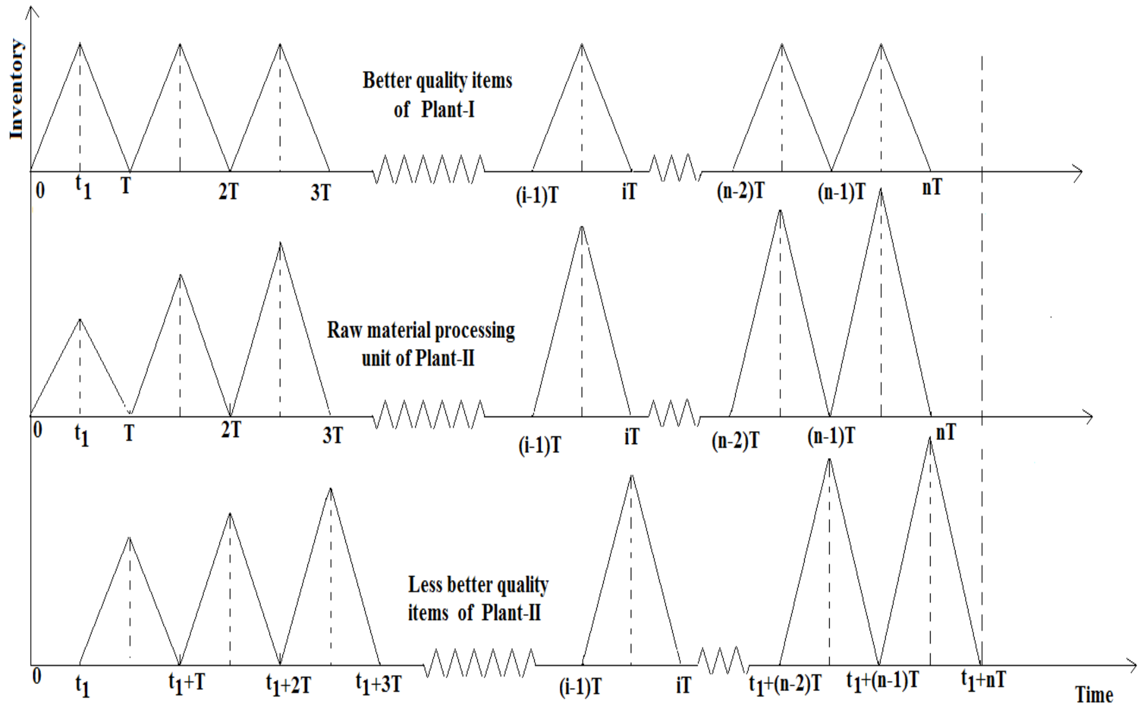

Figure 9.2: Pictorial representation of inventory situation of the integrated model

$[(i-1)T, iT]$ $(i = 1, 2, \ldots, n)$ is given by

$$\frac{dq_{1i}}{dt} = \begin{cases} \beta_1 P_1 + \delta_1(1-\beta_1)P_1 - D_1, & (i-1)T \le t \le (i-1)T + t_1 \\ -D_1, & (i-1)T + t_1 \le t \le iT \end{cases} \tag{9.1}$$

with $q_{1i}[(i-1)T] = 0$ and $q_{1i}[iT] = 0$.
The solution of the above differential equation is

$$q_{1i}(t) = \begin{cases} [\beta_1 P_1 + \delta_1(1-\beta_1)P_1 - D_1][t-(i-1)T], & (i-1)T \le t \le (i-1)T + t_1 \\ D_1(iT - t), & (i-1)T + t_1 \le t \le iT \end{cases}$$

From continuity condition at $t = (i-1)T + t_1$, we have

$$[\beta_1 + \delta_1(1-\beta_1)]P_1 t_1 = D_1 T \tag{9.2}$$

The total holding cost of plant-I is given by

$$\begin{aligned} THC_1 &= \sum_{i=1}^{n} h_c\left[\int_{(i-1)T}^{(i-1)T+t_1} q_1(t)dt + \int_{(i-1)T+t_1}^{iT} q_1(t)dt\right] \\ &= \frac{nh_c}{2}[\{\beta_1 + \delta_1(1-\beta_1)\}P_1 t_1^2 + D_1(T^2 - 2Tt_1)] \end{aligned}$$

Amount of water pollution in each production cycle in plant-I=$\epsilon_w^{p_1} P_1 t_1$.
Cost incurred for water pollution in plant-I= $c_w \epsilon_w^{p_1}(nP_1 t_1) = nc_w \epsilon_w P_1^{\eta_1 w} t_1$.
Amount of GHG emission in each production cycle in plant-I = $\epsilon_g^{p_1} P_1 t_1$.
Cost of GHG emission control in plant-I=$c_g \epsilon_g^{p_1}(nP_1 t_1) = nc_g \epsilon_g P_1^{\eta_1 g} t_1$.

The total cost of the plant-I over the time horizon is

$$TC_{p1} = n\Big\{c_p + r_c\delta_1(1-\beta_1) + c_{sr}\Big\}P_1 t_1 + THC_1 + nA_s + nc_w\epsilon_w P_1^{\eta_1 w} t_1 + nc_g \epsilon_g P_1^{\eta_1 g} t_1$$

The total sales revenue from plant-I during $(0, nT)$ is given by

$$TSR_{p1} = \sum_{i=1}^{n} s\int_{(i-1)T}^{iT} D_1 dt = nsD_1 T$$

9.3.2 Formulation for plant-II

For ith **cycle:**$(i = 1, 2, \ldots, n)$
In this case, the initial stock of the ith cycle is zero and starts production with rate P_{2i} at time $(i-1)T + t_1$. As production with screening and reworking continues, inventory begins to pile up continuously after meeting demand with rate D_{2i} and deterioration. Production of ith cycle stops at time iT. The accumulated inventory of ith cycle is just sufficient enough to account for demand and deterioration over the interval $[iT, iT + t_1]$. Production restarts at time $iT + t_1$ for next cycle. The cycle ends with zero inventory. It then repeats itself. Our problem may be

precisely defined as follows.
The differential equation of the item-II for ith cycle during $[(i-1)T+t_1, iT+t_1]$ is given by

$$\frac{dq_{2i}}{dt} = \begin{cases} \beta_2 P_{2i} + \delta_2(1-\beta_2)P_{2i} - D_{2i}, & (i-1)T+t_1 \le t \le iT \\ -D_{2i}, & iT \le t \le iT+t_1 \end{cases} \tag{9.3}$$

with $q_{2i}[(i-1)T+t_1] = 0$ and $q_{2i}[iT+t_1] = 0$.
The solution of the above differential equation is

$$q_{2i}(t) = \begin{cases} [\beta_2 P_{2i} + \delta_2(1-\beta_2)P_{2i} - D_{2i}][t-(i-1)T-t_1], & (i-1)T+t_1 \le t \le iT \\ D_{2i}(iT+t_1-t), & iT \le t \le iT+t_1 \end{cases}$$

From continuity condition at $t = iT$, we have

$$[\beta_2 + \delta_2(1-\beta_2)](T-t_1)P_{2i} = D_{2i}T \tag{9.4}$$

The total production cost of plant-II during $(t_1, nT+t_1)$ is given by

$$TPC_2 = \sum_{i=1}^{n} c'_p \int_{(i-1)T+t_1}^{iT} P_{2i}\, dt = c'_p(T-t_1)\sum_{i=1}^{n} P_{2i} = c'_p(T-t_1)P_2\frac{\lambda^n-1}{\lambda-1}$$

The total screening cost of plant-II during $(t_1, nT+t_1)$ is given by

$$SC_2 = \sum_{i=1}^{n} c'_{sr} \int_{(i-1)T+t_1}^{iT} P_{2i}\, dt = c'_{sr}(T-t_1)\sum_{i=1}^{n} P_{2i} = c'_{sr}(T-t_1)P_2\frac{\lambda^n-1}{\lambda-1}$$

The total reworked cost of plant-II during $(t_1, nT+t_1)$ is given by

$$RWC_2 = \sum_{i=1}^{n} r'_c \int_{(i-1)T+t_1}^{iT} \delta_2(1-\beta_2)P_{2i}\, dt = r'_c\delta_2(1-\beta_2)(T-t_1)P_2\frac{\lambda^n-1}{\lambda-1}$$

The total holding cost of plant-II during $(t_1, nT+t_1)$ is given by

$$\begin{aligned} THC_2 &= \sum_{i=1}^{n} h'_c[\int_{(i-1)T+t_1}^{iT} q_{2i}(t)\, dt + \int_{iT}^{iT+t_1} q_{2i}(t)\, dt] \\ &= \frac{h'_c}{2}\Big[\{\beta_2+\delta_2(1-\beta_2)\}(T-t_1)^2\sum_{i=1}^{n} P_{2i} + (2t_1T-T^2)\sum_{i=1}^{n} D_{2i}\Big] \\ &= \frac{h'_c}{2}\Big[\{\beta_2+\delta_2(1-\beta_2)\}(T-t_1)^2 P_2\frac{\lambda^n-1}{\lambda-1} + (2t_1T-T^2)D_2\frac{\mu^n-1}{\mu-1}\Big] \end{aligned}$$

Amount of water pollution in ith production cycle in plant-II $=\epsilon_w^{p_2} P_{2i}(T-t_1)$.
Cost incurred for water pollution in plant-II= $c_w\epsilon_w(T-t_1)\sum_{i=1}^{n} P_{2i}^{\eta_{2w}}$.
Amount of GHG emission in ith production cycle in from plant-II $=\epsilon_g^{p_2} P_{2i}(T-t_1)$.

Cost incurred for GHG emission control in plant-II= $c_g \epsilon_g (T - t_1) \sum_{i=1}^{n} P_{2i}^{\eta_{2g}}$.
The total cost of the plant-II during $(t_1, nT + t_1)$ is

$$TCP_{p2} \;=\; TPC_2 + RWC_2 + TSC_2 + THC_2 + nA'_s + \left(c_w \epsilon_w \sum_{i=1}^{n} P_{2i}^{\eta_{2w}} + c_g \epsilon_g \sum_{i=1}^{n} P_{2i}^{\eta_{2g}} \right)$$

The total sales revenue of plant-II during $(t_1, nT + t_1)$ is given by

$$TSR_{p2} = \sum_{i=1}^{n} s' \int_{(i-1)T+t_1}^{iT+t_1} D_{2i}\, dt = s'T \sum_{i=1}^{n} D_{2i} = s'TD_2 \frac{\mu^n - 1}{\mu - 1}$$

9.3.3 Formulation for Raw Material Processing Unit of Plant-II

For 1st cycle:
The differential equation in the raw material processing unit during $[0, T]$ is given by

$$\frac{dq_{31}}{dt} = \begin{cases} \gamma_1[\alpha_{11} D_1 + (1 - \delta_1)(1 - \beta_1)P_1], & 0 \le t \le t_1 \\ \gamma_1[(\alpha'_{11} + \alpha_{11})D_1 + \beta_{11} D_{21} + (1 - \delta_2)(1 - \beta_2)P_{21}] - P_{21}, & t_1 \le t \le T \end{cases} \tag{9.5}$$

with $q_{31}(0) = 0$ and $q_{31}(T) = 0$.
The solution of the above differential equation is

$$q_{31}(t) = \begin{cases} \gamma_1[\alpha_1 D_1 + (1 - \delta_1)(1 - \beta_1)P_1]t, & 0 \le t \le t_1 \\ \left[\gamma_1\{\alpha'_1 D_1 + \alpha_1 D_1 + \alpha_2 D_2 + (1 - \delta_2)(1 - \beta_2)P_2\} - P_2\right](t - T), & t_1 \le t \le T \end{cases}$$

From continuity condition at $t = t_1$, we have
$\gamma_1[\alpha_1 D_1 + (1 - \delta_1)(1 - \beta_1)P_1]t_1 = \left[\gamma_1\{\alpha'_1 D_1 + \alpha_1 D_1 + \alpha_2 D_2 + (1 - \delta_2)(1 - \beta_2)P_2\} - P_2\right](t_1 - T)$
The holding cost in raw material processing unit for first cycle is given by

$$\begin{aligned} HC_{31} &= h''_c \int_0^{t_1} q_{31}(t)\, dt + h''_c \int_{t_1}^{T} q_{31}(t)\, dt \\ &= \frac{h''_c}{2} \Big[\gamma_1 \Big\{\alpha_1 D_1 (2Tt_1 - T^2) + (1 - \delta_1)(1 - \beta_1)P_1 t_1^2 - (\alpha'_1 D_1 + \alpha_2 D_2)(t_1 - T)^2\Big\} \\ &\quad + \Big\{1 - \gamma_1(1 - \delta_2)(1 - \beta_2)\Big\} P_2 (t_1 - T)^2\Big] \end{aligned}$$

For 2nd cycle :
The differential equation in the raw material processing unit during $[T, 2T]$ is given by

$$\frac{dq_{32}}{dt} = \begin{cases} \gamma_1[(\alpha_{21} + \alpha_{22})D_1 + (1 - \delta_1)(1 - \beta_1)P_1 + (\beta_{21} + \beta'_{21})D_{21}], & T \le t \le T + t_1 \\ \gamma_1[(\alpha_{21} + \alpha_{22} + \alpha'_{22})D_1 + \beta_{21} D_{21} + \beta_{22} D_{22} & \\ +(1 - \delta_2)(1 - \beta_2)P_{22}] - P_{22}, & T + t_1 \le t \le 2T \end{cases} \tag{9.6}$$

with boundary conditions $q_{32}[T] = 0$ and $q_{32}[2T] = 0$
The solution of the above differential equation is

$$q_{32}(t) = \begin{cases} \gamma_1[(\alpha_{21} + \alpha_1)D_1 + (1 - \delta_1)(1 - \beta_1)P_1 + (\beta_{21} + \alpha'_2)D_2](t - T), & T \le t \le T + t_1 \\ \left[\gamma_1\{\alpha_{21} D_1 + \alpha_1 D_1 + \alpha'_1 D_1 + \beta_{21} D_{21} + \alpha_2 D_{22} \right. & \\ \left. +(1 - \delta_2)(1 - \beta_2)P_{22}\} - P_{22}\right](t - 2T), & T + t_1 \le t \le 2T \end{cases}$$

From continuity condition at $t = T + t_1$, we have

$$\gamma_1[\alpha_1 D_1 + (1-\delta_1)(1-\beta_1)P_1 + \alpha_{21}D_1 + \alpha_2' D_2 + \alpha_2 D_2]t_1 =$$
$$\Big[\gamma_1\{\alpha_1' D_1 + \alpha_1 D_1 + \alpha_2 D_{22} + (1-\delta_2)(1-\beta_2)P_{22} + \alpha_{21}D_1 + \beta_{21}D_{21}\} - P_{22}\Big](t_1 - T)$$

The holding cost in raw material processing unit is given by

$$\begin{aligned} HC_{32} &= h_c'' \int_T^{T+t_1} q_{32}(t)\,dt + h_c'' \int_{T+t_1}^{2T} q_{32}(t)\,dt \\ &= \frac{h_c''}{2}\gamma_1\Big[\{\alpha_1 D_1 + \alpha_{21}D_1 + \alpha_2 D_2\}(2Tt_1 - T^2) + \{\alpha_2' D_2 + (1-\delta_1)(1-\beta_1)P_1\}t_1^2\Big] \\ &\quad + \frac{h_c''}{2}\Big[\{1-\gamma_1(1-\delta_2)(1-\beta_2)\}P_{22} - \gamma_1(\alpha_1' D_1 + \beta_{21}D_2)\Big](t_1 - T)^2 \end{aligned}$$

For ith cycle ($i = 3, 4, \ldots, k$):
The differential equation in the raw material processing unit during $[(i-1)T, iT]$ is given by

$$\frac{dq_{3i}}{dt} = \begin{cases} \gamma_1\big[\sum_{j=1}^{i}\alpha_{ij}D_1 + (1-\delta_1)(1-\beta_1)P_1 \\ +\sum_{j=1}^{i-2}\beta_{ij}D_{2j} + \{\beta_{i(i-1)} + \beta_{i(i-1)}'\}D_{2(i-1)}\big], & (i-1)T \le t \le (i-1)T + t_1 \\ \gamma_1\big[\{\sum_{j=1}^{i}\alpha_{ij} + \alpha_{ii}'\}D_1 + \sum_{j=1}^{i-1}\beta_{ij}D_{2j} \\ +\beta_{ii}D_{2i} + (1-\delta_2)(1-\beta_2)P_{2i}\big] - P_{2i}, & (i-1)T + t_1 \le t \le iT \end{cases} \tag{9.7}$$

with $q_{3i}[(i-1)T] = 0$ and $q_{3i}[iT] = 0$
The solution of the above differential equation is

$$q_{3i}(t) = \begin{cases} \gamma_1\Big[\sum_{j=1}^{i}\alpha_{ij}D_1 + (1-\delta_1)(1-\beta_1)P_1 + \sum_{j=1}^{i-2}\beta_{ij}D_{2j} \\ +\beta_{i(i-1)}D_{2(i-1)} + \beta_{i(i-1)}'D_{2(i-1)}\Big][t-(i-1)T], & (i-1)T_1 \le t \le (i-1)T + t_1 \\ \Big[\gamma_1\Big\{\sum_{j=1}^{i}\alpha_{ij}D_1 + \alpha_{ii}'D_1 + \sum_{j=1}^{i-1}\beta_{ij}D_{2j} + \beta_{ii}D_{2i} \\ +(1-\delta_2)(1-\beta_2)P_{2i}\Big\} - P_{2i}\Big](t - iT), & (i-1)T + t_1 \le t \le iT \end{cases}$$

From continuity condition at $t = (i-1)T + t_1$, we have
$\gamma_1\big[\alpha_1 D_1 + \alpha_2 D_{2i} + \alpha_2' D_{2(i-1)} + \sum_{j=1}^{i-1}\alpha_{ij}D_1 + \sum_{j=1}^{i-2}\beta_{ij}D_{2j} + (1-\delta_1)(1-\beta_1)P_1\big]t_1 =$
$\Big[\gamma_1\big\{\alpha_1 D_1 + \alpha_2 D_{2i} + \alpha_{ii}'D_1 + \sum_{j=1}^{i-1}\alpha_{ij}D_1 + \sum_{j=1}^{i-1}\beta_{ij}D_{2j} + (1-\delta_2)(1-\beta_2)P_{2i}\big\} - P_{2i}\Big](t_1 - T)$
Holding cost of raw material processing unit for ith cycle ($i = 3, 4, \ldots, k$) is given by

$$\begin{aligned} HC_{3i} &= h_c'' \int_{(i-1)T}^{(i-1)T+t_1} q_{3i}(t)\,dt + h_c'' \int_{(i-1)T+t_1}^{iT} q_{3i}(t)\,dt \\ &= \frac{h_c''}{2}\gamma_1\Big[\Big\{\alpha_1 D_1 + \alpha_2 D_{2i} + \sum_{j=3}^{i-1}\alpha_{ij}D_1 + \sum_{j=3}^{i-2}\beta_{ij}D_{2j}\Big\}(2Tt_1 - T^2) + \Big\{\alpha_2' D_{2(i-1)} \\ &\quad +(1-\delta_1)(1-\beta_1)P_1\Big\}t_1^2 - \gamma_1\Big\{\alpha_1' D_1 + \beta_{i,i-1}D_{2(i-1)}\Big\}(t_1 - T)^2\Big] \\ &\quad + \frac{h_c''}{2}\Big\{1 - \gamma_1(1-\delta_2)(1-\beta_2)\Big\}P_{2i}(t_1 - T)^2 \end{aligned}$$

Holding cost of raw material processing unit for $(k-2)$ cycles is given by

$$HCN = \sum_{i=3}^{k} HC_{3i}$$
$$= \frac{h''_c}{2}\gamma_1\Big[\Big\{(k-2)\alpha_1 D_1 + \alpha_2 D_2\frac{\mu^{k-2}-1}{\mu-1} + \sum_{i=3}^{k}\sum_{j=3}^{i-1}\alpha_{ij}D_1 + \sum_{i=3}^{k}\sum_{j=3}^{i-2}\beta_{ij}D_{2j}\Big\}(2Tt_1 - T^2)$$
$$+\Big\{\sum_{i=3}^{k}\alpha'_2 D_{2(i-1)} + (k-2)(1-\delta_1)(1-\beta_1)P_1\Big\}t_1^2\Big] + \frac{h''_c}{2}\Big[-\gamma_1\Big\{(k-2)\alpha'_1 D_1$$
$$+\sum_{i=3}^{k}\beta_{i,i-1}D_{2(i-1)}\Big\} + \Big\{1-\gamma_1(1-\delta_2)(1-\beta_2)\Big\}P_2\frac{\lambda^{k-2}-1}{\lambda-1}\Big](t_1-T)^2$$

For ith cycle ($i = k+1, k+2, \ldots, n$):
The differential equation in the raw material processing unit during $[(i-1)T, iT]$ is given by

$$\frac{dq_{3i}}{dt} = \begin{cases} \gamma_1\big[\sum_{j=i-k+1}^{i}\alpha_{ij}D_1 + (1-\delta_1)(1-\beta_1)P_1 \\ +\sum_{j=i-k+1}^{i-2}\beta_{ij}D_{2j} + \{\beta_{i(i-1)} + \beta'_{i,i-1}\}D_{2(i-1)}\big], & (i-1)T \le t \le (i-1)T + t_1 \\ \gamma_1\big[\{\sum_{j=i-k+1}^{i}\alpha_{ij} + \alpha'_{ii}\}D_1 + \sum_{j=i-k+1}^{i-1}\beta_{ij}D_{2j} \\ +\beta_{ii}D_{2i} + (1-\delta_2)(1-\beta_2)P_{2i}] - P_{2i}, & (i-1)T + t_1 \le t \le iT \end{cases} \tag{9.8}$$

with $q_{3i}[(i-1)T] = 0$ and $q_{3i}[iT] = 0$
The solution of the above differential equation is

$$q_{3i}(t) = \begin{cases} \gamma_1\big[\sum_{j=i-k+1}^{i}\alpha_{ij}D_1 + (1-\delta_1)(1-\beta_1)P_1 + \sum_{j=i-k+1}^{i-2}\beta_{ij}D_{2j} \\ \beta_{i(i-1)}D_{2(i-1)} + \beta'_{i,i-1}D_{2(i-1)}\big][t-(i-1)T], & (i-1)T \le t \le (i-1)T + t_1 \\ \Big[\gamma_1\{\sum_{j=i-k+1}^{i}\alpha_{ij}D_1 + \alpha'_{ii}D_1 + \sum_{j=i-k+1}^{i-1}\beta_{ij}D_{2j} + \beta_{ii}D_{2i} \\ +(1-\delta_2)(1-\beta_2)P_{2i}\} - P_{2i}\Big](t-iT), & (i-1)T + t_1 \le t \le iT \end{cases}$$

From continuity condition at $t = (i-1)T + t_1$,
$\gamma_1\big[\alpha_1 D_1 + \alpha_2 D_{2i} + \alpha'_2 D_{2(i-1)} + \sum_{j=i-k+1}^{i-1}\alpha_{ij}D_1 + \sum_{j=i-k+1}^{i-2}\beta_{ij}D_{2j} + (1-\delta_1)(1-\beta_1)P_1\big]t_1 = \Big[\gamma_1\big\{(\alpha_1 + \alpha'_1 + \sum_{j=i-k+1}^{i-1}\alpha_{ij})D_1 + \alpha_2 D_{2i} + \sum_{j=i-k+1}^{i-1}\beta_{ij}D_{2j} + (1-\delta_2)(1-\beta_2)P_{2i}\big\} - P_{2i}\Big](t_1 - T)$
Holding cost of raw material processing unit for ith cycle ($i = k+1, k+2, \ldots, n$) is given by

$$HC'_{3i} = h''_c\int_{(i-1)T}^{(i-1)T+t_1} q_{3i}(t)dt + h''_c\int_{(i-1)T+t_1}^{iT} q_{3i}(t)dt$$
$$= \frac{h''_c}{2}\gamma_1\Big[\Big\{\alpha_1 D_1 + \alpha_2 D_{2i} + \sum_{j=i-k+1}^{i-1}\alpha_{ij}D_1 + \sum_{j=i-k+1}^{i-2}\beta_{ij}D_{2j}\Big\}(2Tt_1 - T^2)$$
$$+\Big\{\beta'_{i,i-1}D_{2(i-1)} + (1-\delta_1)(1-\beta_1)P_1\Big\}t_1^2 - \gamma_1\Big\{\alpha'_1 D_1 + \beta_{i,i-1}D_{2(i-1)}\Big\}(t_1-T)^2\Big]$$
$$+\frac{h''_c}{2}\Big\{1-\gamma_1(1-\delta_2)(1-\beta_2)\Big\}P_{2i}(t_1-T)^2$$

Holding cost in raw material processing unit for $(n-k)$ cycles is given by

$$\begin{aligned} HCN' &= \sum_{i=k+1}^{n} HC'_{3i} \\ &= \frac{h''_c}{2}\gamma_1\Big[\Big\{(n-k)\alpha_1 D_1 + \alpha_2 D_2 \frac{\mu^{n-k}-1}{\mu-1} + \sum_{i=k+1}^{n}\sum_{j=i-k+1}^{i-1}\alpha_{ij}D_1 \\ &\quad + \sum_{i=k+1}^{n}\sum_{j=i-k+1}^{i-2}\beta_{ij}D_{2j}\Big\}(2Tt_1 - T^2\Big) + \Big\{(n-k)(1-\delta_1)(1-\beta_1)P_1 \\ &\quad + \sum_{i=k+1}^{n}\beta'_{i,i-1}D_{2(i-1)}\Big\}t_1^2 + (n-k)\alpha'_1 D_1 (t_1-T)^2\Big] + \frac{h''_c}{2}\Big[-\gamma_1\Big\{(n-k)\alpha'_1 D_1 \\ &\quad + \sum_{i=k+1}^{n}\beta_{i,i-1}D_{2(i-1)}\Big\} + \Big\{1-\gamma_1(1-\delta_2)(1-\beta_2)\Big\}P_2\frac{\lambda^{n-k}-1}{\lambda-1}\Big](t_1-T)^2 \end{aligned}$$

Therefore, total holding cost in raw material processing unit is given by

$$THC_3 = HC_{31} + HC_{31} + HCN + HCN' \tag{9.9}$$

9.3.4 Formulation of Disposal Units

For 1st cycle:
The differential equation in the disposal units during $[0, T]$ is given by

$$\frac{dq_{41}}{dt} = \begin{cases} (1-\gamma_1)[\alpha_{11}D_1 + (1-\delta_1)(1-\beta_1)P_1], & 0 \le t \le t_1 \\ (1-\gamma_1)[\alpha'_{11}D_1 + \alpha_{11}D_1 + \beta_{11}D_{21} + (1-\delta_2)(1-\beta_2)P_{21}], & t_1 \le t \le T \end{cases}$$

$$\begin{aligned} Q_{41} &= (1-\gamma_1)\Big\{\alpha_1 D_1 + (1-\delta_1)(1-\beta_1)P_1\Big\}t_1 \\ &\quad +(1-\gamma_1)\Big\{\alpha'_{11}D_1 + \alpha_1 D_1 + \alpha_2 D_2 + (1-\delta_2)(1-\beta_2)P_2\Big\}(T-t_1) \\ &= (1-\gamma_1)\Big[\alpha_1 D_1 T + (1-\delta_1)(1-\beta_1)P_1 t_1 \\ &\quad +\Big\{\alpha'_1 D_1 + \alpha_2 D_2 + (1-\delta_2)(1-\beta_2)P_2\Big\}(T-t_1)\Big] \end{aligned}$$

For 2nd cycle :
The differential equation in the the disposal units during $[T, 2T]$ is given by

$$\frac{dq_{42}}{dt} = \begin{cases} (1-\gamma_1)[\alpha_{21}D_1 + \alpha_{22}D_1 + (1-\delta_1)(1-\beta_1)P_1 \\ +\beta_{21}D_{21} + \beta'_{21}D_{21}], & T \le t \le T+t_1 \\ (1-\gamma_1)[\alpha_{21}D_1 + \alpha_{22}D_1 + \alpha'_{22}D_1 + \beta_{21}D_{21} \\ +\beta_{22}D_{22} + (1-\delta_2)(1-\beta_2)P_{22}], & T+t_1 \le t \le 2T \end{cases}$$

$$
\begin{aligned}
Q_{42} &= (1-\gamma_1)\Big\{\alpha_{21}D_1+\alpha_{22}D_1+(1-\delta_1)(1-\beta_1)P_1+\beta_{21}D_{21}+\beta^{'}_{21}D_{21}\Big\}t_1\\
&+(1-\gamma_1)\Big\{(\alpha_{21}+\alpha_{22}+\alpha'_{22})D_1+\beta_{21}D_{21}+\beta_{22}D_{22}+(1-\delta_2)(1-\beta_2)P_{22}\Big\}(T-t_1)\\
&= (1-\gamma_1)\Big[\Big\{\alpha_1D_1+\alpha_2D_2+\alpha_{21}D_1\Big\}T+\Big\{\alpha'_2D_2+(1-\delta_1)(1-\beta_1)P_1\Big\}t_1\\
&+\Big\{\alpha'_1D_1+(1-\delta_2)(1-\beta_2)P_{22}\Big\}(T-t_1)\Big]
\end{aligned}
$$

For ith cycle ($i=3,4,\ldots,k$):
The differential equation in the the disposal units during $[(i-1)T, iT]$ is given by

$$
\frac{dq_{4i}}{dt}=\begin{cases}(1-\gamma_1)\Big[\sum_{j=1}^{i}\alpha_{ij}D_1+(1-\delta_1)(1-\beta_1)P_1\\ +\sum_{j=1}^{i-2}\beta_{ij}D_{2j}+\beta_{i(i-1)}D_{2(i-1)}+\beta'_{i(i-1)}D_{2(i-1)}\Big], & (i-1)T\le t\le (i-1)T+t_1\\ (1-\gamma_1)\Big[\sum_{j=1}^{i}\alpha_{ij}D_1+\alpha^{'}_{ii}D_1+\sum_{j=1}^{i-1}\beta_{ij}D_{2j}\\ +\beta_{ii}D_{2i}+(1-\delta_2)(1-\beta_2)P_{2i}\Big], & (i-1)T+t_1\le t\le iT\end{cases}
$$

$$
\begin{aligned}
Q_{4i} &= (1-\gamma_1)\Big[\alpha_1D_1+\alpha_2D_{2i}+\alpha'_2D_{2(i-1)}+\sum_{j=1}^{i-1}\alpha_{ij}D_1\\
&+\sum_{j=1}^{i-2}\beta_{ij}D_{2j}+(1-\delta_1)(1-\beta_1)P_1\Big]t_1+(1-\gamma_1)\Big[\alpha_1D_1+\alpha_2D_{2i}+\alpha'_1D_1\\
&+\sum_{j=1}^{i-1}\alpha_{ij}D_1+\sum_{j=1}^{i-1}\beta_{ij}D_{2j}+(1-\delta_2)(1-\beta_2)P_{2i}\Big](T-t_1)\\
&= (1-\gamma_1)\Big[\Big\{\alpha_1D_1+\alpha_2D_{2i}+\sum_{j=1}^{i-1}\alpha_{ij}D_1+\sum_{j=1}^{i-2}\beta_{ij}D_{2j}\Big\}T+\Big\{(1-\delta_1)(1-\beta_1)P_1\\
&+\alpha'_2D_{2(i-1)}\Big\}t_1+\Big\{\alpha'_1D_1+\beta_{i,i-1}D_{2(i-1)}+(1-\delta_2)(1-\beta_2)P_{2i}\Big\}(T-t_1)\Big]
\end{aligned}
$$

For ith cycle ($i=k+1,k+2,\ldots,n$):
The differential equation in the the disposal units during $[(i-1)T, iT]$ is given by

$$
\frac{dq_{4i}}{dt}=\begin{cases}(1-\gamma_1)\Big[\sum_{j=i-k+1}^{i}\alpha_{ij}D_1+(1-\delta_1)(1-\beta_1)P_1\\ +\sum_{j=i-k+1}^{i-2}\beta_{ij}D_{2j}\beta_{i(i-1)}D_{2(i-1)}+\beta^{'}_{i,i-1}D_{2(i-1)}\Big], & (i-1)T\le t\le (i-1)T+t_1\\ (1-\gamma_1)\Big[\sum_{j=i-k+1}^{i}\alpha_{ij}D_1+\alpha'_{ii}D_1\\ +\sum_{j=i-k+1}^{i-1}\beta_{ij}D_{2j}+\beta_{ii}D_{2i}+(1-\delta_2)(1-\beta_2)P_{2i}\Big], & (i-1)T+t_1\le t\le iT\end{cases}
$$

$$\begin{aligned}
Q_{4i} &= (1-\gamma_1)\Big[\alpha_1 D_1 + \alpha_2 D_{2i} + \alpha_2' D_{2(i-1)} + \sum_{j=i-k+1}^{i-1} \alpha_{ij} D_1 \\
&+ \sum_{j=i-k+1}^{i-2} \beta_{ij} D_{2j} + (1-\delta_1)(1-\beta_1)P_1\Big] t_1 + (1-\gamma_1)\Big[\alpha_1 D_1 + \alpha_2 D_{2i} + \alpha_1' D_1 \\
&+ \sum_{j=i-k+1}^{i-1} \alpha_{ij} D_1 + \sum_{j=i-k+1}^{i-1} \beta_{ij} D_{2j} + (1-\delta_2)(1-\beta_2)P_2\Big](T-t_1) \\
&= (1-\gamma_1)\Big[\Big\{\alpha_1 D_1 + \alpha_2 D_{2i} + \sum_{j=i-k+1}^{i-1} \alpha_{ij} D_1 + \sum_{j=i-k+1}^{i-2} \beta_{ij} D_{2j}\Big\} T + \Big\{\alpha_2' D_{2(i-1)} \\
&+ (1-\delta_1)(1-\beta_1)P_1\Big\} t_1 + \Big\{\alpha_1' D_1 + \beta_{i,i-1} D_{2(i-1)} + (1-\delta_2)(1-\beta_2)P_2\Big\}(T-t_1)\Big] \\
Q_4 &= \text{Total disposal amount during [0, nT]} \\
&= Q_{41} + Q_{42} + \sum_{i=3}^{k} Q_{4i} + \sum_{i=k+1}^{n} Q_{4i} \\
&= (1-\gamma_1)\Big[n\alpha_1 D_1 T + n(1-\delta_1)(1-\beta_1)P_1 t_1 + \Big\{\alpha_2 D_2 \frac{\mu^{n-1}-1}{\mu-1} \\
&+ \sum_{i=3}^{k}\sum_{j=1}^{i-1} \alpha_{ij} D_1 + \sum_{i=k+1}^{n}\sum_{j=i-k+1}^{i-1} \alpha_{ij} D_1 + \sum_{i=3}^{k}\sum_{j=1}^{i-1} \beta_{ij} D_{2j} + \sum_{i=k+1}^{n}\sum_{j=i-k+1}^{i-2} \beta_{ij} D_{2j}\Big\} T \\
&+ t_1 \sum_{i=2}^{n} \alpha_2' D_{2(i-1)} + \Big\{n\alpha_1' D_1 + \sum_{i=3}^{n} \alpha_2' D_{2(i-1)} + (1-\delta_2)(1-\beta_2)P_2 \frac{\lambda^n - 1}{\lambda - 1}\Big\}(T-t_1)\Big]
\end{aligned}$$

Transportation cost to transport the disposal amount Q_4 for landfill= $c_t Q_4$.
Total GHG emission from landfill= $ER_g Q_4$.
Total cost to control GHG emission from landfill= $c_g ER_g Q_4$.

9.4 Objectives of the Proposed Model

The total profit of the system is given by

$TP = TSR_{p1} + TSR_{p2} - TC_{p1} - TC_{p2} - THC_3 - c_g ER_g Q_4 - c_t Q_4$

Total GHG emission from the system is given by

$$Z_g = n\epsilon_g P_1^{\eta_{1g}} t_1 + \epsilon_g (T-t_1) \sum_{i=1}^{n} P_{2i}^{\eta_{2g}} + ER_g Q_4$$

Lemma 9.1. *For $0 < \alpha_1 < 1$, $h > 1$ and positive integer k, the following relation holds:* $\alpha_1(h^k - 1) \leq (h-1), \quad i = 1, 2, \ldots, n-k+1.$ *and* $\alpha_1(h^{n-i+1} - 1) \leq (h-1), \quad i = n-k+2, \ldots, n-1.$

Proof. $\alpha_{ri}D =$ Return rate of items from rth cycle to the ith cycle.

Now $\sum_{r=i}^{k+i-1} \alpha_{ri} = \alpha_1 \frac{h^k - 1}{h-1}$, $h > 1,\ i = 1, 2, \ldots, n-k+1$ (see Appendix F).

and $\sum_{r=i}^{n} \alpha_{ri} = \alpha_1 \frac{h^{n-i+1} - 1}{h-1}$, $h > 1,\ i = n-k+2, \ldots, n-1$ (see Appendix F).

Since we consider the return of the items from a cycle to at most k consecutive cycles, so we must have

$\sum_{r=i}^{k+i-1} \alpha_{ri}D < D,\ i = 1, 2, \ldots, n-k+1$ and $\sum_{r=i}^{n} \alpha_{ri}D < D,\ i = n-k+2, \ldots, n-1$

i.e., $\sum_{r=i}^{k+i-1} \alpha_{ri} < 1,\ i = 1, 2, \ldots, n-k+1$ and $\sum_{r=1}^{n} \alpha_{ri} < 1,\ i = n-k+2, \ldots, n-1$

i.e., $\alpha_1 \frac{h^k-1}{h-1} < 1,\ i = 1, 2, \ldots, n-k+1$ and $\alpha_1 \frac{h^{n-i+1}-1}{h-1} < 1,\ i = n-k+2, \ldots, n-1$

i.e., $\alpha_1(h^k - 1) < (h-1),\ i = 1, 2, \ldots, n-k+1$ and $\alpha_1(h^{n-i+1} - 1) < (h-1),\ i = n-k+2, \ldots, n-1$

Hence the proof. ☐

Lemma 9.2. *For $0 < \alpha_2 < 1$, $h' > 1$ and positive integer k, the following relation holds:* $\alpha_2(h'^k - 1) \leq (h' - 1),\ \ i = 1, 2, \ldots, n-k+1$ *and* $\alpha_2(h'^{n-i+1} - 1) \leq (h' - 1),\ \ i = n-k+2, \ldots, n-1$.

Proof. Similar to Lemma 9.1. ☐

Lemma 9.3. *For positive integer k, $0 < \alpha_1, \alpha_2 < 1$, and $h,\ h' > 1$ then optimal value of k is given by $k = min\{[X_1], [X_2]\}$, where $X_1 = \frac{\log(1+\frac{h-1}{\alpha_1})}{\log h}$, $X_2 = \frac{\log(1+\frac{h'-1}{\alpha_2})}{\log h'}$ and [ ] denote the greatest integer function.*

Proof. From Lemma 9.1, we get

$$\alpha_1(h^k - 1) < (h-1)\ i.e.,\ \alpha_1 < \frac{(h-1)}{(h^k - 1)},\ i = 1, 2, \ldots, n-k+1. \tag{9.10}$$

$$\text{and } \alpha_1(h^{n-i+1} - 1) < (h-1)\ i.e.,\ \alpha_1 < \frac{(h-1)}{(h^{n-i+1} - 1)},\ i = n-k+2, \ldots, n-1 \tag{9.11}$$

Substituting $i = (n-k+2), (n-k+3), \ldots, (n-2), (n-1)$ in (9.11), we get respectively

$$\alpha_1 < \frac{(h-1)}{(h^{k-2} - 1)}, \alpha_1 < \frac{(h-1)}{(h^{k-3} - 1)}, \ldots, \alpha_1 < \frac{(h-1)}{(h^3 - 1)}, \alpha_1 < \frac{(h-1)}{(h^2 - 1)} \tag{9.12}$$

Since $\frac{(h-1)}{(h^k-1)} < \frac{(h-1)}{(h^{k-2}-1)} < \frac{(h-1)}{(h^{k-3}-1)} < \cdots < \frac{(h-1)}{(h^2-1)}$, so from (9.10), we gate
$\alpha_1 < \frac{(h-1)}{(h^k-1)}$ i.e., $k < X_1$, where $X_1 = \frac{1}{\log h}\log(1+\frac{h-1}{\alpha_1})$
Again from Lemma 9.2, we get

$$\alpha_2(h'^k - 1) < (h'-1) \; i.e., \alpha_2 < \frac{(h'-1)}{(h'^k-1)}, \; i = 1, 2, \ldots, n-k+1 \tag{9.13}$$

$$\text{and } \alpha_2(h'^{n-i+1} - 1) < (h'-1) \; i.e., \alpha_2 < \frac{(h'-1)}{(h'^{n-i+1}-1)}, \; i = n-k+2, \ldots, n-1 \tag{9.14}$$

Substituting $i = (n-k+2), (n-k+3), \ldots, (n-2), (n-1)$ in (9.14), we get respectively

$$\alpha_2 < \frac{(h'-1)}{(h'^{k-2}-1)}, \; \alpha_2 < \frac{(h'-1)}{(h'^{k-3}-1)}, \ldots, , \; \alpha_2 < \frac{(h'-1)}{(h'^3-1)}, \; \alpha_2 < \frac{(h'-1)}{(h'^2-1)} \tag{9.15}$$

Since $\frac{(h'-1)}{(h'^k-1)} < \frac{(h'-1)}{(h'^{k-2}-1)} < \frac{(h'-1)}{(h'^{k-3}-1)} < \cdots < \frac{(h'-1)}{(h'^2-1)}$, so from (9.14), we get
$\alpha_2 < \frac{(h'-1)}{(h'^k-1)}$ i.e., $k < X_2$, where $X_2 = \frac{1}{\log h'}\log(1+\frac{h'-1}{\alpha_2})$
Since $k < X_1$, $k < X_2$ and k is positive integer, so the optimal value of k is $min\{[X_2], [X_2]\}$.
Hence the proof. □

Lemma 9.4. *If* $k_1 = \beta_1 + \delta_1(1-\beta_1)$ *and* $k_2 = \beta_2 + \delta_2(1-\beta_2)$ *then the positive real parameters* α_1, α_2*,* $\beta_1, \beta_2, \delta_1, \delta_2$*,* D_1, D_2, P_1, P_2 *satisfying the following relations* $\frac{D_1}{P_1} < k_1$*,* $\frac{D_2}{P_2} < k_2$ *and* $\frac{D_1}{k_1P_1} + \frac{D_2}{k_2P_2} = 1$

Proof. From equation (9.1), we get
$\beta_1P_1 + \delta_1(1-\beta_1)P_1 - D_1 > 0$ i.e., $\{\beta_1 + \delta_1(1-\beta_1)\}P_1 > D_1$ i.e., $\frac{D_1}{P_1} < k_1$
For $i = 1$, we have from equation (9.3),
$\beta_2P_2 + \delta_2(1-\beta_2)P_2 - D_2 > 0$ i.e., $\{\beta_2 + \delta_2(1-\beta_2)\}P_2 > D_2$ i.e., $\frac{D_2}{P_2} < k_2$
From equation (9.2), we get
$\{\beta_1 + \delta_1(1-\beta_1)\}P_1t_1 = D_1T$ i.e., $k_1P_1t_1 = D_1T$ i.e., $\frac{t_1}{T} = \frac{D_1}{k_1P_1}$
For $i = 1$, we have from equation (9.4), we get
$\{\beta_2 + \delta_2(1-\beta_2)\}P_2(T-t_1) = D_2T$ i.e., $k_2P_2(T-t_1) = D_2T$ i.e., $1 - \frac{t_1}{T} = \frac{D_2}{k_2P_2}$ i.e., $1 - \frac{D_1}{k_1P_1} = \frac{D_2}{k_2P_2}$ i.e., $\frac{D_1}{k_1P_1} + \frac{D_2}{k_2P_2} = 1$. Hence the proof. □

Lemma 9.5. *If the positive real parameters* λ *and* μ *satisfying the following relations* $P_{2i} = \lambda P_{2(i-1)}$*,* $D_{2i} = \mu D_{2(i-1)}$*,* $i = 1, 2, \ldots, k$ *then* $\lambda = \mu$ *holds.*

Proof. From the continuity condition (9.4), we get

$\{\beta_2 + \delta_2(1-\beta_2)\}P_{2i}(T-t_1) = D_{2i}T$, $i = 1, 2, \ldots, n$

For $i = 1$, $k_2(1-\frac{t_1}{T})P_2 = D_2$, where $k_2 = \{\beta_2 + \delta_2(1-\beta_2)\}$

Also $k_2(T-t_1)\lambda^{i-1}P_2 = \mu^{i-1}D_2T$, $i = 2, 3, \ldots, k$

$\Rightarrow (\frac{\lambda}{\mu})^{i-1}D_2 = D_2 \ \Rightarrow (\frac{\lambda}{\mu})^{i-1} = 1$, $i = 1, 2, \ldots, k$

$\Rightarrow \lambda = \mu$. Hence the proof. □

Lemma 9.6. *P_2 and λ satisfying the following relations:*

(i) $\{1-\gamma_1(1-\delta_2)(1-\beta_2)\}P_2 > \frac{\gamma_1}{\lambda^{k-1}}[\alpha_1 D_1\sum_{i=1}^{k} h^{i-1} + \alpha_1' D_1 \alpha_2 D_2 \sum_{i=1}^{k} h_1^{i-1}\mu^{k-1}]$ *when* $\lambda \geq 1 + \frac{\alpha_1 D_1 h + \alpha_2 D_2 h_1}{(\alpha_1+\alpha_1')D_1}$

and (ii) $\{1-\gamma_1(1-\delta_2)(1-\beta_2)\}P_2 < \gamma_1[\alpha_1 D_1 + \alpha_1' D_1 + \alpha_2 D_2]$ *when* $\lambda \leq 1 + \frac{\alpha_1 D_1 h^{k-1} + \alpha_2 D_2 h_1^k}{\{(1+h+h^2+\cdots+h^{k-1})\alpha_1+\alpha_1'\}D_1}$.

Proof. From equation (9.5), we get

$\gamma_1\{\alpha_{11}' D_1 + \alpha_{11}D_1 + \beta_{11}D_{21} + (1-\delta_2)(1-\beta_2)P_{21}\} - P_{21} < 0$

i.e., $\{1-\gamma_1(1-\delta_2)(1-\beta_2)\}P_2 > \gamma_1\{\alpha_1' D_1 + \alpha_1 D_1 + \alpha_2 D_2\}$

i.e., $MP_2 > g_1$, where $M = \{1-\gamma_1(1-\delta_2)(1-\beta_2)\}$ and $g_1 = \gamma_1\{\alpha_1' D_1 + \alpha_1 D_1 + \alpha_2 D_2\}$.

From equation (9.6), we get

$\gamma_1\{\alpha_{21}D_1 + \alpha_{22}D_1 + \alpha_{22}'D_1 + \beta_{21}D_{21} + \beta_{22}D_{22} + (1-\delta_2)(1-\beta_2)P_{22}\} - P_{22} < 0$

i.e., $\{1-\gamma_1(1-\delta_2)(1-\beta_2)\}P_2 > \frac{\gamma_1}{\lambda}\{\alpha_1 D_1 \sum_{i=1}^{2} h^{i-1} + \alpha_1' D_1 + \alpha_2 D_2 \sum_{i=1}^{2} h_1^{i-1}\mu^{2-i}\}$

i.e., $MP_2 > g_2$, where $g_2 = \frac{\gamma_1}{\lambda}\{\alpha_1 D_1 \sum_{i=1}^{2} h^{i-1} + \alpha_1' D_1 + \alpha_2 D_2 \sum_{i=1}^{2} h_1^{i-1}\mu^{2-i}\}$.

Put $i = 3$ in equation (9.7), we get

$\gamma_1\{\sum_{j=1}^{3}\alpha_{3j}D_1 + \alpha_{33}'D_1 + \sum_{j=1}^{2}\beta_{3j}D_{3j} + \beta_{33}D_2 + (1-\delta_2)(1-\beta_2)P_{23}\} - P_{23} < 0$

i.e., $\{1-\gamma_1(1-\delta_2)(1-\beta_2)\}P_2 > \frac{\gamma_1}{\lambda^2}\{\alpha_1 D_1 \sum_{i=1}^{3} h^{i-1} + \alpha_1' D_1 + \alpha_2 D_2 \sum_{i=1}^{3} h_1^{i-1}\mu^{3-i}\}$

i.e., $MP_2 > g_2$, where $g_3 = \frac{\gamma_1}{\lambda^2}\{\alpha_1 D_1 \sum_{i=1}^{3} h^{i-1} + \alpha_1' D_1 + \alpha_2 D_2 \sum_{i=1}^{3} h_1^{i-1}\mu^{3-i}\}$.

Proceeding in the way, if we put $i = k$ in equation (9.7) we get

$MP_2 > g_k$ where $g_k = \frac{\gamma_1}{\lambda^{k-1}}\{\alpha_1 D_1 \sum_{i=1}^{k} h^{i-1} + \alpha_1' D_1 + \alpha_2 D_2 \sum_{i=1}^{k} h_1^{i-1}\mu^{k-i}\}$.

Put $i = k+1$ in equation (9.7), we get

$\gamma_1\{\sum_{j=2}^{k+1}\alpha_{(k+1)j}D_1 + \alpha_{(k+1)(k+1)}'D_1 + \sum_{j=2}^{k}\beta_{(k+1)j}D_{2j} + \beta_{(k+1)(k+1)}D_{2(k+1)} + (1-\delta_2)(1-\beta_2)P_{2(k+1)}\} - P_{2(k+1)} < 0$

i.e., $MP_2 > g_{(k+1)}$, where $g_{(k+1)} = \frac{\gamma_1}{\lambda_1\lambda^{(k+1)}}\{\alpha_1 D_1 \sum_{i=1}^{k} h^{i-1} + \alpha_1' D_1\} + \frac{\gamma_1\alpha_2 D_2}{\lambda^{(k+1)}}(\frac{\mu}{\lambda_1})\{\sum_{i=1}^{k} h_1^{i-1}\mu^{k-i} - \mu^{k-2}(\mu-\mu_1)\}$.

Put $i = k+2$ in equation (9.7), we get

$\gamma_1\{\sum_{j=3}^{k+2}\alpha_{(k+2)j}D_1 + \alpha'_{(k+2)(k+2)}D_1 + \sum_{j=3}^{k+1}\beta_{(k+2)j}D_{2j} + \beta_{(k+2)(k+2)}D_{2(k+2)} + (1-\delta_2)(1-\beta_2)P_{2(k+2)}\} - P_{2(k+2)} < 0$

i.e., $MP_2 > g_(k+2)$, where $g_(k+2) = \frac{\gamma_1}{\lambda_1^2\lambda^(k-1)}\{\alpha_1 D_1 \sum_{i=1}^{k} h^{i-1} + \alpha_1' D_1\} + \frac{\gamma_1\alpha_2 D_2}{\lambda^(k-1)}(\frac{\mu}{\lambda_1})^2[\sum_{i=1}^{k} h_1^{i-1}\mu^{k-i} - \mu^{k-3}\{h_1(\mu-\mu_1) + (\mu^2-\mu_1^2)\}]$.

Proceeding in the way, if we put $i = n$ in equation (9.7) we get

$MP_2 > g_n$ where $g_n = \frac{\gamma_1}{\lambda_1^{n-k}\lambda^{k-1}}\{\alpha_1 D_1 \sum_{i=1}^{k} h^{i-1} + \alpha_1' D_1\} + \frac{\gamma_1\alpha_2 D_2}{\lambda^(k-1)}(\frac{\mu}{\lambda_1})^{n-k}\{\sum_{i=1}^{k} h_1^{i-1}\mu^{k-i} - \mu^{2k-n-1}\sum_{i=1}^{n-k} h_1^{n-k-i}(\mu^i - \mu_1^i)\}$.

Case-I: When $\lambda \geq 1 + \frac{\alpha_1 D_1 h + \alpha_2 D_2 h_1}{(\alpha_1+\alpha_1')D_1}$ then we have

$g_1 \leq g_2 \leq g_3 \leq \cdots \leq g_{k-1} \leq g_k \geq g_{k+1} \geq g_{k+2} \geq \cdots \geq g_n$

Therefore $g_k = max\{g_1, g_2, \ldots, g_n\}$

Hence g_k is maximum when $\lambda \geq 1 + \frac{\alpha_1 D_1 h + \alpha_2 D_2 h_1}{(\alpha_1+\alpha_1')D_1}$

Here $MP_2 > g_1, MP_2 > g_2, \ldots, MP_2 > g_n$ gives $MP_2 > max\{g_1, g_2, \ldots, g_n\}$

$\Rightarrow \; MP_2 > g_k$ when $\lambda \geq 1 + \frac{\alpha_1 D_1 h + \alpha_2 D_2 h_1}{(\alpha_1+\alpha_1')D_1}$

$\Rightarrow \; \{1-\gamma_1(1-\delta_2)(1-\beta_2)\}P_2 > \frac{\gamma_1}{\lambda^{k-1}}[\alpha_1 D_1 \sum_{i=1}^{k} h^{i-1} + \alpha_1' D_1 + \alpha_2 D_2 \sum_{i=1}^{k} h_1^{i-1}\mu^{k-1}]$

when $\lambda \geq 1 + \frac{\alpha_1 D_1 h + \alpha_2 D_2 h_1}{(\alpha_1+\alpha_1')D_1}$. Hence the proof.

Case-II: when $\lambda \leq 1 + \frac{\alpha_1 D_1 h^{k-1} + \alpha_2 D_2 h_1^k}{\{(1+h+h^2+\cdots+h^{k-1})\alpha_1 + \alpha_1'\}D_1}$ then we have

$g_n \leq g_{n-1} \leq \cdots \leq g_{k+1} \leq g_k \geq g_{k-1} \geq g_{k-2} \geq \cdots \geq g_1$

Therefore $g_k = min\{g_1, g_2, \ldots, g_n\}$

Hence g_k is minimum when $\lambda \leq 1 + \frac{\alpha_1 D_1 h^{k-1} + \alpha_2 D_2 h_1^k}{\{(1+h+h^2+\cdots+h^{k-1})\alpha_1 + \alpha_1'\}D_1}$

Here $MP_2 > g_1, MP_2 > g_2, \ldots, MP_2 > g_n$ gives $MP_2 > min\{g_1, g_2, \ldots, g_n\}$

$\Rightarrow \; MP_2 > g_1$ when $\lambda \leq 1 + \frac{\alpha_1 D_1 h^{k-1} + \alpha_2 D_2 h_1^k}{\{(1+h+h^2+\cdots+h^{k-1})\alpha_1 + \alpha_1'\}D_1}$

$\Rightarrow \; \{1-\gamma_1(1-\delta_2)(1-\beta_2)\}P_2 < \gamma_1[\alpha_1 D_1 + \alpha_1' D_1 + \alpha_2 D_2]$ when $\lambda \leq 1 + \frac{\alpha_1 D_1 h^{k-1} + \alpha_2 D_2 h_1^k}{\{(1+h+h^2+\cdots+h^{k-1})\alpha_1 + \alpha_1'\}D_1}$. Hence the proof. □

9.4.1 Model in Crisp Environment

Maximize TP
Minimize Z_g
subject to the conditions,

$$[\beta_1 P_1 + \delta_1(1-\beta_1)P_1]t_1 = D_1 T \tag{9.16}$$
$$\Big[\beta_2 + \delta_2(1-\beta_2)\Big](T-t_1)P_{2i} = D_{2i}T, \quad i = 1,2,\ldots,n. \tag{9.17}$$
$$\alpha_1(h^k - 1) < (h-1), \; i = 1,2,\ldots,n-k+1 \tag{9.18}$$
$$\alpha_1(h^{n-i+1} - 1) < (h-1), \; i = n-k+2,\ldots,n-1. \tag{9.19}$$
$$\alpha_2(h'^k - 1) < (h'-1), \; i = 1,2,\ldots,n-k+1 \tag{9.20}$$
$$\alpha_2(h'^{n-i+1} - 1) < (h'-1), \; i = n-k+2,\ldots,n-1 \tag{9.21}$$
$$n, k, \lambda, t_1 > 0$$

9.4.2 Model in Fuzzy Environment

Here we take the parameters ϵ_g, ϵ_w and ER_g as interval valued fuzzy number. Then the above problem in fuzzy environment is
Maximize $\widetilde{TP} = TSR_{p1} + TSR_{p2} - \widetilde{TC}_{p1} - \widetilde{TC}_{p2} - THC_3 - c_g\widetilde{ER}_g Q_4 - c_t Q_4$
Minimize $\widetilde{Z}_g = n\tilde{\epsilon}_g P_1^{\eta_{1g}} t_1 + \tilde{\epsilon}_g (T-t_1)\sum_{i=1}^{n} P_{2i}^{\eta_{2g}} + \widetilde{ER}_g Q_4$
subject to, the conditions stated above in (9.16) to (9.21), where
$\widetilde{TC}_{p1} = n\Big\{c_p + r_c\delta_1(1-\beta_1) + c_{sr}\Big\}P_1 t_1 + THC_1 + nA_s + nc_w\tilde{\epsilon}_w P_1^{\eta_{1w}} t_1 + nc_g\tilde{\epsilon}_g P_1^{\eta_{1g}} t_1$
$\widetilde{TC}_{p2} = TPC_2 + RWC_2 + TSC_2 + THC_2 + nA'_s + c_w\tilde{\epsilon}_w (T-t_1)\sum_{i=1}^{n} P_{2i}^{\eta_{2g}}$
$+c_g\tilde{\epsilon}_g (T-t_1)\sum_{i=1}^{n} P_{2i}^{\eta_{2g}}$.

Taking $\tilde{\epsilon}_w = [\tilde{\epsilon}_w^L, \tilde{\epsilon}_w^R]$, $\tilde{\epsilon}_g = [\tilde{\epsilon}_g^L, \tilde{\epsilon}_g^R]$, $\widetilde{ER}_g = [\widetilde{ER}_g^L, \widetilde{ER}_g^R]$, we have
$\widetilde{TC}_{p1} = [\widetilde{TC}_{p1}^L, \widetilde{TC}_{p1}^R]$, $\widetilde{TC}_{p2} = [\widetilde{TC}_{p2}^L, \widetilde{TC}_{p2}^R]$, $\widetilde{Z}_g = [\widetilde{Z}_g^L, \widetilde{Z}_g^R]$. (see Appendix F)

Then the above multi-objective fuzzy interval valued problem can be stated as
Maximize $\widetilde{TP} = [\widetilde{TP}^L, \widetilde{TP}^R]$
Minimize $\widetilde{Z}_g = [\tilde{Z}_g^L, \tilde{Z}_g^R]$
subject to the conditions stated in (9.16) to (9.21), where
$\widetilde{TP}^L = TSR_{p1} + TSR_{p2} - \widetilde{TC}_{p1}^R - \widetilde{TC}_{p2}^R - THC_3 - c_g\widetilde{ER}_g^R Q_4 - c_t Q_4$,
$\widetilde{TP}^R = TSR_{p1} + TSR_{p2} - \widetilde{TC}_{p1}^L - \widetilde{TC}_{p2}^L - THC_3 - c_g\widetilde{ER}_g^L Q_4 - c_t Q_4$.

The interval valued multi-objective problem is transformed into following:
Maximize $\widetilde{TP} = \omega_1\widetilde{TP}^L + (1-\omega_1)\widetilde{TP}^R, \quad 0 < \omega_1 < 1$
Minimize $\tilde{Z}_g = \omega_2\tilde{z}_g^L + (1-\omega_2)\tilde{z}_g^R, \quad 0 < \omega_2 < 1$
subject to, the conditions stated in (9.16) to (9.21) (see Appendix F).

Equivalent Crisp Model
Using Lemma (2.3), above fuzzy multi-objective problem is transformed into crisp multi-objective problem:
Maximize $E[\widetilde{TP}] = \omega_1 E[\widetilde{TP}^L] + (1-\omega_1)E[\widetilde{TP}^R], \quad 0 < \omega_1 < 1$
Minimize $E[\tilde{Z}_g] = \omega_2 E[\tilde{Z}_g^L] + (1-\omega_2)E[\tilde{Z}_g^R], \quad 0 < \omega_2 < 1$
subject to, the conditions stated in (9.16) to (9.21). (see Appendix F).

9.5 Numerical Illustration

The proposed model of the production and reproduction inventory system has been developed with the help of following numerical example in this section. The values of the parameters of the model, considered in these numerical examples are not elected from any real life case study, but these values have been seems to be realistic. The example have been solved to find optimal values of n, k, λ, t_1 and T along with the optimal expected total profit of the system. To illustrate the solution, we consider an inventory system with the following input data:
Crisp input data: $\beta_1 = 0.89$, $\beta_2 = 0.85$, $\delta_1 = 0.74$, $\delta_2 = 0.75$, $\gamma_1 = 0.81$, $\alpha_1 = 0.09$, $\alpha_2 = 0.11$, $\alpha'_1 = 0.08$, $\alpha'_2 = 0.07$, $h = 1.05$, $h = 1.07$, $\eta_{1g} = 1.02$, $\eta_{2g} = 1.09$, $\eta_{1w} = 1.06$, $\eta_{2w} = 1.08$, $c_g = 1.05$, $c_w = 1.02$, $c_p = 19.0$, $c'_p = 15.0$, $c_t = 0.62$, $h_c = 1.50$, $h'_c = 1.40$, $h''_c = 1.30$, $A_s = 120.0$, $A'_s = 130.0$, $c_{sr} = 0.40$, $c'_{sr} = 0.50$, $r_c = 1.10$, $r'_c = 1.20$, $s = 38.0$, $s' = 36.0$, $P_1 = 56$, $P_2 = 51$, $D_1 = 29$, $D_2 = 26$.
Fuzzy input data: $\tilde{\epsilon}_g^L = (0.75, 0.76, 0.78)$, $\tilde{\epsilon}_g^R = (0.79, 0.81, 0.82)$, $\tilde{\epsilon}_w^L = (0.68, 0.69, 0.70)$, $\tilde{\epsilon}_w^R = (0.70, 0.71, 0.73)$, $\widetilde{ER}_g^L = (0.02, 0.03, 0.05)$, $\widetilde{ER}_g^R = (0.06, 0.08, 0.09)$.
Applying the solution procedure, the optimal solution given in the following Table 9.1.

Table 9.1: Optimal result of the illustrated model when $\rho = 0.4$, $w_1 = 0.3$, $w_2 = 0.6$

Number of cycle (n)	Number of consecutive cycles (k)	λ	Production time t_1	Time length of each cycle (T)	Expected total profit ($E[\widetilde{TP}]$)	Expected GHG emission ($E[\tilde{Z}_g]$)
8	3	1.3135	3.5250	6.6122	29658.1329	1213.7692

• Table 9.1 shows that the optimal production time of each cycle is 3.5250 unit, time length of each cycle is 6.6122 unit, expected total profit is 29658.1329 unit and expected GHG emission 1213.7692 unit.

9.5.1 Sensitivity Analysis

Using the same data as that in above Example, we next study the sensitivity of the optimal production time of each cycle, time length of each cycle, expected total profit and expected

GHG emission to change the values of the different parameters associated with the model. The computational results are reported in the following Table 9.2, 9.3 and 9.4.

Table 9.2: Sensitivity analysis of $E[\widetilde{TP}]$ and $E[\tilde{Z}_g]$ w.r.t. β_1, β_2

Parameters (β_1, β_2)	Production time (t_1)	Time length of each cycle (T)	Expected total profit ($E[\widetilde{TP}]$)	Expected GHG emission ($E[\tilde{Z}_g]$)
$\beta_1 = 0.86, \beta_2 = 0.83$	3.5128	6.5931	28514.7215	1426.3527
$\beta_1 = 0.89, \beta_2 = 0.85$	3.5250	6.6122	29658.1329	1213.7692
$\beta_1 = 0.92, \beta_2 = 0.87$	3.5364	6.6414	30142.1276	1134.5192

• Table 9.2 shows that when the fraction of the better quality item β_1 and β_2 in plant-I and plant-II are respectively increases, expected total profit ($E[\widetilde{TP}]$) increase and expected GHG emission ($E[\tilde{Z}_g]$) decrease.

Table 9.3: Sensitivity analysis of $E[\widetilde{TP}]$ and $E[\tilde{Z}_g]$ w.r.t. δ_1, δ_2, r_c, r'_c

Parameters (δ_1, δ_2)	Reworked cost (r_c, r'_c)	Production time (t_1)	Time length of each cycle (T)	Expected total profit ($E[\widetilde{TP}]$)	Expected GHG emission ($E[\tilde{Z}_g]$)
$\delta_1 = 0.70, \delta_2 = 0.72$	$r_c = 1.05, r'_c = 1.10$	3.5250	6.5813	28564.2143	1328.2314
$\delta_1 = 0.74, \delta_2 = 0.75$	$r_c = 1.10, r'_c = 1.20$	3.5250	6.6122	29658.1329	1213.7692
$\delta_1 = 0.77, \delta_2 = 0.78$	$r_c = 1.22, r'_c = 1.34$	3.5250	6.7529	31041.3597	1159.4237
$\delta_1 = 0.82, \delta_2 = 0.85$	$r_c = 1.45, r'_c = 1.50$	3.5250	6.7684	30219.0425	1046.2139

• Table 9.3 shows that when the reworked rate and average reworked cost (δ_1, r_c) and (δ_2, r'_c) in plant-I and plant-II are respectively simultaneously increase, initially expected total profit ($E[\widetilde{TP}]$) increases, after that expected total profit ($E[\widetilde{TP}]$) decrease due to some portion of defective item to be reworked with a minimum rework cost and rest portion of defective item to be reworked with a large amount rework cost. Also expected GHG emission ($E[\tilde{Z}_g]$) decrease. So any manufacturer company can find the optimal reworked rate from this study.

Table 9.4: Sensitivity analysis of $E[\widetilde{TP}]$ and $E[\tilde{Z}_g]$ w.r.t. γ_1, c'_p

Parameters (γ_1, c'_p)	Production time (t_1)	Time length of each cycle (T)	Expected total profit ($E[\widetilde{TP}]$)	Expected GHG emission ($E[\tilde{Z}_g]$)
$\gamma_1 = 0.76, c'_p = 12$	3.5214	6.5834	28746.5823	1347.8479
$\gamma_1 = 0.81, c'_p = 15$	3.5250	6.6122	29658.1329	1213.7692
$\gamma_1 = 0.84, c'_p = 19$	3.5276	6.6371	31254.8579	1140.2584
$\gamma_1 = 0.89, c'_p = 24$	3.5295	6.6497	28521.2314	1047.1725

• Table 9.4 shows that when percentage of amount of disposal unit $(1-\gamma_1)$ decreases and the corresponding raw material producing cost per unit (c_p) simultaneously increases, initially the expected total profit $(E[\widetilde{TP}])$ increases, after that the expected total profit $(E[\widetilde{TP}])$ decreases. Because at first increasing raw material producing cost per unit more increasing the amount of raw material unit. As a result, the expected total profit $(E[\widetilde{TP}])$ increases. But latter, though raw material producing cost per unit increases, the amount of raw material unit does not increase as much as previous due to more defectiveness. Also expected GHG emission $(E[\tilde{Z}_g])$ decrease.So from this study, any manufacturer company can find the optimal percentage of amount of disposal unit and the corresponding raw material producing cost per unit.

9.6 Conclusion

This chapter focused on the combined effect of imperfect production and reproduction over finite time horizon. Both the manufactured and re-manufactured used items are returned from the market and some portion of them are used as raw-materials for the re-manufacturing process to save the natural resources for future and protect environment from pollution by the used items after the end of their life cycle. Some portion of non-reworkable defective units from both the plants is also used as raw-materials of plant-II for the same purpose. Generally, the items produced in re-manufacturing process are less quality than the items produced in normal manufacturing process. So with respect to quality measure, two types of items are produced in manufacturing (plant-I) and re-manufacturing process (plant-II). In the proposed model, we maximized the expected total profit from both the plants and simultaneously minimized the GHG emission from the industrial waste during production and from the used and non-reworkable defective units by industrial waste management. Our main aim of this model is to minimize the disposal cost by minimizing the disposal amount (ISW) from both the plants so that GHG emission is less to the environment. Here, we consider the parameters relating to the environmental pollution due to industrial waste as interval valued fuzzy number since they are not fixed by nature. Finally, a numerical example has been illustrated to study the feasibility of the model. Also sensitivity analysis has been carried out to draw some managerial insights of the model.

Part V

Studies on Imperfect Production Inventory System in Fuzzy Stochastic Environment

Chapter 10

Multi-item EPQ model with learning effect on imperfect production over fuzzy-random planning horizon[1]

10.1 Introduction

It is more reasonable to describe some parameters as fuzzy variables due to unreliability and scarcity of historical data. Analogous to chance-constrained programming with stochastic parameters, in a fuzzy environment it is assumed that some constraints may be satisfied with a least optimistic and/or pessimistic level (i.e., here the 'chance' is represented by either by 'possibility' or 'necessity' see Charnes and Cooper [27, 28]). Zadeh [240], Dubois and Prade [61, 62] first introduced the necessity and possibility constraints which are very relevant to real-life decision-making problems and presented the process of defuzzification for these constraints. Recently, Das et al. [51] developed an integrated model with a new type of fuzzy credit period. **But, till now, no one has considered the imprecise parameter of the probability density function of random time horizon and the expressions are converted into a deterministic by using possibility, necessity and credibility measures following Liu and Iwamura [140].**

Normally, production inventory models are developed in an infinite planning horizon because it is assumed that the production-inventory process including thevarious inventory parameters, demand, etc., remain constant over the time horizon. In reality, it is not so due to

[1] This model published in **Journal of ManagementAnalytics, (2016), DOI 10.1080/23270012.2016.1217755,** Taylor & Francis.

several reasons such as variation in inventory costs, changes in product specifications and designs, technological changes due to environmental conditions, availability of product, etc. Moreover, for seasonal products like fruits, vegetables, warm garments, etc., the business period cannot be infinite. Das et al. [51], Palanivel & Uthayakumar [162] and many others have supported this idea. Again, the assumption of a finite planning horizon is not appropriate if it is fixed (in crisp nature); for seasonal products, it fluctuates in every season depending upon the environment effect. Moon and Yun [155] considered this type of horizon with exponential distribution for an economic order quantity (EOQ) model. Then, a lot of research work has been done in this field by several researchers (see Roy et al. [182]). **But none has considered a multi-item multi-cycle imperfect production inventory model with reworking in fuzzy random environment**.

In real-world manufacturing systems, due to reliability and/or other factors, the generation of defective items is inevitable. Practically, these defective items sometimes can be reworked at a cost to make perfect-quality items. Schrady [190] was the first researcher who concentrated on rework and remanufacturing processes. Khouja [114] considered direct rework for the economic lot sizing and delivery scheduling problem (ELDSP). Teunter [205] developed an EPQ models with rework in two policies. Widyadana and Wee [224] developed an EPQ model with multiple production setups and rework. Hence, many studies (cf. Chiu and Chiu [37], Chiu et al. [40], Jaber [100], Jamal et al., [104], Rahim and Ben-Daya [170], Taleizadeh et al. [202]) have been carried out to address the issues of imperfect production and reduction of its corresponding quality costs. **In this chapter, the rework of the imperfect product has been made, but on a certain percentage of defective amount.**

In financial market, the effect of time value of money and inflation in todays financial market is well established. Due to high inflation, the economic situation of most countries may change rigorously, and so, it is not possible to ignore the effect of inflation, because inflation causes the purchasing power of money to decline sharply. In this direction, some remarkable work has been done by Chandra and Bahner [23], Ray and Chaudhuri [173], Chen [31], Chung and Lin [43], and Yang [231]. Dey, Mondal and Maity [57] discussed a two-storage inventory problem with dynamic demand under inflation and time-value of money.

According to Dutton and Thomas [69], defect (imperfect quality) levels are high in a firm may devote additional effort to investigate the cause of defects and this leads to additional knowledge of the process which, in turn, increases both quality and productivity. Jaber et al. [101] showed that the fraction of imperfect quantity (defective items) reduces due to the learning knowledge of the system. Konstantaras et al. [123] observed an EOQ model for imperfect quality items with shortages under the learning of inspection. Inspection of several learning curve model (cf. Balkhi [7]) leads to complete an EPQ model with shortages and learning inspection over a learning random planing horizon. This chapter extends the model of Manna et al. [148] by assuming stock dependent demands of the items, shortages, learning effect on imperfect production rate in fuzzy random environment. **But, till now, none has**

considered the learning knowledge to reduce the defective rate as well as to increase the production cost, screening cost in imperfect production inventory model. Summary of related literature for multi-item EOQ/EPQ models with shortage is shown in Table 10.1.

Table 10.1: Summary of related literature for multi-item EOQ/EPQ models with shortage

Author(s) and year	EOQ /EPQ	Learning effect	Inflation	Shortage allow	Demand rate	Uncertain environment	Multi -item
Roy et al. [182]	EOQ	No	No	No	stock dependent	Yes	No
Yang et al. [231]	EOQ	No	No	Yes	stock dependent	No	No
Taleizadeh et al. [202]	EPQ	No	No	No	uniform	No	No
Dey et al. [57]	EOQ	No	Yes	No	dynamic	No	No
Moon & Yun [155]	EOQ	No	No	No	constant	No	No
Dutton & Thomas [69]	-	Yes	No	-	-	-	-
Jaber et al. [101]	EPQ	Yes	No	-	No	No	No
Chen [31]	EOQ	No	Yes	Yes	time proportional	No	
Chung & Lin [43]	EOQ	No	Yes	No	-	Yes	Yes
Schrady [190]	EPQ	No	No	No	constant	No	No
Khouja [114]	EPQ	No	No	No	constant	No	No
Widyadana & Wee [224]	EPQ	No	No	No	constant	No	Yes
Chiu & Chiu [37]	EPQ	No	No	Yes	-	No	No
Chiu et al. [40]	EPQ	No	No	No	-	No	Yes
Manna et al. [148]	EPQ	No	Yes	No	promotional	No	Yes
Present model	EPQ	Yes	Yes	Yes, make an effect on demand	stock dependent	Yes(Fuzzy and Stochastic)	Yes

So, in spite of all these developments in imperfect EPQ models, there are still some lacunas. Making the model more realistic, the main contributions in this chapter are

- The time horizon of the imperfect production inventory model is considered to be randomly distributed following the exponential distribution with an imprecise parameter.
- The production process is defective in nature, the rate of defective units is reduced due to the learning effect of the employer, and the defective units are reworked at a constant rate.
- The model is formulated with an imprecise space constraint which is defuzzyfied using possibility, necessity and credibility measures.
- The model maximizes the expected profit in different uncertainty measurements following a simulating genetic algorithm (GA).
- Finally, the model is illustrated through a numerical example for different levels of possibility, necessity and credibility measures. The obtained results are verified through sensitivity analysis.

10.2 Notations and assumptions

The proposed model have been developed on the basis of following notations and assumptions:

10.2.1 Notations

The following notations have been used to develop the chapter (for the j-th item).

$q_i^j(t)$	:	On hand inventory of ith cycle at time t for perfect quality, $(i = 1, 2, ..., N^j)$.
$q_L^j(t)$	:	On hand inventory of last cycle at time t for perfect quality.
$S^j(t)$	:	Shortage for perfect quality items.
Q^j	:	Maximum inventory level.
P^j	:	Production rate.
D^j	:	Demand rate of customers.
T^j	:	Duration of a complete cycle.
t_1^j	:	Duration of production in each cycle.
t_2^j	:	Duration of exhaust of inventory in each cycle.
α^j	:	Increasing learning parameter.
β^j	:	Decreasing learning parameter.
$\theta^j e^{-(i-1)\alpha^j}$	:	Defective rate of items in ith cycle.
δ^j	:	Percentage of disposal from imperfect quality items.
$c_p^j e^{(i-1)\beta^j}$	:	Production cost per unit in ith cycle.
$c_{sr}^j e^{(i-1)\beta^j}$	:	Screening cost per unit item in ith cycle.
r_c^j	:	Average reworking cost per unit item.
h_c^j	:	The inventory holding cost per unit time for perfect quality.
s^j	:	Selling price per unit for perfect quality.
N^j	:	Number of fully accommodated cycles to be made during the prescribed time horizon.
M	:	Total number of items.
ρ^j	:	Space required for storing unit jth item.
$\widetilde{W}$	:	Total space available in the system (fuzzy parameter).
R	:	Difference between inflation rate and time value of money.
H	:	The length of time horizon.

10.2.2 Assumptions

The proposed model based on the following assumptions:

(i) In this production system, multiple items are produced with imperfect quality also, the perfect quality items are directly ready for sale and imperfect quality items are reworked to make as good as perfect.

(ii) Demand rate (D^j) of jth items $(j = 1, 2, ..., M)$ is assumed to be depend on the displayed stock level, which is of the form:

$$D^j(q^j) = \begin{cases} a^j + b^j q^j(t), & q^j(t) \geq 0 \\ a^j - b^j S^j(t), & q^j(t) \leq 0 \end{cases} \tag{10.1}$$

where, a^j and b^j are positive constants.

(iii) The time horizon H is finite and randomly distributed. Here, it is assumed that H follows an exponential distribution with the following probability density function (p.d.f)

$$\widetilde{f}(h) = \begin{cases} \widetilde{\lambda} e^{-\widetilde{\lambda} h}, & h \geq 0 \\ 0, & otherwise \end{cases} \tag{10.2}$$

where $\widetilde{\lambda}(> 0)$, is an imprecise parameter, $E[\widetilde{\lambda}] = \bar{\lambda}$ and h is the real value of H.

(iv) The time horizon completely accommodates first N^j cycles and end during $(N^j + 1)$th cycle.

(v) Lead time is negligible and stock-out is allowed, where the customers in stock out period are bulk away.

(vi) The initial and terminal inventory levels in each cycle are zero.

(vii) The effects of inflation and time value of money have been considered.

(viii) The manual inspection of the process reduces the defective rate of the items due to learning effect of the inspectors. This higher learning effect influence the decision maker to charge more production cost and screening cost to the next cycle.

(viii) After occurring shortage, due to substitute item, the customer will not wait in the stock out period. So, the shortage during stock out period is not back logging.

10.3 Mathematical Formulation of the Proposed Model

In the development of the model, we consider that there are N^j full cycles during the random time horizon H. In this case, the initial stock of the cycle is zero and it starts production with rate P^j. As the production continues screening and reworking, the inventory begins to pile up continuously after meeting the demand with the rate D^j. Production and reworking of the cycle stop at time t_1^j and restart at time T^j. The quantity received at $(i-1)T^j$ is used partly to meet the accumulated back-orders in the previous cycle. The inventory at $(i-1)T^j$ gradually reduces to zero at $(i-1)T^j + t_2^j$. Also in this production process, both perfect and imperfect quality items are produced. The production process is 100% screened during the production run time. The perfect quality items are ready for sell. When the imperfect items are found, some imperfect quality products (which are reworkable at a minimum cost) are continuously reworked during the production run time to make as perfect and restore its to the original quality. The employees learning effort reduce the defective rate θ^j and influence the decision maker to increase the production and screening cost to the next cycle.

10.3.1 Formulation for ith $(1 \leq i \leq N^j)$ Cycle for jth Item

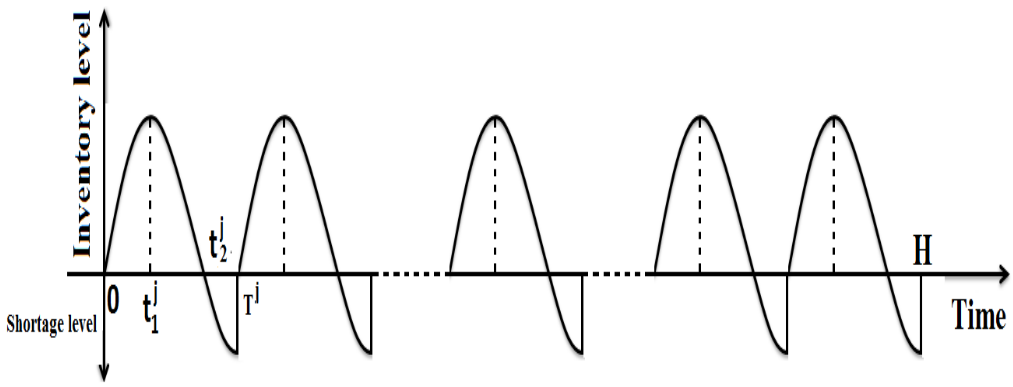

Figure 10.1: Logistic diagram of inventory model for first N^j cycles

The differential equation to describe the inventory level $q^j(t)$ in the interval $[(i-1)T^j, (i-1)T^j + t_2^j]$ is given by

$$\frac{dq^j(t)}{dt} = \begin{cases} \{1 - \delta^j\theta^j e^{-(i-1)\alpha^j}\}P^j - D^j, & (i-1)T^j \leq t \leq (i-1)T^j + t_1^j \\ -D^j, & (i-1)T^j + t_1^j \leq t \leq (i-1)T^j + t_2^j \end{cases}$$
$$= \begin{cases} \{1 - \delta^j\theta^j e^{-(i-1)\alpha^j}\}P^j - a^j - b^j q^j(t), & (i-1)T^j \leq t \leq (i-1)T^j + t_1^j \\ -a^j - b^j q^j(t), & (i-1)T^j + t_1^j \leq t \leq (i-1)T^j + t_2^j \end{cases}$$

where δ^j is the disposal part of the defective units $\theta^j P^j e^{-(i-1)\alpha^j}$ in the ith cycle, this unit is reduced with every shipment in conformance with learning parameter α^j.
The variation of inventory level $q^j(t)$ over the time interval $[(i-1)T^j, (i-1)T^j + t_2^j]$ satisfies the conditions $q^j[(i-1)T^j] = 0$ and $q^j[(i-1)T^j + t_2^j] = 0$.
During the time interval $[(i-1)T^j + t_2^j, iT^j]$, the demand rate is $D^j(S^j) = a^j - b^j S^j(t)$ and this period is the shortage period. Hence the shortage level $S^j(t)$ is governed by the following differential equation:

$$\frac{dS^j(t)}{dt} = a^j - b^j S^j(t), \qquad [(i-1)T^j + t_2^j, iT^j]$$

subject to the conditions that, $S^j[(i-1)T^j + t_2^j] = 0$.

The solution of the above differential equations are

$$q^j(t) = \begin{cases} \frac{1}{b^j}\left\{\{1 - \delta^j\theta^j e^{-(i-1)\alpha^j}\}P^j - a^j\right\}\Big[1 & \\ -e^{-b^j\{t-(i-1)T^j\}}\Big], & (i-1)T^j \leq t \leq (i-1)T^j + t_1^j \\ -\frac{a^j}{b^j}\Big[1 - e^{-b^j\{t-(i-1)T^j - t_2^j\}}\Big], & (i-1)T^j + t_1^j \leq t \leq (i-1)T^j + t_2^j \end{cases}$$

$$\text{and } S^j(t) = \frac{a^j}{b^j}\Big[1 - e^{-b^j\{t-(i-1)T^j-t_2^j\}}\Big], \qquad [(i-1)T^j + t_2^j, iT^j]$$

The present value of production cost of the ith cycle is given by

$$PC_i^j = c_p^j e^{(i-1)\beta^j}\int_{(i-1)T^j}^{(i-1)T^j+t_1^j} P^j e^{-Rt}\,dt = \frac{c_p^j}{R}P^j(1-e^{-Rt_1^j})e^{-(i-1)(RT^j-\beta^j)}$$

The term $e^{(i-1)\beta^j}$ increases the production cost from the next cycle with increasing learning rate β^j and the term e^{-Rt} reduce the present cost with inflation rate R.
Present value of screening cost of the ith cycle is given by

$$SC_i^j = c_{sr}^j e^{(i-1)\beta^j}\int_{(i-1)T^j}^{(i-1)T^j+t_1^j} P^j e^{-Rt}\,dt = \frac{c_{sr}^j}{R}P^j(1-e^{-Rt_1^j})e^{-(i-1)(RT^j-\beta^j)}$$

Present value of reworked cost of the ith cycle is given by

$$\begin{aligned} RC_i^j &= r_c^j\int_{(i-1)T^j}^{(i-1)T^j+t_1^j}(1-\delta^j)\theta^j P^j e^{-(i-1)\alpha^j}e^{-Rt}\,dt \\ &= \frac{1}{R}r_c^j(1-\delta^j)\theta^j P^j e^{-(i-1)\alpha^j}\Big\{1-e^{-Rt_1^j}\Big\}e^{-R(i-1)T^j} \end{aligned}$$

Present value of holding cost of the inventory for the ith cycle is given by

$$\begin{aligned} HC_i^j &= h_c^j\Big[\int_{(i-1)T^j}^{(i-1)T^j+t_1^j} q^j(t)e^{-Rt}\,dt + \int_{(i-1)T^j+t_1^j}^{(i-1)T^j+t_2^j} q^j(t)e^{-Rt}\,dt\Big] \\ &= \frac{h_c^j}{b^j}\Big\{(1-\delta^j\theta^j e^{-(i-1)\alpha^j})P^j - a^j\Big\}\Big[\frac{1-e^{-Rt_1^j}}{R} - \frac{1-e^{-(R+b^j)t_1^j}}{R+b^j}\Big]e^{-(i-1)RT^j} \\ &\quad -\frac{h_c^j a^j}{b^j}\Big[\frac{1}{R}(e^{-Rt_1^j}-e^{-Rt_2^j}) - \frac{e^{b^j t_2^j}}{R+b^j}\{e^{-(R+b^j)t_1^j}-e^{-(R+b^j)t_2^j}\}\Big]e^{-R(i-1)T^j} \end{aligned}$$

Present value of shortage cost of the ith cycle is given by

$$\begin{aligned} SH_i^j &= c_{sh}^j\int_{(i-1)T^j+t_2^j}^{iT^j} S^j(t)e^{-Rt}\,dt \\ &= c_{sh}^j\frac{a^j}{b^j}\Big[\frac{1}{R}(e^{-Rt_2^j}-e^{-RT^j}) - \frac{e^{b^j t_2^j}}{R+b^j}\{e^{-(R+b^j)t_2^j}-e^{-(R+b^j)T^j}\}\Big]e^{-R(i-1)T^j} \end{aligned}$$

Present value of sales revenue for the ith cycle is given by

$$\begin{aligned} SR_i^j &= s^j\int_{(i-1)T^j}^{(i-1)T^j+t_2^j} D^j e^{-Rt}\,dt = s^j\int_{(i-1)T^j}^{(i-1)T^j+t_2^j}\{a^j+b^j q^j(t)\}e^{-Rt}\,dt \\ &= s^j a^j\Big[\frac{1}{R}(1-e^{-Rt_1^j}) + \frac{e^{b^j t_2^j}}{R+b^j}\{e^{-(R+b^j)t_1^j}-e^{-(R+b^j)t_2^j}\}\Big]e^{-R(i-1)T^j} \\ &\quad + s^j\Big\{(1-\delta^j\theta^j e^{-(i-1)\alpha^j})P^j - a^j\Big\}\Big[\frac{1-e^{-Rt_1^j}}{R} - \frac{1-e^{-(R+b^j)t_1^j}}{R+b^j}\Big]e^{-(i-1)RT^j} \end{aligned}$$

Total profit after completing first N^j fully accommodated cycles is given by

$$
\begin{aligned}
TP(t_1^j, T^j) &= \sum_{i=1}^{N^j} \Big[SR_i^j - PC_i^j - SC_i^j - RC_i^j - HC_i^j - SH_i^j \Big] \\
&= \Big[\Big\{ s^j P^j - \frac{h_c^j}{b^j}(P^j - a^j) \Big\} \frac{(1 - e^{-Rt_1^j})}{R} - (s^j - \frac{h_c^j}{b^j})(P^j - a^j) \frac{\{1 - e^{-(R+b^j)t_1^j}\}}{R + b^j} \\
&+ (s^j a^j - \frac{h_c^j a^j}{b^j}) \frac{e^{b^j t_2^j}}{R + b^j} \Big\{ e^{-(R+b^j)t_1^j} - e^{-(R+b^j)t_2^j} \Big\} + \frac{c_{sh}^j a^j e^{b^j t_2^j}}{b^j (R + b^j)} \Big\{ e^{-(R+b^j)t_2^j} - e^{-(R+b^j)T^j} \Big\} \\
&+ \frac{h_c^j a^j}{R b^j} e^{-Rt_1^j} - \frac{(h_c^j + c_{sh}^j) a^j}{R b^j} e^{-Rt_2^j} + \frac{c_{sh}^j a^j}{R b^j} e^{-RT^j} \Big] \frac{1 - e^{-RN^j T^j}}{1 - e^{-RT^j}} \\
&- \frac{(c_p^j + c_{sr}^j)}{R} P^j (1 - e^{-Rt_1^j}) \frac{1 - e^{-(RT^j - \beta^j)N^j}}{1 - e^{-(RT^j - \beta^j)}} - \Big[r_c^j (1 - \delta^j) \theta^j P^j \Big\{ \frac{1 - e^{-Rt_1^j}}{R} \Big\} \\
&+ (s^j - \frac{h_c^j}{b^j}) \delta^j \theta^j P^j \Big\{ \frac{1 - e^{-Rt_1^j}}{R} - \frac{1 - e^{-(R+b^j)t_1^j}}{R + b^j} \Big\} \Big] \frac{1 - e^{-(\alpha^j + RT^j)N^j}}{1 - e^{-(\alpha^j + RT^j)}}
\end{aligned}
$$

Since $\widetilde{f}(h)$ is the p.d.f of the planning horizon H, therefore the expected total profit from the N^j complete cycles is given by,

$$
\begin{aligned}
E[\widetilde{TP(t_1^j, T^j)}] &= \sum_{N^j=0}^{\infty} \int_{N^j T^j}^{(N^j+1)T^j} TP(t_1^j, T^j) \widetilde{f}(h)\, dh \\
&= \Big[\Big\{ s^j P^j - \frac{h_c^j}{b^j}(P^j - a^j) \Big\} \frac{(1 - e^{-Rt_1^j})}{R} - (s^j - \frac{h_c^j}{b^j})(P^j - a^j) \frac{\{1 - e^{-(R+b^j)t_1^j}\}}{R + b^j} \\
&+ (s^j a^j - \frac{h_c^j a^j}{b^j}) \frac{e^{b^j t_2^j}}{R + b^j} \Big\{ e^{-(R+b^j)t_1^j} - e^{-(R+b^j)t_2^j} \Big\} + \frac{c_{sh}^j a^j e^{b^j t_2^j}}{b^j (R + b^j)} \Big\{ e^{-(R+b^j)t_2^j} - e^{-(R+b^j)T^j} \Big\} \\
&+ \frac{h_c^j a^j}{R b^j} e^{-Rt_1^j} - \frac{(h_c^j + c_{sh}^j) a^j}{R b^j} e^{-Rt_2^j} + \frac{c_{sh}^j a^j}{R b^j} e^{-RT^j} \Big] \frac{e^{-\widetilde{\lambda} T^j}}{1 - e^{-(R+\widetilde{\lambda})T^j}} \\
&- \frac{(c_p^j + c_{sr}^j)}{R} P^j (1 - e^{-Rt_1^j}) \frac{e^{-\widetilde{\lambda} T^j}}{1 - e^{-(RT^j + \widetilde{\lambda} T^j - \beta^j)}} - \Big[r_c^j (1 - \delta^j) \theta^j P^j \Big\{ \frac{1 - e^{-Rt_1^j}}{R} \Big\} \\
&+ (s^j - \frac{h_c^j}{b^j}) \delta^j \theta^j P^j \Big\{ \frac{1 - e^{-Rt_1^j}}{R} - \frac{1 - e^{-(R+b^j)t_1^j}}{R + b^j} \Big\} \Big] \frac{e^{-\widetilde{\lambda} T^j}}{1 - e^{-(\alpha^j + RT^j + \widetilde{\lambda} T^j)}}
\end{aligned}
$$

10.3.2 Formulation for Last Cycle

The differential equations describing the inventory level $q^j(t)$ in the interval $[N^j T^j, (N^j + 1)T^j]$ are given by,

$$
\begin{aligned}
\frac{dq^j(t)}{dt} &= \begin{cases} (1 - \delta^j \theta^j e^{-N^j \alpha^j}) P^j - D^j, & N^j T^j \le t \le N^j T^j + t_1^j \\ -D^j, & N^j T^j + t_1^j \le t \le N^j T^j + t_2^j \end{cases} \\
&= \begin{cases} (1 - \delta^j \theta^j e^{-N^j \alpha^j}) P^j - a^j - b^j q^j(t), & N^j T^j \le t \le N^j T^j + t_1^j \\ -a^j - b^j q^j(t), & N^j T^j + t_1^j \le t \le N^j T^j + t_2^j \end{cases}
\end{aligned}
$$

subject to the conditions that, $q^j[N^jT^j] = 0$ and $q^j[N^jT^j + t_2^j] = 0$.

During the time interval $[(N^jT^j + t_2^j, (N^j+1)T^j]$, the demand rate $D^j = a^j - b^jS^j(t)$ and this period is the shortage period. Hence the shortage level $S^j(t)$ is governed by the following differential equation:

$$\frac{dS^j(t)}{dt} = a^j - b^jS^j(t)$$

subject to the conditions that, $S^j[N^jT^j + t_2^j] = 0$.

The solution of the above differential equation is

$$q^j(t) = \begin{cases} \frac{1}{b^j}\Big\{(1-\delta^j\theta^j e^{-N^j\alpha^j})P^j - a^j\Big\}\Big[1-e^{-b^j\{t-N^jT^j\}}\Big], & N^jT^j \le t \le N^jT^j + t_1^j \\ -\frac{a^j}{b^j}\Big[1-e^{-b^j\{t-N^jT^j-t_2^j\}}\Big], & N^jT^j + t_1^j \le t \le N^jT^j + t_2^j \end{cases}$$

$$\text{and } S^j(t) = \frac{a^j}{b^j}\Big[1-e^{-b^j\{t-N^jT^j-t_2^j\}}\Big], \qquad [N^jT^j + t_2^j, iT^j]$$

In last cycle, for simplicity we consider three cases only depending upon the cycle length. Let h be the real value corresponding to the random variable H.

Case-I: when $N^jT^j \le h \le N^jT^j + t_1^j$

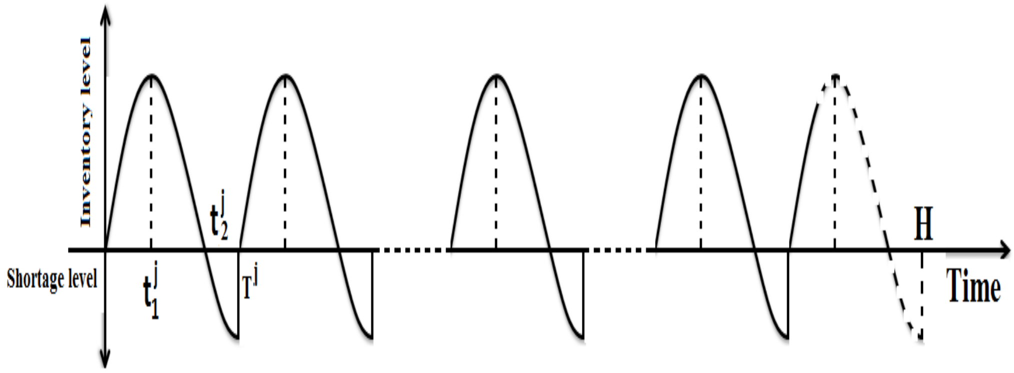

Figure 10.2: Logistic diagram of inventory model for case-I

Present value of production cost of the last cycle is given by,

$$PC_{L_1}^j = c_p^j e^{N^j\beta^j}\Big[\int_{N^jT^j}^{h} P^j e^{-Rt}\,dt\Big] = \frac{c_p^j}{R}P^j e^{N^j\beta^j}(e^{-RN^jT^j} - e^{-Rh})$$

Present value of screening cost of the last cycle is given by,

$$SC^j_{L_1} = c^j_{sr}e^{N^j\beta^j}\Big[\int_{N^jT^j}^{h} P^j e^{-Rt}\,dt\Big] = \frac{c^j_{sr}}{R}P^j e^{N^j\beta^j}(e^{-RN^jT^j} - e^{-Rh})$$

Present value of reworked cost of the last cycle is given by,

$$RC^j_{L_1} = r^j_c\Big[\int_{N^jT^j}^{h}(1-\delta^j)\theta^j P^j e^{-N^j\alpha^j}e^{-Rt}\,dt\Big] = \frac{r^j_c(1-\delta^j)\theta^j P^j e^{-N^j\alpha^j}}{R}\Big\{e^{-RN^jT^j} - e^{-Rh}\Big\}$$

Present value of sales revenue for the last cycle is given by

$$\begin{aligned} SR^j_{L_1} &= s^j\int_{N^jT^j}^{h} D^j e^{-Rt}\,dt = s^j\int_{N^jT^j}^{h}\{a^j + b^j q^j(t)\}e^{-Rt}\,dt \\ &= \frac{s^j a^j}{R}(e^{-RN^jT^j} - e^{-Rh}) + s^j\{(P^j - a^j) - \delta^j\theta^j P^j e^{-N^j\alpha^j}\}\Big[\frac{1}{R}(e^{-RN^jT^j} - e^{-Rh}) \\ &\quad + \frac{e^{b^jN^jT^j}}{R+b^j}\Big\{e^{-(b^j+R)N^jT^j} - e^{-(b^j+R)h}\Big\}\Big] \end{aligned}$$

Present value of holding cost of the inventory for the last cycle is given by,

$$\begin{aligned} HC^j_{L_1} &= h^j_c\Big[\int_{N^jT^j}^{h} q^j(t)e^{-Rt}\,dt\Big] \\ &= \frac{h^j_c}{b^j}\{(P^j - a^j) - \delta^j\theta^j P^j e^{-N^j\alpha^j}\}\Big[\frac{1}{R}(e^{-RN^jT^j} - e^{-Rh}) \\ &\quad + \frac{e^{b^jN^jT^j}}{R+b^j}\Big\{e^{-(b^j+R)N^jT^j} - e^{-(b^j+R)h}\Big\}\Big] \end{aligned}$$

Case-II: when $N^jT^j + t^j_1 \le h \le N^jT^j + t^j_2$

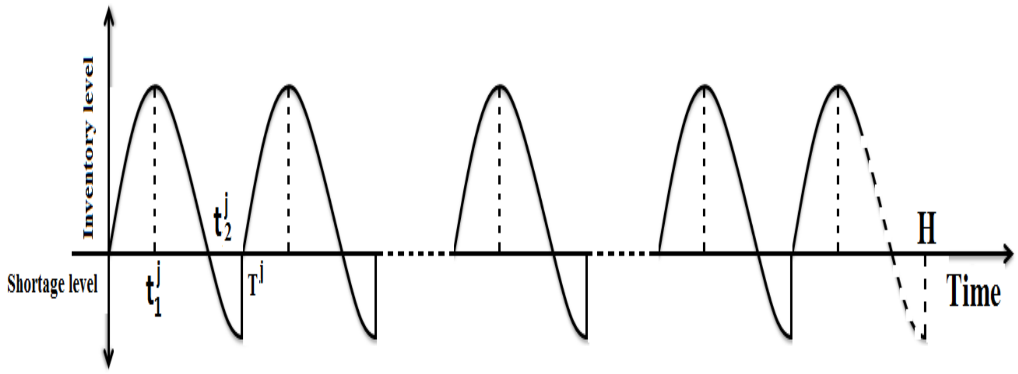

Figure 10.3: Logistic diagram of Inventory Model for case-II

Present value of production cost of the last cycle is given by,

$$PC^j_{L_2} = c^j_p e^{N^j\beta^j}\int_{N^jT^j}^{N^jT^j+t^j_1} P^j e^{-Rt}\,dt = \frac{c^j_p}{R}P^j(1-e^{-Rt^j_1})e^{-(RT^j-\beta^j)N^j}$$

Present value of screening cost of the last cycle is given by,

$$SC^j_{L_2} = c^j_{sr} e^{N^j\beta^j}\int_{N^jT^j}^{N^jT^j+t^j_1} P^j e^{-Rt}\,dt = \frac{c^j_{sr}}{R}P^j(1-e^{-Rt^j_1})e^{-(RT^j-\beta^j)N^j}$$

Present value of reworked cost of the last cycle is given by,

$$\begin{aligned} RC^j_{L_2} &= r^j_c\int_{N^jT^j}^{N^jT^j+t^j_1}(1-\delta^j)\theta^j P^j e^{-N^j\alpha^j}e^{-Rt}\,dt \\ &= \frac{r^j_c}{R}(1-\delta^j)\theta^j P^j(1-e^{-Rt^j_1})e^{-(RT^j+\alpha^j)N^j} \end{aligned}$$

Present value of holding cost of the inventory for the last cycle is given by,

$$\begin{aligned} &HC^j_{L_2} = h^j_c\Big[\int_{N^jT^j}^{N^jT^j+t^j_1} q^j(t)e^{-Rt}\,dt + \int_{N^jT^j+t^j_1}^{h} q^j(t)e^{-Rt}\,dt\Big] \\ &= \frac{h^j_c}{b^j}\Big\{(1-\delta^j\theta^j e^{-N^j\alpha^j})P^j - a^j\Big\}\Big[\frac{1}{R}\Big(1-e^{-Rt^j_1}\Big) - \frac{1}{R+b^j}\Big\{1-e^{-(R+b^j)t^j_1}\Big\}\Big]e^{-RN^jT^j} \\ &-\frac{h^j_c a^j}{b^j}\Big[\frac{1}{R}\Big\{e^{-R(N^jT^j+t^j_1)} - e^{-Rh}\Big\} - \frac{e^{b^j(N^jT^j+t^j_2)}}{R+b^j}\Big\{e^{-(R+b^j)(N^jT^j+t^j_1)} - e^{-(R+b^j)h}\Big\}\Big] \end{aligned}$$

Present value of sales revenue for the last cycle is given by

$$\begin{aligned} &SR^j_{L_2} = s^j\Big[\int_{N^jT^j}^{N^jT^j+t^j_1} D^j e^{-Rt}\,dt + \int_{N^jT^j+t^j_1}^{h} D^j e^{-Rt}\,dt\Big] \\ &= \frac{s^j a^j}{R}(e^{-RN^jT^j} - e^{-Rh}) \\ &+s^j\Big\{(1-\delta^j\theta^j e^{-N^j\alpha^j})P^j - a^j\Big\}\Big[\frac{1}{R}\Big(1-e^{-Rt^j_1}\Big) - \frac{1}{R+b^j}\Big\{1-e^{-(R+b^j)t^j_1}\Big\}\Big]e^{-RN^jT^j} \\ &-s^j a^j\Big[\frac{1}{R}\Big\{e^{-R(N^jT^j+t^j_1)} - e^{-Rh}\Big\} - \frac{e^{b^j(N^jT^j+t^j_2)}}{R+b^j}\Big\{e^{-(R+b^j)(N^jT^j+t^j_1)} - e^{-(R+b^j)h}\Big\}\Big] \end{aligned}$$

Case-III: when $N^jT^j + t_2^j \leq h \leq (N^j+1)T^j$

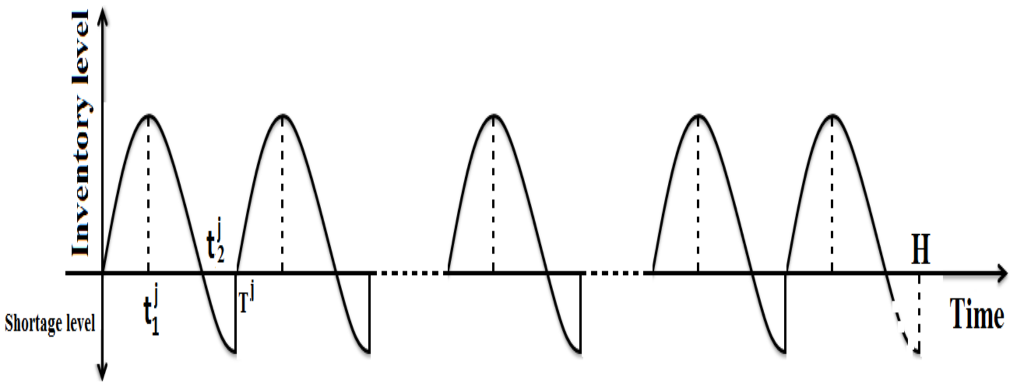

Figure 10.4: Graphical Representation of inventory model for case-III

Present value of production cost of the last cycle is given by,

$$PC_{L_3}^j = c_p^j e^{N^j\beta^j} \int_{N^jT^j}^{N^jT^j+t_1^j} P^j e^{-Rt}\, dt = \frac{c_p^j}{R} P^j(1-e^{-Rt_1^j})e^{-(RT^j-\beta^j)N^j}$$

Present value of screening cost of the last cycle is given by,

$$SC_{L_3}^j = c_{sr}^j e^{N^j\beta^j} \int_{N^jT^j}^{N^jT^j+t_1^j} P^j e^{-Rt}\, dt = \frac{c_{sr}^j}{R} P^j(1-e^{-Rt_1^j})e^{-(RT^j-\beta^j)N^j}$$

Present value of reworked cost of the last cycle is given by,

$$\begin{aligned} RC_{L_3}^j &= r_c^j \int_{N^jT^j}^{N^jT^j+t_1^j} (1-\delta^j)\theta^j P^j e^{-N^j\alpha^j} e^{-Rt}\, dt \\ &= \frac{r_c^j}{R}(1-\delta^j)\theta^j P^j e^{-N^j\alpha^j}(1-e^{-Rt_1^j})e^{-(RT^j+\alpha^j)N^j} \end{aligned}$$

Present value of holding cost of the inventory for the last cycle is given by,

$$\begin{aligned} HC_{L_3}^j &= h_c^j \Big[\int_{N^jT^j}^{N^jT^j+t_1^j} q^j(t)e^{-Rt}\, dt + \int_{N^jT^j+t_1^j}^{N^jT^j+t_2^j} q^j(t)e^{-Rt}\, dt \Big] \\ &= \frac{h_c^j}{b^j}\Big\{(1-\delta^j\theta^j e^{-N^j\alpha^j})P^j - a^j\Big\}\Big[\frac{1-e^{-Rt_1^j}}{R} - \frac{1-e^{-(R+b^j)t_1^j}}{R+b^j}\Big]e^{-RN^jT^j} \\ &\quad -\frac{h_c^j a^j}{b^j}\Big[\frac{1}{R}(e^{-Rt_1^j}-e^{-Rt_2^j}) - \frac{e^{b^jt_2^j}}{R+b^j}\{e^{-(R+b^j)t_1^j} - e^{-(R+b^j)t_2^j}\}\Big]e^{-RN^jT^j} \end{aligned}$$

Present value of sales revenue for the last cycle is given by,

$$\begin{aligned}
SR_{L_3}^j &= s^j\Big[\int_{N^jT^j}^{N^jT^j+t_1^j} D^j e^{-Rt}\,dt + \int_{N^jT^j+t_1^j}^{N^jT^j+t_2^j} D^j e^{-Rt}\,dt\Big] \\
&= s^j a^j\Big[\frac{1}{R}(1-e^{-Rt_1^j}) + \frac{e^{b^jt_2^j}}{R+b^j}\{e^{-(R+b^j)t_1^j} - e^{-(R+b^j)t_2^j}\}\Big]e^{-RN^jT^j} \\
&\quad + s^j\Big\{(1-\delta^j\theta^j e^{-N^j\alpha^j})P^j - a^j\Big\}\Big[\frac{1-e^{-Rt_1^j}}{R} - \frac{1-e^{-(R+b^j)t_1^j}}{R+b^j}\Big]e^{-RN^jT^j}
\end{aligned}$$

Present value of shortage cost for the last cycle is given by,

$$\begin{aligned}
SH_3^j &= c_{sh}^j\int_{N^jT^j+t_2^j}^{h} S^j(t)e^{-Rt}\,dt \\
&= \frac{c_{sh}^j a^j}{b^j}\Big[\frac{1}{R}(e^{-R(N^jT^j+t_2^j)} - e^{-Rh}) - \frac{e^{b^j(N^jT^j+t_2^j)}}{R+b^j}\{e^{-(R+b^j)(N^jT^j+t_2^j)} - e^{-(R+b^j)h}\}\Big]
\end{aligned}$$

Expected production cost for the last cycle is given by,

$$\begin{aligned}
&\widetilde{E[PC_L^j]} = \sum_{N^j=0}^{\infty}\int_{N^jT^j}^{(N^j+1)T^j} PC_L^j\widetilde{f(h)}\,dh \\
&= \frac{c_p^jP^j}{R\{1-e^{-(RT^j+\widetilde{\lambda}T^j-\beta^j)}\}}\Big[(1-e^{-\widetilde{\lambda}T^j}) - \frac{\widetilde{\lambda}}{R+\widetilde{\lambda}}\{1-e^{-(R+\widetilde{\lambda})t_1^j}\} - e^{-Rt_1^j}(e^{-\widetilde{\lambda}t_1^j} - e^{-\widetilde{\lambda}T^j})\Big]
\end{aligned}$$

Expected screening cost for the last cycle is given by,

$$\begin{aligned}
&\widetilde{E[SC_L^j]} = \sum_{N^j=0}^{\infty}\int_{N^jT^j}^{(N^j+1)T^j} SC_L^j\widetilde{f(h)}\,dh \\
&= \frac{c_{sr}^jP^j}{R\{1-e^{-(RT^j+\widetilde{\lambda}T^j-\beta^j)}\}}\Big[(1-e^{-\widetilde{\lambda}T^j}) - \frac{\widetilde{\lambda}}{R+\widetilde{\lambda}}\{1-e^{-(R+\widetilde{\lambda})t_1^j}\} - e^{-Rt_1^j}(e^{-\widetilde{\lambda}t_1^j} - e^{-\widetilde{\lambda}T^j})\Big]
\end{aligned}$$

Expected reworked cost for the last cycle is given by,

$$\begin{aligned}
&\widetilde{E[RC_L^j]} = \sum_{N^j=0}^{\infty}\int_{N^jT^j}^{(N^j+1)T^j} RC_L^j\widetilde{f(h)}\,dh \\
&= \frac{r_c^j(1-\delta^j)\theta^jP^j}{R\{1-e^{-\{(R+\widetilde{\lambda})T^j+\alpha^j\}}\}}\Big[(1-e^{-\widetilde{\lambda}T^j}) - \frac{\widetilde{\lambda}}{R+\widetilde{\lambda}}\{1-e^{-(R+\widetilde{\lambda})t_1^j}\} - e^{-Rt_1^j}(e^{-\widetilde{\lambda}t_1^j} - e^{-\widetilde{\lambda}T^j})\Big]
\end{aligned}$$

Expected holding cost for the last cycle is given by,

$$\widetilde{E[HC_L^j]} = \sum_{N^j=0}^{\infty} \int_{N^jT^j}^{(N^j+1)T^j} HC_L^j \widetilde{f(h)}\, dh$$
$$= \frac{h_c^j}{b^j}\Big\{\frac{(P^j-a^j)}{\{1-e^{-(R+\widetilde{\lambda})T^j}\}} - \frac{\delta^j\theta^jP^j}{\{1-e^{-\{(R+\widetilde{\lambda})T^j+\alpha^j\}}\}}\Big\}\Big[\frac{1}{R}(1-e^{-\widetilde{\lambda}t_1^j}) + \frac{1}{R+b^j}(1-e^{-\widetilde{\lambda}t_1^j})$$
$$-\frac{\widetilde{\lambda}}{R(\widetilde{\lambda}+R)}\{1-e^{-(R+\widetilde{\lambda})t_1^j}\} - \frac{\widetilde{\lambda}}{(R+b^j)(\widetilde{\lambda}+R+b^j)}\{1-e^{-(R+b^j+\widetilde{\lambda})t_1^j}\}$$
$$+\Big\{\frac{1}{R}(1-e^{-Rt_1^j}) - \frac{1}{R+b^j}\{1-e^{-(R+b^j)t_1^j}\}\Big\}(e^{-\widetilde{\lambda}t_1^j}-e^{-\widetilde{\lambda}T^j})\Big]$$
$$-\frac{h_c^ja^j}{b^j\{1-e^{-(R+\widetilde{\lambda})T^j}\}}\Big[\frac{e^{-Rt_1^j}}{R}(e^{-\widetilde{\lambda}t_1^j}-e^{-\widetilde{\lambda}t_2^j}) - \frac{\widetilde{\lambda}}{R(\widetilde{\lambda}+R)}(e^{-(R+\widetilde{\lambda})t_1^j}-e^{-(R+\widetilde{\lambda})t_2^j})$$
$$-\frac{e^{b^jt_2}}{R+b^j}\Big\{e^{-(R+b^j)t_1^j}(e^{-\widetilde{\lambda}t_1^j}-e^{-\widetilde{\lambda}t_2^j}) - \frac{\widetilde{\lambda}}{(\widetilde{\lambda}+R+b^j)}\{e^{-(R+b^j+\widetilde{\lambda})t_1^j}-e^{-(R+b^j+\widetilde{\lambda})t_2^j}\}\Big\}$$
$$+\Big\{\frac{1}{R}(e^{-Rt_1^j}-e^{-Rt_2^j}) - \frac{e^{b^jt_2^j}}{R+b^j}\{e^{-(R+b^j)t_1^j}-e^{-(R+b^j)t_2^j}\}\Big\}(e^{-\widetilde{\lambda}T^j}-e^{-\widetilde{\lambda}t_2^j})\Big]$$

Expected sales revenue from the last cycle is given by,

$$\widetilde{E[SR_L^j]} = \sum_{N^j=0}^{\infty} \int_{N^jT^j}^{(N^j+1)T^j} SR_L^j \widetilde{f(h)}\, dh$$
$$= \frac{s^jP^j}{R}\Big\{\frac{1}{1-e^{-(R+\widetilde{\lambda})T^j}} - \frac{\delta^j\theta^j}{1-e^{-\{(R+\widetilde{\lambda})T^j+\alpha^j\}}}\Big\}\Big[(1-e^{-\widetilde{\lambda}t_2^j}) - \frac{\widetilde{\lambda}}{\widetilde{\lambda}+R}\{1-e^{-(R+\widetilde{\lambda})t_2^j}\}\Big]$$
$$+s^j\Big\{\frac{(P^j-a^j)}{1-e^{-(R+\widetilde{\lambda})T^j}} - \frac{\delta^j\theta^jP^j}{1-e^{-\{(R+\widetilde{\lambda})T^j+\alpha^j\}}}\Big\}\Big[\frac{1}{R+b^j}(1-e^{-\widetilde{\lambda}t_1^j}) + \Big\{\frac{1}{R}(1-e^{-Rt_1^j})$$
$$-\frac{\widetilde{\lambda}\{1-e^{-(\widetilde{\lambda}+R+b^j)t_1^j}\}}{(R+b^j)(\widetilde{\lambda}+R+b^j)} - \frac{1}{R+b^j}\{1-e^{-(R+b^j)t_1^j}\}\Big\}(e^{-\widetilde{\lambda}t_1^j}-e^{-\widetilde{\lambda}T^j})\Big]$$
$$-\frac{s^ja^j}{1-e^{-(R+\widetilde{\lambda})T^j}}\Big[\frac{e^{-Rt_1^j}}{R}(e^{-\widetilde{\lambda}t_1^j}-e^{-\widetilde{\lambda}t_2^j}) - \frac{\widetilde{\lambda}}{R(\widetilde{\lambda}+R)}(e^{-(R+\widetilde{\lambda})t_1^j}-e^{-(R+\widetilde{\lambda})t_2^j})$$
$$-\frac{e^{b^jt_2}}{R+b^j}\Big\{e^{-(R+b^j)t_1^j}(e^{-\widetilde{\lambda}t_1^j}-e^{-\widetilde{\lambda}t_2^j}) - \frac{\widetilde{\lambda}}{(\widetilde{\lambda}+R+b^j)}\{e^{-(R+b^j+\widetilde{\lambda})t_1^j}-e^{-(R+b^j+\widetilde{\lambda})t_2^j}\}\Big\}$$
$$+\Big\{\frac{1}{R}(e^{-Rt_1^j}-e^{-Rt_2^j}) - \frac{e^{b^jt_2^j}}{R+b^j}\{e^{-(R+b^j)t_1^j}-e^{-(R+b^j)t_2^j}\}\Big\}(e^{-\widetilde{\lambda}T^j}-e^{-\widetilde{\lambda}t_2^j})\Big]$$

Expected shortage cost for the last cycle is given by,

$$\widetilde{E[SH_L^j]} = \sum_{N^j=0}^{\infty} \int_{N^jT^j}^{(N^j+1)T^j} SH_L^j \widetilde{f(h)}\, dh$$
$$= \frac{c_{sh}^j a^j}{b^j\{1 - e^{-RT^j} e^{-\widetilde{\lambda}T^j}\}} \Big[\Big\{ \frac{1}{R} - \frac{1}{R+b^j} \Big\} (e^{-\widetilde{\lambda}t_2^j} - e^{-\widetilde{\lambda}T^j}) e^{-Rt_2^j}$$
$$+ \frac{\widetilde{\lambda} e^{bt_2^j}}{(R+b^j)(R+\widetilde{\lambda}+b^j)} \Big\{ e^{-(R+\widetilde{\lambda}+b^j)t_2^j} - e^{-(R+\widetilde{\lambda}+b^j)T^j} \Big\}$$
$$- \frac{\widetilde{\lambda}}{R(R+\widetilde{\lambda})} \Big\{ e^{-(R+\widetilde{\lambda})t_2^j} - e^{-(R+\widetilde{\lambda})T^j} \Big\} \Big]$$

Expected total profit from last cycle is given by,

$$E[\widetilde{TP_L^j(t_1^j, T^j)}] = \widetilde{E[SR_L^j]} - \widetilde{E[PC_L^j]} - \widetilde{E[SC_L^j]} - \widetilde{E[RC_L^j]} - \widetilde{E[HC_L^j]} - \widetilde{E[SH_L^j]}$$

10.3.3 Objective of the Proposed Model

The expected total profit from the complete time horizon is given by,

$$\widetilde{E[TP]} = \sum_{j=1}^{M} E[\widetilde{TP^j(t_1^j, T^j)}] + \sum_{j=1}^{M} E[\widetilde{TP_L^j(t_1^j, T^j)}] \tag{10.3}$$

subject to the space constraint:

$$\sum_{j=1}^{M} \rho^j Q^j \leq \widetilde{W} \tag{10.4}$$

Then for predefined confidence levels μ (for objective function) and μ_1 (for constraint function) the deterministic problem (followed Liu and Iwamura [140]) is:

$$\begin{aligned} &Maximize \quad Z && (10.5)\\ &\text{subject to} \quad Pos\Big(\widetilde{E[TP]} \leq Z\Big) \geq \mu \text{ and } Pos\Big(\sum_{j=1}^{M} \rho^j Q^j \leq \widetilde{W}\Big) \geq \mu_1 \\ &\text{or} \quad Nec\Big(\widetilde{E[TP]} \leq Z\Big) \geq \mu \text{ and } Nec\Big(\sum_{j=1}^{M} \rho^j Q^j \leq \widetilde{W}\Big) \geq \mu_1 \\ &\text{or} \quad Cr\Big(\widetilde{E[TP]} \leq Z\Big) \geq \mu \text{ and } Cr\Big(\sum_{j=1}^{M} \rho^j Q^j \leq \widetilde{W}\Big) \geq \mu_1 \end{aligned}$$

which is equivalent to

$$\begin{aligned}
&Maximize \quad Z && (10.6)\\
&\text{subject to} \quad \frac{Z - E[TP]_{min}}{E[TP]_{mid} - E[TP]_{min}} \geq \mu \text{ and } \sum_{j=1}^{M} \rho^j Q^j \leq W_{max} - \mu_1(W_{max} - W_{mid})\\
&\text{or} \quad \frac{E[TP]_{max} - Z}{E[TP]_{max} - E[TP]_{mid}} \leq 1 - \mu \text{ and } \frac{\sum_{j=1}^{M} \rho^j Q^j - W_{min}}{W_{mid} - W_{min}} \leq 1 - \mu_1\\
&\text{or} \quad \frac{1}{2}\left(\frac{Z - E[TP]_{min}}{E[TP]_{mid} - E[TP]_{min}} + \frac{Z - E[TP]_{mid}}{E[TP]_{max} - E[TP]_{mid}}\right) \geq \mu\\
&\text{and } \frac{1}{2}\left(\frac{W_{max} - \sum_{j=1}^{M} \rho^j Q^j}{W_{max} - W_{mid}} + \frac{\sum_{j=1}^{M} a^j Q^j - W_{min}}{W_{mid} - W_{min}}\right) \geq \mu_1
\end{aligned}$$

10.4 Solution Procedure

To determine the feasible solution for the decision variables t_1^j, T^j (j=1, 2) of the above fuzzy model, the fuzzy simulation based genetic algorithm (FSGA) have been used which described in section 2.4.5. To implementing FSGA we set the following parametric values: POPSIZE=500, PCROSS=0.3, PMUT=0.2, MAXGEN=2000.

10.5 Numerical Illustration

A company manufactures its productions at a rate of 55 units and 60 units per unit time with the defectiveness 15% and 12% respectively. The rework system recoveries 84% and 82% of the defective units for 1st and 2nd item respectively. Due to learning effect the company reduces 12% and 11% of the defectiveness from one to another cycle respectively. The per unit selling price of the items are \$92 and \$90 respectively. The difference between inflation rate and time value of money is $R = 0.10$. The different costs of the items are given in Table 10.2.

Table 10.2: Values of different type of cost parameters

	Production cost (c_p^j)	Screening cost (c_{sr}^j)	Rework cost (r_c^j)	Holding cost (h_c^j)	Shortage cost (c_{sh}^j)
Item-1	\$30	\$1.5	\$ 5	\$2.1	\$ 2.5
Item-2	\$28	\$1.4	\$4	\$2.5	\$2.0

The input values of the demand parameters and parameters relative to defectiveness in appropriate unit are given in Table 10.3.

Table 10.3: Values of different type of other parameters

Parameters	a^j	b^j	θ^j	δ^j	ρ^j	β^j
Item-1	40	0.1	0.15	0.16	1.4	0.010
Item-2	43	0.12	0.1	0.16	1.6	0.012

The imprecise parameters are taken as: $\widetilde{\lambda} = [0.095, 0.102, 0.112], \widetilde{W} = [45, 50, 55]$. The problem is to determine the optimal policies under different management system. For the empirical parametric values the optimum results are obtained by using simulation based FSGA and presented in the following tables.

Table 10.4: Optimum results of illustrated model for possibility measure

Possibility measure	$Item$	t_1^j (unit)	t_2^j (unit)	T^j (unit)	Q^j (unit)	$\widetilde{E[TP]}$ ($)	Z
$\mu = 0.8$	Item-1	1.12	1.51	3.51	14.65	[9406.00, 9683.77, 9966.69]	0.99
$\mu_1 = 0.9$	Item-2	1.16	1.58	3.65	14.45		
$\mu = 0.6$	Item-1	1.06	1.43	3.17	13.91	[8286.36, 8549.99, 8818.21]	0.98
$\mu_1 = 0.9$	Item-2	1.27	1.72	4.95	15.74		

Table 10.5: Optimum results of illustrated model for credibility measure

Credibility measure	$Item$	t_1^j (unit)	t_2^j (unit)	T^j (unit)	Q^j (unit)	$\widetilde{E[TP]}$ ($)	Z
$\mu = 0.8$	Item-1	1.09	1.47	5.84	14.28	[8230.24, 8510.60, 8791.57]	0.98
$\mu_1 = 0.9$	Item-2	1.96	2.62	4.97	23.22		
$\mu = 0.6$	Item-1	1.17	1.58	3.97	15.27	[6967.86, 7209.59, 7455.51]	0.98
$\mu_1 = 0.9$	Item-2	1.14	1.55	4.94	14.22		

Table 10.6: Optimum results of illustrated model for necessity measure

Necessity measure	*Item*	t_1^j (unit)	t_2^j (unit)	T^j (unit)	Q^j (unit)	$\widetilde{E[TP]}$ ($)	Z
$\mu = 0.8$	Item-1	1.01	1.36	4.75	13.29	[6333.46, 6559.85, 6788.76]	0.98
$\mu_1 = 0.9$	Item-2	1.19	1.62	4.37	14.81		
$\mu = 0.6$	Item-1	1.18	1.59	5.92	15.39	[5804.19, 6028.78, 6256.18]	0.99
$\mu_1 = 0.9$	Item-2	1.12	1.52	4.22	13.99		

The results in Tables 10.4, 10.5 and 10.6 show the optimum production run-times, cycle times as well as ordered quantities for two different items. From these tables, we conclude that:

• From Table 10.4, 10.5 and 10.6 one can see that the level of uncertainty has a similar effect in each management, i.e., as the level of uncertainty decreases profit is also decrease.

• Table 10.4 and 10.5 show that the level of uncertainty has a reverse impact on maximized parameter Z.

10.5.1 Sensitivity Analysis

The change in the values of system parameters can take place due to uncertainties and dynamic market conditions in any decision-making (DM) situation. In order to examine the impacting of these changes in the values of parameters, the sensitivity analysis will be of great help in a decision-making process.

Table 10.7: Sensitivity analysis of $\widetilde{E[TP]}$ w.r.t. R

R	$\widetilde{E[TP]}$ in possibility	$\widetilde{E[TP]}$ in credibility	$\widetilde{E[TP]}$ in necessity
0.06	$[10966.1, 11347.9, 11740.8]$	$[9390.19, 9731.67, 10082.99]$	$[8018.71, 8326.32, 8641.49]$
0.08	$[9010.91, 9314.41, 9624.25]$	$[7597.33, 7871.85, 8152.04]$	$[6367.76, 6618.99, 6874.18]$
0.10	$[7980.17, 8236.59, 8497.19]$	$[6683.14, 6917.93, 7156.52]$	$[5551.88, 5769.92, 5990.40]$
0.12	$[7231.85, 7453.19, 7677.48]$	$[6040.58, 6245.64, 6453.47]$	$[4998.76, 5191.72, 5386.38]$
0.14	$[6655.50, 6849.79, 7046.32]$	$[5558.11, 5740.12, 5924.31]$	$[4595.77, 4769.11, 4943.81]$

There is a noticeable deviation in expected total profit in each measures (possibility, necessity and credibility) as the difference between inflation and time value money moves away. Increasing the difference between inflation and time value money (R), the average profit can be decreased.

Table 10.8: Sensitivity analysis of $\widetilde{E[TP]}$ w.r.t. α^1 and α^2

Increasing learning parameter	$\widetilde{E[TP]}$ in possibility	$\widetilde{E[TP]}$ in credibility	$\widetilde{E[TP]}$ in necessity
$\alpha^1 = 0.08,\ \alpha^2 = 0.07$	$[6126.71, 6342.33, 6558.57]$	$[5882.73, 6099.00, 6316.64]$	$[4261.45, 4444.06, 4626.99]$
$\alpha^1 = 0.10,\ \alpha^2 = 0.09$	$[6935.75, 7179.69, 7426.70]$	$[6894.24, 7134.68, 7377.87]$	$[4906.56, 5112.13, 5319.06]$
$\alpha^1 = 0.12,\ \alpha^2 = 0.11$	$[7961.99, 8218.09, 8478.36]$	$[7922.75, 8196.29, 8470.05]$	$[5546.78, 5764.69, 5985.03]$
$\alpha^1 = 0.14,\ \alpha^2 = 0.13$	$[6719.87, 6968.22, 7221.63]$	$[6295.96, 6539.69, 6787.63]$	$[5010.66, 5218.97, 5429.10]$
$\alpha^1 = 0.16,\ \alpha^2 = 0.15$	$[6133.87, 6369.00, 6609.38]$	$[5948.81, 6184.67, 6425.04]$	$[4699.96, 4896.02, 5093.10]$

Table 10.8 shows the optimal results of the expected total profit in different managements for different values of the learning parameters α^1 and α^2. This table shows that the expected total profit is concave with respect to the learning effect parameters, i.e., at initial increment of learning effect reduce the defectiveness of the items and increase the profit, but after a certain level of that increment, the learning effect reduce the defectiveness as well as increases the production cost and screening cost.

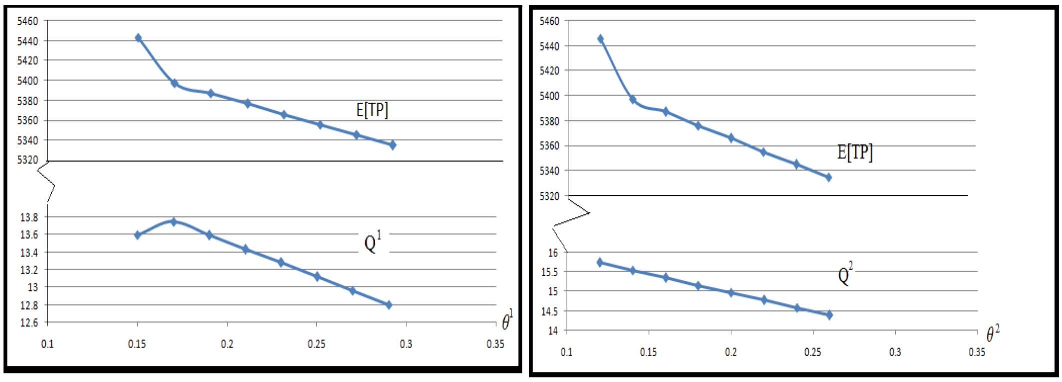

Figure 10.5: Variations of θ^1 and θ^2 effects on Q^1, Q^2 and ETP

Finally, from the Figure 10.5 it is concluded that there is a decrease (increase) in the stock amounts Q^1 and Q^2 as well as expected total profit when the rate of defective rates θ^1 and θ^2 are increased (decreased).

10.5.2 Practical Implications

A manufacturing system may be illustrated as follows: Let a mobile phone company produce two types of mobiles with different features, having a random length of business period for a lot of items. At the time of production, a few defective units (e.g. scratched, disordered shape, etc.) are repaired to sell at the market. Each item has a different demand rate depending on the displayed amount. The company has a space capacity, say about 500 acres, which appears in an imprecise sense. The supporting staff of the company gathers experience to reduce the defectiveness of the mobile phones. The decision managers of the company decide how much of each item to produce, and what will be the length or frequency of the production cycle. For such a real-life problem, the present model can be implemented. The solved model gives a managerial insight to the decision maker of any newly established company where more than one item is produced.

10.6 Conclusion

In this chapter, a production-lot sizing model has been presented that incorporates some realistic features in a imperfect production system. In practice, the production processes of a manufacturer are not perfect. Hence, a production system produces some perfect and some defective items. The production inventory model has been developed over a random planning horizon with fuzzy parameters. The modern soft computing method of genetic algorithm based on simulation process investigates the optimal solution of the model. This chapter suggests the optimal time period, production period and ordered quantity as well as the effect of the learning parameters for different measures such as possibility, necessity and credibility. So, from this study, the following conclusions can be drawn:

(i) The expected profit in an optimistic sense is larger than that in the pessimistic sense.

(ii) More production run time implies a higher quantity of maximum inventory, and profit.

(iii) When increasing the difference between inflation and the time value of money, the average profit can be decreased.

(iv) The initial increment of the learning effect reduces the defectiveness of the items and increases the expected profit. But after a certain level of that increment, expected profit decreases due to more increases in the production cost and screening cost.

Chapter 11

Two layers supply chain imperfect production inventory model with fuzzy credit period, time and production rate dependent imperfectness

11.1 Introduction

In today's highly competitive business world, the supply chain management (SCM) is a vital issue for manufacturers, retailers and customers. It is a methodology to improve the business performance. As a result, Supply Chain Management (SCM) is in the form to enhance the revenue and to reduce operational costs, to improve flow of supplies, to reduction delays of production and increase customer satisfaction. Researchers as well as practitioners in manufacturing industries have given importance to develop inventory control problems in supply chain management. Adoption of supply chain management practices in industries has steadily increased since the 1980s. However, Manna et al. [147], Munson and Rosenblatt [157], Yang and Wee [230], Khouja [117], Yao et al. [234], Chaharsooghi et al. [16], Wang et al. [216] and others provided excellent review on supply chain management literature. These articles define the concept, principals, nature and development of SCM and indicate that there is an intense research being conducted around the world in this field.

In present business culture, usually a supplier offers a permissible delay in payments to a manufacturer, a manufacturer offers a permissible delay in payments to a retailer and a retailer offers a permissible delay in payments to customers, known as trade credit period, in paying for purchasing cost, which is a very common business practice. Suppliers often offer trade credit as a marketing strategy to increase sales and reduce on hand stock level. Once a trade credit has been offered, the amount of period that the retailer's capital tied up in stock is reduced, and that leads to a reduction in the retailer's holding cost of finance. In addition, during trade credit period, the retailer can accumulate revenues by selling items and by earning interests. As a matter of fact, retailers, especially small businesses which tend to have a limited number of financing opportunities, rely on trade credit as a source of short-term funds. In this research field, Goyal [78] was the first who established an EOQ model with a constant demand rate under the condition of permissible delay in payments. Chung and Liao [46] studied a lot-sizing problem under a supplier's trade credit depending on the retailer's order quantity. Abad et al. [2] developed a seller-buyer model under permissible delay in payments by game theory to determine the optimal unit price with trade credit period, considering that the demand rate is a function of the retail price. Recently Das et al. [51] developed an integrated model under trade credit policy. Summary of related literature for multi-retailer EOQ/EPQ models with credit period is shown in Table 11.1.

Table 11.1: Summary of related literature for EPQ/EOQ models with Credit period

Author(s)	EOQ /EPQ	Defective rate	Demand rate depend on	Environment	Credit period	Retailer /Agent
Abad & Jaggi [2]	EOQ	-	Price sensitive	Crisp	Decision variables	Single
Annadurai & Uthayakumar [6]	EOQ	Random	Stochastic lead-time	Stochastic	-	Single
Chaharsooghi et al. [16]	EPQ	-	Stochastic lead-time	Stochastic	Crisp (fixed)	Multiple
Chang et al. [26]	EOQ	-	Fuzzy random-lead time	Fuzzy-stochastic	-	Single
Chung & Liao [46]	EOQ	Constant	Constant	Crisp	Crisp (fixed)	Single
Das et al. [51]	EPQ	Constant	Constant	Fuzzy	Crisp and fuzzy	Multiple
Datta & Pal [52]	EOQ	-	Stock level	Crisp	-	Single
Dey et al. [56]	EOQ	-	Dynamic	Fuzzy	-	Single
Jaber et al. [103]	EOQ	Random	Constant	Stochastic	-	Single
Khouja [117]	EPQ	-	Constant	Crisp	-	Multiple
Manna et al. [147]	EPQ	Constant	Stock level	Fuzzy	-	Single
Panda & Maiti [163]	EPQ	-	Price dependent	Fuzzy	-	Single
Yang & Wee [230]	EPQ	-	Constant	Stochastic	-	Multiple
Present model	EPQ	Production rate & time dependent	Stock and credit linked	Fuzzy-stochastic	Bi-level with fuzzy & crisp	Multiple

In recent years, the green house effect and global warming have gained much attention due to strong and more frequent extreme weather events. In every developing countries, there is a scope of measuring and maintaining such carbon-emission. Benjaafar et al. [9] first presented a series of model formulations that illustrate how carbon emission considerations can be incorporated in to decision-making problem. Dye and Yang [70] study a deteriorating inventory system under various carbon emissions policies.

Uncertainty of the parameters in a decision making is a well established phenomenon in recent years. Estimation of such parameters in the objective functions using traditional econometric methods is not always possible due to the insufficient historical data, especially for newly launched products. Generally, nature of uncertainties can be classified into three major groups such as random (stochastic), fuzzy (imprecise) and rough (approximation). Several research works on fuzzy inventory problem [26, 56] have been done in the existing literature. Panda et al. [163] extended the single period inventory problem in a multi-product manufacturing system under chance and imprecise constraints. Chang [25] developed an EOQ model with fuzzy defective rate and demand. Recently Manna et al. [147] and Das et al. [51] considered different imprecise parameters in their supply chain models.

This chapter considers a manufacturer-retailer-customer supply chain model involving bi-level trade credit and random carbon emission. In this chapter, a two-echelon supply chain system with several markets, by taking the imperfect production process is considered. Here, we establish a bi-level trade credit model to enhance the demand of the customers, which actually is a Stackelberg model with the customer's satisfaction and whole system being the leader in the management. Here,we introduce an integrated production-inventory model with rework policy. Here, the manufacturer offers trade credit period to retailer and as well as retailer offers trade credit to the customers. Here, both the credit periods are fuzzy in nature and the model is defuzzified using the expression of expectation. The demand of the customers is considered as stock dependent.

11.2 Notations and Assumptions

The following notations and assumptions have been used to develop the proposed model:

11.2.1 Notations

The following notations have been used to develop the model.

$q_m(t)$	:	Inventory level of the manufacturer at any time t of perfect quality items.
$q_r^i(t)$	:	Inventory level of the ith retailer at any time t of perfect quality items.
n	:	Number of retailers.
P	:	Production rate in units, $P > D$.
θ	:	Rate of produced defective item which depend on time and production rate.
$\hat{\eta}^{\varepsilon}$	:	Rate of carbon emissions associated per unit produce item, a random variable.
$\hat{\eta}^{\gamma}$	:	Rate of carbon emissions associated per unit rework item, a random variable.
$\hat{\eta}^{\rho}$	:	Rate of carbon emissions associated per unit disposal item, a random variable.
$g(\eta)$	:	The probability density function of $\hat{\eta}$, $\eta \in [0, 1)$.
δ	:	Percentage of rework of defective units per unit time.

d_r^i	:	Demand rate of perfect quality items of the ith retailer.
d_c^i	:	Demand rate of perfect quality items of the customers from ith retailer.
D	:	Selling rate of perfect quality items of the manufacturer, where $D = \sum_{i=1}^{n} d_r^i$.
t_1	:	Production run-time in one period.
t_2	:	Manufacturer business period.
T^i	:	Time at which the selling season ends for ith retailer.
$\widetilde{M}$	:	Imprecise credit period offered by the the manufacturer to the retailers.
N^i	:	Credit period offered by the ith retailer to the customers, $0 < N^i < \widetilde{M}$.
I_{em}	:	Rate of interest per year earned manufacturer from retailer.
I_{er}	:	Rate of interest per year earned retailer from customer.
s_c	:	Screening cost per unit item.
A_m	:	Set up cost of manufacturer, $A_m = A_{m0} + A_{m1}P^k$, $k > 0$.
h_m	:	Holding cost per unit for per unit time for perfect item in manufacturer.
h_r^i	:	Holding cost per unit for per unit time of perfect quality items of the ith retailer.
r_{cm}	:	Reworking cost per unit for manufacturer.
c_p	:	Production cost per unit.
e_c	:	Cost per unit carbon emission.
c_d	:	Disposal cost per unit.
s_m	:	Selling price per unit of perfect quality items for manufacturer.
A_r^i	:	Set up cost of the ith retailer.
s_r^i	:	Selling price per unit of perfect quality items of the ith retailer $s_r^i \in [s_{min}, s_{max}]$.

11.2.2 Assumptions

The following assumptions have been used to develop the model.

(i) Manufacturer produces a mixture of perfect and imperfect quality items. Some portion of imperfect items are reworked and transformed into a perfect quality items.

(ii) The defective rate is not constant, it increased with time and production rate. So, the defective rate depend on time and production rate, it is defined as follows: $\theta = \beta + \lambda P + \xi t$, where β, λ and ξ are positive constants as well as taken suitable values.

(iii) The demand rate of the customers depend on displayed stock/inventory of the item and credit period offered them. i.e., $d_c^i = d_{c0}^i + d_{c1}^i e^{\mu N^i} + d_1^i q_r^i(t)$, $d_{c0}^i > 0, d_{c1}^i > 0, d_1^i > 0, \mu > 0$.

(iv) The production-project might involve rolling out clean energy technologies or soaking up carbon emission from the production, that need to include in a production problem as a carbon emission cost.

(v) Set up cost of manufacturer has been considered as production rate dependent.

(vi) It is assumed that the fuzzy credit period $(\widetilde{M})$ offered by supplier must be within replenishment period (T), i.e., $\widetilde{M} < T$.

(vii) The ith retailer provide a down-stream credit period (N^i) to his / her customers, where $N^i < \widetilde{M}$.

11.3 Mathematical Formulation of the Proposed Model

We consider a manufacturing system which produces both perfect and imperfect item in each production run at a rate $(1-\theta)P$ and θP respectively. Among the imperfect item few items are repaired at a rate $\delta\theta P$ portion We consider a manufacturing system which produces the lot size Q in each production run, with constant production and demand rates denoted by p, and d, respectively. In these process of production, screening and repaired unavoidable carbons are emission at a rate η^{ϵ}, η^{ρ} and η^{γ} respectively. The fresh units are transported to several market with their individual demand along with an imprecise trade credit $\widetilde{M}$. The retailers sold the units in their respective markets as per customers demand $d_c^i(t) = d_{c0}^i + d_{c1}^i e^{\mu^i N^i} + d_1^i q_r^i(t)$. Here it necessary to mention that the demand depend on the displayed stock of the retailer and the credit period N^i offered to the customers. Such supply chain inventory model is derived to formulate different cost expression.

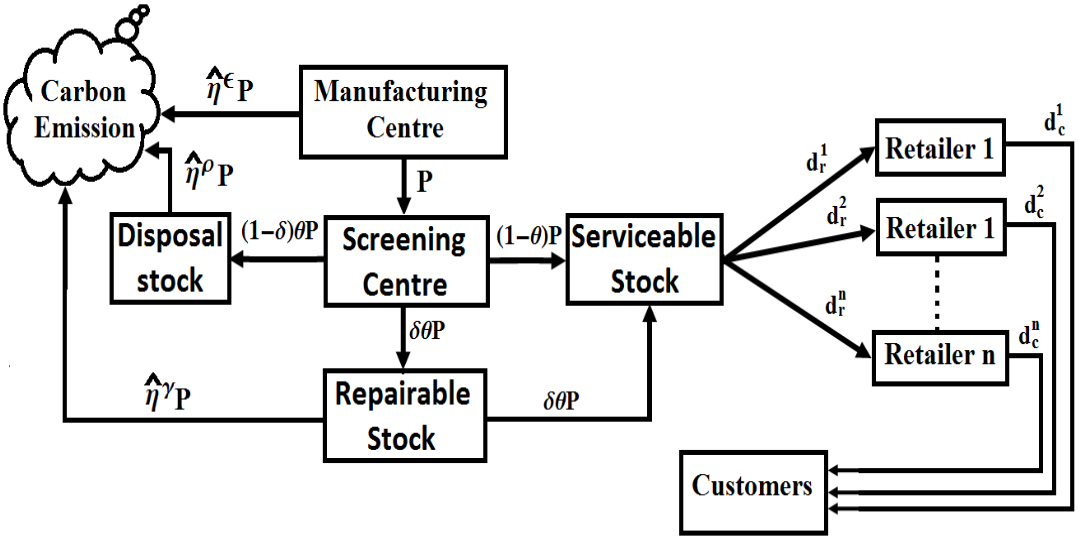

Figure 11.1: The flow of the produce items of the integrated model

11.3.1 Formulation of the Manufacturer

The rate of change of inventory level of manufacturer for perfect quality items can be represented by the following differential equations:

$$\frac{dq_m}{dt} = \begin{cases} P - D - (1-\delta)(\beta + \lambda P + \xi t)P, & 0 \le t \le t_1 \\ -D, & t_1 \le t \le t_2 \end{cases}$$

with boundary conditions $q_m(0) = 0,\ q_m(t_2) = 0.$
The solution of above differential equations are given by

$$q_m(t) = \begin{cases} \{P - D - (1-\delta)(\beta + \lambda P)P\}t - (1-\delta)\frac{\xi}{2}Pt^2, & 0 \le t \le t_1 \\ -D(t - t_2), & t_1 \le t \le t_2 \end{cases}$$

Lemma 11.1. *The manufacturer's production time length (t_1) and production rate (P) must satisfy the condition* $t_2 = \frac{1}{D}\{P - (1-\delta)(\beta + \lambda P)P\}t_1 - (1-\delta)\frac{\xi P}{2D}t_1^2$.

Proof. From the continuity conditions of $q_m(t)$ at $t = t_1$, the following is obtained,

$\{P - D - (1-\delta)(\beta + \lambda P)P\}t_1 - (1-\delta)\frac{\xi}{2}Pt_1^2 = -D(t_1 - t_2)$

$\Rightarrow Pt_1 - (1-\delta)(\beta + \lambda P)Pt_1 - (1-\delta)\frac{\xi}{2}Pt_1^2 = Dt_2$

$\Rightarrow t_2 = \frac{1}{D}\{P - (1-\delta)(\beta + \lambda P)P\}t_1 - (1-\delta)\frac{\xi P}{2D}t_1^2$ □

Inventory holding cost for perfect items is:

$$\begin{aligned} HCM &= h_m\Big[\int_0^{t_1} q_m(t)\,dt + \int_{t_1}^{t_2} q_m(t)\,dt\Big] \\ &= h_m\Big[\int_0^{t_1}\Big\{\{P - D - (1-\delta)(\beta + \lambda P)P\}t - (1-\delta)\frac{\xi}{2}Pt^2\Big\}\,dt - \int_{t_1}^{t_2} D(t - t_2)\,dt\Big] \\ &= \frac{h_m}{2}\Big[\{P - D - (1-\delta)(\beta + \lambda P)P\}t_1^2 - (1-\delta)\frac{\xi}{3}Pt_1^3 + D(t_1 - t_2)^2\Big] \end{aligned}$$

Production cost for the manufacturer= c_pPt_1.
Inspection cost = s_cPt_1.
Reworking cost for manufacture = $r_{cm}\int_0^{t_1}\delta(\beta + \lambda P + \xi t)P\,dt = r_{cm}\delta\Big(\beta + \lambda P + \frac{\xi}{2}t_1\Big)Pt_1$
Revenue of perfect quality items for the manufacturer = $s_md_rt_2$.
Disposal cost during $(0, t_2)$ = $c_d(1-\delta)\Big(\beta + \lambda P + \frac{\xi}{2}t_1\Big)Pt_1$
The total amount of carbon emissions during the production run time can be calculated as follows:

$$\begin{aligned} CE(t_1) &= \hat{\eta}^{\varepsilon}\int_0^{t_1} P\ dt + \hat{\eta}^{\gamma}\int_0^{t_1}\delta(\beta + \lambda P + \xi t)P\ dt + \hat{\eta}^{\rho}\int_0^{t_1}(1-\delta)(\beta + \lambda P + \xi t)P\ dt \\ &= \hat{\eta}^{\varepsilon}Pt_1 + \Big\{\hat{\eta}^{\gamma}\delta + \hat{\eta}^{\rho}(1-\delta)\Big\}\Big(\beta + \lambda P + \frac{\xi}{2}t_1\Big)Pt_1 \end{aligned}$$

The expected carbon emission cost during the production run time is given by

$$E[CE(t_1;\hat{\eta})] \;=\; e_c\Big[E[\hat{\eta}^{\varepsilon}]Pt_1 + \Big\{E[\hat{\eta}^{\gamma}]\delta + E[\hat{\eta}^{\rho}](1-\delta)\Big\}\Big(\beta + \lambda P + \frac{\xi}{2}t_1\Big)Pt_1\Big]$$

Expected total profit $E[\Pi_m(t_1)]$ of manufacturer during the period $(0,T)$ is given by

$$\begin{aligned} E[\Pi_m(t_1;\hat{\eta})] \;=\; & s_m Dt_2 - (c_p + s_c + e_c E[\hat{\eta}^{\varepsilon}])Pt_1 - (r_{cm} + e_c E[\hat{\eta}^{\gamma}])\delta\Big(\beta + \lambda P + \frac{\xi}{2}t_1\Big)Pt_1 \\ & -(c_d + e_c E[\hat{\eta}^{\rho}])(1-\delta)\Big(\beta + \lambda P + \frac{\xi}{2}t_1\Big)Pt_1 - A_m \\ & -\frac{h_m}{2}\Big[\{P - D - (1-\delta)(\beta + \lambda P)P\}t_1^2 - (1-\delta)\frac{\xi}{3}Pt_1^3 + D(t_1 - t_2)^2\Big] \end{aligned}$$

11.3.2 Formulation of the ith Retailer

The ith retailer receives his/her required quantity per unit time d_r^i from the manufacturer and fulfill the customers' demand rate d_c^i. Those retailer start their business on or before the production run time t_1, pay r portion of the price amount payable initially and the remaining $(1-r)$ portion pay at the end of his/her business period. But those retailer's arrive after the production run time t_1, pay the total amount at their business starting time. They pay the initial amount by getting loan from a bank at the rate of interest of I_p per year. Every retailer's earns interest at the rate of I_e by depositing sales revenue continuously. The inventory level $q_r^i(t)$ for the ith retailer's is governed by the following differential equation:

$$\frac{dq_r^i(t)}{dt} \;=\; \begin{cases} (d_r^i - d_c^i), & 0 \le t \le t_2 \\ -d_c^i, & t_2 \le t \le T^i \end{cases}$$

with boundary conditions $q_r^i(0) = 0$ and $q_r^i(T^i) = 0$.
The customer demand is $d_c^i(t) = d_{c0}^i + d_{c1}^i e^{\mu^i N^i} + d_1^i q_r^i(t) = d_0^i + d_1^i q_r^i(t)$,
where $d_0^i = d_{c0}^i + d_{c1}^i e^{\mu N^i}$. Therefore the solutions of above differential equations are given by

$$q_r^i(t) \;=\; \begin{cases} \frac{(d_r^i - d_0^i)}{d_1^i}(1 - e^{-d_1^i t}), & 0 \le t \le t_2 \\ -\dfrac{d_0^i}{d_1^i}[1 - e^{-d_1^i(t - T^i)}], & t_2 \le t \le T^i \end{cases}$$

Lemma 11.2. *The retailer time length of inventory (T^i) is given by*
$T^i = t_2 + \frac{1}{d_1^i} log\{1 + \frac{(d_r^i - d_0^i)}{d_0^i}(1 - e^{-d_1^i t_2})\}$

Proof. From the continuity conditions of $q_r^i(t)$ at $t = t_2$, we have

$\frac{(d_r^i - d_0^i)}{d_1^i}(1 - e^{-d_1^i t_2}) = -\frac{d_0^i}{d_1}\{1 - e^{-d_1(t_2 - T^i)}\}$

$\Rightarrow (d_r^i - d_0^i)(1 - e^{-d_1^i t_2^i}) = -d_0^i\{1 - e^{-d_1^i(t_2^i - T^i)}\}$

$\Rightarrow e^{-d_1^i(t_2 - T^i)} = 1 + \frac{(d_r^i - d_0^i)}{d_0^i}(1 - e^{-d_1^i t_2})$

$\Rightarrow T^i = t_2 + \frac{1}{d_1^i} log\{1 + \frac{(d_r^i - d_0^i)}{d_0^i}(1 - e^{-d_1^i t_2})\}$ □

Holding cost of the ith retailer is given by

$$\begin{aligned} HCR^i &= h_r^i\Big[\int_0^{t_2} q_r^i(t)\,dt + \int_{t_2}^{T^i} q_r^i(t)\,dt\Big] \\ &= h_r^i\Big[\frac{(d_r^i - d_0^i)}{d_1^i}\Big\{t_2 + \frac{1}{d_1^i}(e^{-d_1^i t_2} - 1)\Big\} - \frac{d_0^i}{d_1^i}\Big\{(T^i - t_2) + \frac{1}{d_1^i}\{1 - e^{-d_1^i(t_2 - T^i)}\}\Big\}\Big] \end{aligned}$$

Holding cost (HCR) for all retailers' is given by

$$HCR = \sum_{i=1}^{n} h_r^i\Big[\frac{(d_r^i - d_0^i)}{d_1^i}\Big\{t_2 + \frac{1}{d_1^i}(e^{-d_1^i t_2} - 1)\Big\} - \frac{d_0^i}{d_1^i}\Big\{(T^i - t_2) + \frac{1}{d_1^i}\{1 - e^{-d_1^i(t_2 - T^i)}\}\Big\}\Big]$$

Sales revenue from perfect quality items of the ith retailer is given by

$$\begin{aligned} SRR^i &= s_r^i\Big[\int_0^{t_2} (d_0^i + d_1^i q_r^i(t))\,dt + \int_{t_2}^{T} (d_0^i + d_1^i q_r^i(t))\,dt\Big] \\ &= s_r^i\Big[d_0^i t_2 + (d_r^i - d_0^i)\Big\{t_2 + \frac{1}{d_1^i}(e^{-d_1^i t_2} - 1)\Big\} - \frac{d_0^i}{d_1^i}\{1 - e^{-d_1^i(t_2 - T^i)}\}\Big] \\ &= s_r^i\Big[d_r^i t_2 + \frac{(d_r^i - d_0^i)}{d_1^i}(e^{-d_1^i t_2} - 1) - \frac{d_0^i}{d_1^i}\{1 - e^{-d_1^i(t_2 - T^i)}\}\Big] \end{aligned}$$

All retailers' total sales revenue (SRR) is given by

$$SRR = \sum_{i=1}^{n} s_r^i\Big[d_r^i t_2 + \frac{(d_r^i - d_0^i)}{d_1^i}(e^{-d_1^i t_2} - 1) - \frac{d_0^i}{d_1^i}\{1 - e^{-d_1^i(t_2 - T^i)}\}\Big]$$

All retailers' total purchase cost (PCR) is given by

$$PCR = \sum_{i=1}^{n} s_m d_r^i t_2$$

Here it is assumed that the retailer's trade credit period offered by the manufacturer is M and that of customer's offered by the retailer is N^i (where $N^i < M$). The retailer is charged by the manufacturer, an interest at the rate of I_p per year per unit for the unpaid amount after the delay period and can earn an interest at the rate of I_e ($I_e > I_p$) per year per unit for the amount sold during the period (N^i, M) respectively. Depending on the cycle times of the retailer and offering as well as receiving credit periods, three different cases may arise, which have been discussed separately.

Case-I: when $N^i < M < t_2 < T^i$

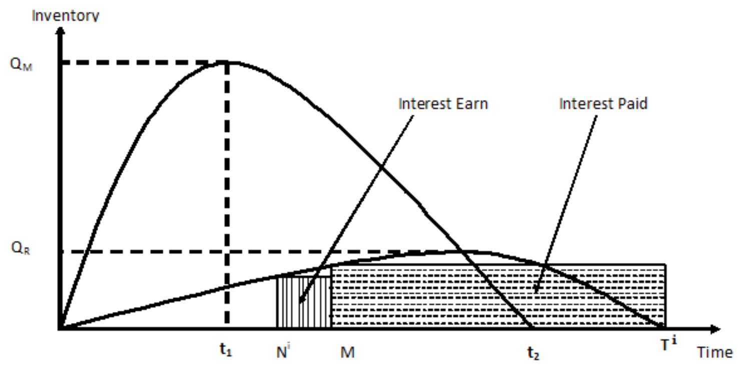

Figure 11.2: Total interest earned and paid representation when $N^i < M < t_2 < T^i$

Interest paid by the ith retailer (IP_1^i),

$$
\begin{aligned}
IP_1^i &= s_m I_p \int_M^{T^i} q_r^i(t)\,dt \\
&= s_m I_p \Big[\int_M^{t_2} q_r^i(t)\,dt + \int_{t_2}^{T^i} q_r^i(t)\,dt \Big] \\
&= s_m I_p \Big[\frac{(d_r^i - d_0^i)}{d_1^i} \Big\{ (t_2 - M) + \frac{1}{d_1^i}(e^{-d_1^i t_2} - e^{-d_1^i M}) \Big\} \\
&\quad - \frac{d_0^i}{d_1^i} \Big\{ (T^i - t_2) + \frac{1}{d_1^i}\{1 - e^{-d_1^i(t_2 - T^i)}\} \Big\} \Big]
\end{aligned}
$$

Interest earned by the ith retailer (IE_1^i),

$$
\begin{aligned}
IE_1^i &= s_r^i I_e \Big[(T^i - N^i) \int_0^{N^i} d_c^i(t)\,dt + (T^i - M) \int_{N^i}^{M} (M - t) d_c^i(t)\,dt \\
&\quad + (T^i - t_2) \int_M^{t_2} (t_2 - t) d_c^i(t)\,dt + \int_{t_2}^{T^i} (T^i - t) d_c^i(t)\,dt \Big] \\
&= s_r^i I_e \Big[(T^i - N^i) \Big\{ d_0^i N^i + (d_r^i - d_0) \Big\{ N + \frac{1}{d_1^i}(e^{-d_1^i N^i} - 1) \Big\} \Big\} \\
&\quad + (T^i - M) \Big\{ \frac{d_r^i}{2}(M - N^i)^2 - (M - N^i)\frac{e^{-d_1^i N^i}}{d_1^i} + \frac{1}{(d_1^i)^2}(e^{-d_1^i N^i} - e^{-d_1^i M}) \Big\} \\
&\quad + (T^i - t_2) \Big\{ \frac{d_r^i}{2}(M - t_2)^2 - (t_2 - M)\frac{e^{-d_1^i M}}{d_1^i} + \frac{1}{(d_1^i)^2}(e^{-d_1^i M} - e^{-d_1^i t_2}) \Big\} \\
&\quad + \frac{d_0^i}{d_1^i}(T^i - t_2) e^{-d_1^i(t_2 - T^i)} + \frac{d_0^i}{(d_1^i)^2}\{1 - e^{-d_1^i(t_2 - T^i)}\} \Big]
\end{aligned}
$$

All retailers' total interest payable (IP_1) is expressed as

$$\begin{aligned} IP_1 &= \sum_{i=1}^{n} s_m I_p \Big[\frac{(d_r^i - d_0^i)}{d_1^i}\Big\{(t_2 - M) + \frac{1}{d_1^i}(e^{-d_1^i t_2} - e^{-d_1^i M})\Big\} \\ &\quad -\frac{d_0^i}{d_1^i}\Big\{(T^i - t_2) + \frac{1}{d_1^i}\{1 - e^{-d_1^i(t_2 - T^i)}\}\Big\}\Big] \end{aligned}$$

All retailers' total interest earned (IE_1) is obtained as

$$\begin{aligned} IE_1 &= \sum_{i=1}^{n} s_r^i I_e \Big[(T^i - N^i)\Big\{d_0^i N^i + (d_r^i - d_0)\Big\{N + \frac{1}{d_1^i}(e^{-d_1^i N^i} - 1)\Big\}\Big\} \\ &\quad +(T^i - M)\Big\{\frac{d_r^i}{2}(M - N^i)^2 - (M - N^i)\frac{e^{-d_1^i N^i}}{d_1^i} + \frac{1}{(d_1^i)^2}(e^{-d_1^i N^i} - e^{-d_1^i M})\Big\} \\ &\quad +(T^i - t_2)\Big\{\frac{d_r^i}{2}(M - t_2)^2 - (t_2 - M)\frac{e^{-d_1^i M}}{d_1^i} + \frac{1}{(d_1^i)^2}(e^{-d_1^i M} - e^{-d_1^i t_2})\Big\} \\ &\quad +\frac{d_0^i}{d_1^i}(T^i - t_2)e^{-d_1^i(t_2 - T^i)} + \frac{d_0^i}{(d_1^i)^2}\{1 - e^{-d_1^i(t_2 - T^i)}\}\Big] \end{aligned}$$

Therefore, all retailers' total profit is given by

$$\Pi_r^{(1)}(t_1) = SRR - PCR - HCR - IP_1 + IE_1 - \sum_{i=1}^{n} A_r^i$$

So, the total profit (ITP) for this case of the integrated system is written as
$E[ITP_1(t_1;\hat{\eta})] = E[\Pi(t_1;\hat{\eta})] + \Pi_r^{(1)}(t_1)$

Case-II: when $N^i < t_2 < M < T^i$

Interest paid by the retailer (IP_2^i),

$$\begin{aligned} IP_2^i &= s_m I_p \int_M^{T^i} q_r^i(t)\, dt \\ &= s_m I_p \int_M^{T^i} -\frac{d_0^i}{d_1^i}[1 - e^{-d_1^i(t - T^i)}]\, dt \\ &= \frac{d_0^i s_m I_p}{d_1^i}\Big[\frac{1}{d_1^i}\{e^{-d_1^i(M - T^i)} - 1\} - (T^i - M)\Big] \end{aligned}$$

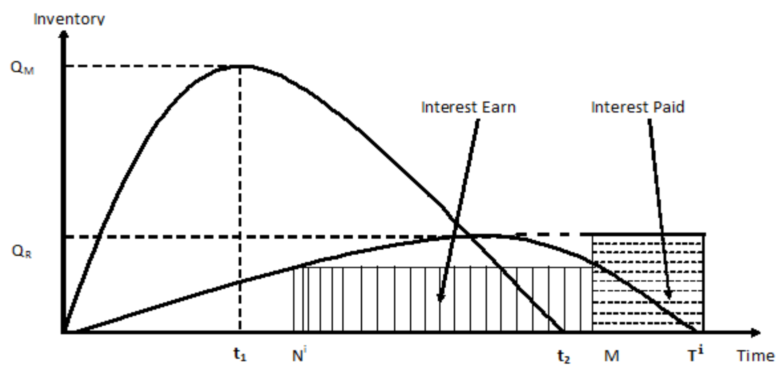

Figure 11.3: Total interest earned and paid representation when $N^i < t_2 < M < T^i$

Interest earned by the retailer (IE_2^i),

$$\begin{aligned} IE_2^i &= s_r^i I_e\Big[(T^i - N^i)\int_0^{N^i} d_c^i(t)\,dt + (T^i - t_2)\int_{N^i}^{t_2}(t_2 - t)d^i{}_c(t)\,dt \\ &+(T^i - M)\int_{t_2}^{M}(M-t)d_c^i(t)\,dt + \int_M^{T^i}(T^i - t)d_c^i(t)\,dt\Big] \\ &= s_r^i I_e\Big[(T^i - N^i)\Big\{d_0^i N^i + (d_r^i - d_0^i)\Big\{N^i + \frac{1}{d_1^i}(e^{-d_1^i N^i} - 1)\Big\}\Big\} \\ &+(T^i - t_2^i)\Big\{\frac{d_r^i}{2}(t_2^i - N^i)^2 - (t_2 - N^i)\frac{e^{-d_1^i N^i}}{d_1^i} + \frac{1}{(d_1^i)^2}(e^{-d_1^i N^i} - e^{-d_1^i t_2})\Big\} \\ &+(T^i - M)\Big\{\frac{d_0^i}{d_1^i}(M - t_2)e^{-d_1^i(t_2 - M)} + \frac{d_0^i}{(d^i)_1^2}\{1 - e^{-d_1^i(t_2 - M)}\}\Big\} \\ &+\frac{d_0^i}{d_1^i}(T^i - M)e^{-d_1^i(M - T^i)} + \frac{d_0}{(d_1^i)^2}\{1 - e^{-d_1^i(M - T^i)}\}\Big] \end{aligned}$$

All retailers' total interest payable (IP_2) is expressed as

$$IP_2 = \sum_{i=1}^{n}\frac{d_0^i s_m I_p}{d_1^i}\Big[\frac{1}{d_1^i}\{e^{-d_1^i(M-T^i)} - 1\} - (T^i - M)\Big]$$

All retailers' total interest earned (IE_2) is obtained as

$$\begin{aligned} IE_2 &= \sum_{i=1}^{n} s_r^i I_e\Big[(T^i - N^i)\Big\{d_r^i N^i + \frac{(d_r^i - d_0^i)}{d_1^i}(e^{-d_1^i N^i} - 1)\Big\} + (T^i - t_2^i)\Big\{\frac{d_r^i}{2}(t_2^i - N^i)^2 \\ &-(t_2 - N^i)\frac{e^{-d_1^i N^i}}{d_1^i} + \frac{1}{(d_1^i)^2}(e^{-d_1^i N^i} - e^{-d_1^i t_2})\Big\} + (T^i - M)\Big\{\frac{d_0^i}{d_1^i}(M - t_2)e^{-d_1^i(t_2 - M)} \\ &+\frac{d_0^i}{(d^i)_1^2}\{1 - e^{-d_1^i(t_2 - M)}\}\Big\} + \frac{d_0^i}{d_1^i}(T^i - M)e^{-d_1^i(M - T^i)} + \frac{d_0}{(d_1^i)^2}\{1 - e^{-d_1^i(M - T^i)}\}\Big] \end{aligned}$$

Therefore, all retailers' total profit is given by

$$\Pi_r^{(2)}(t_1) = SRR - PCR - HCR - IP_2 + IE_2 - \sum_{i=1}^{n} A_r^i$$

So, the total profit (ITP) for this case of the integrated system is written as
$E[ITP_2(t_1;\hat{\eta})] = E[\Pi(t_1;\hat{\eta})] + \Pi_r^{(2)}(t_1)$

Case-III: when $t_2 < N^i < M < T^i$

Interest paid by the retailer (IP_3^i),

$$\begin{aligned} IP_3^i &= s_m I_p \int_M^{T^i} q_r^i(t)\,dt \\ &= s_m I_p \int_M^{T^i} -\frac{d_0^i}{d_1^i}[1 - e^{-d_1^i(t-T^i)}]\,dt \\ &= \frac{d_0 s_m I_p}{d_1^i}\Big[\frac{1}{d_1^i}\{e^{-d_1^i(M-T^i)} - 1\} - (T^i - M)\Big] \end{aligned}$$

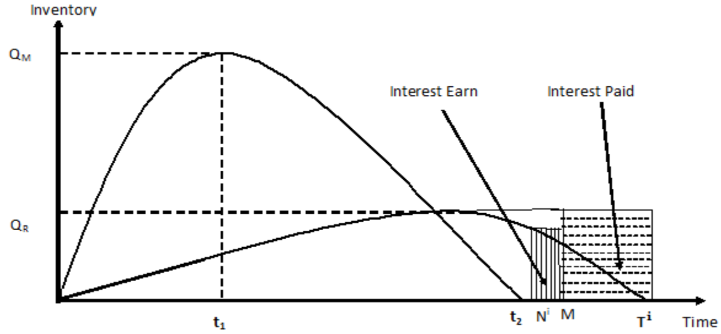

Figure 11.4: Total interest earned and paid representation when $t_2 < N^i < M < T^i$

Interest earned by the retailer (IE_3^i),

$$\begin{aligned} IE_3^i &= s_r^i I_e\Big[(T^i - N^i)\Big\{\int_0^{t_2} d_c^i(t)\,dt + \int_{t_2}^{N^i} d_c^i(t)\,dt\Big\} + \int_{N^i}^{T^i}(T^i - t)d_c^i(t)\,dt\Big] \\ &= s_r^i I_e\Big[(T^i - N^i)\Big\{d_0^i N^i + (d_r^i - d_0^i)\{t_2 + \frac{1}{d_1^i}(e^{-d_1^i t_2} - 1)\} - d_0^i\Big\{(N^i - t_2) \\ &\quad + \frac{1}{d_1^i}\{1 - e^{-d_1^i(t_2 - N^i)}\}\Big\}\Big\} + \frac{d_0^i}{d_1^i}(T^i - N^i)e^{-d_1^i(N^i - T^i)} + \frac{d_0^i}{(d_1^i)^2}\{1 - e^{-d_1^i(N^i - T^i)}\}\Big] \end{aligned}$$

All retailers' total interest payable (IP_3) is expressed as

$$IP_3 = \sum_{i=1}^{n} \frac{d_0 s_m I_p}{d_1^i}\Big[\frac{1}{d_1^i}\{e^{-d_1^i(M-T^i)} - 1\} - (T^i - M)\Big]$$

All retailers' total interest earned (IE_3) is obtained as

$$IE_3 = \sum_{i=1}^{n} s_r^i I_e\Big[(T^i - N^i)\Big\{d_0^i N^i + (d_r^i - d_0^i)\{t_2 + \frac{1}{d_1^i}(e^{-d_1^i t_2} - 1)\} - d_0^i\Big\{(N^i - t_2) + \frac{1}{d_1^i}\{1 - e^{-d_1^i(t_2 - N^i)}\}\Big\}\Big\} + \frac{d_0^i}{d_1^i}(T^i - N^i)e^{-d_1^i(N^i - T^i)} + \frac{d_0^i}{(d_1^i)^2}\{1 - e^{-d_1^i(N^i - T^i)}\}\Big]$$

Therefore, all retailers' total profit is given by

$$\Pi_r^{(3)}(t_1) = SRR - PCR - HCR - IP_3 + IE_3 - \sum_{i=1}^{n} A_r^i$$

So, the total profit (ITP) for this case of the integrated system is written as
$E[ITP_3(t_1;\hat{\eta})] = E[\Pi(t_1;\hat{\eta})] + \Pi_r^{(3)}(t_1)$

11.3.3 Objective of the Proposed Model

The integrated total profit (ITP) for this case of the integrated system is written as
$E[ITP_3(t_1;\hat{\eta})] = E[\Pi(t_1;\hat{\eta})] + \Pi_r^{(3)}(t_1)$

When manufacturer and retailers' have decided to share resources to undertake mutually beneficial cooperation, the joint total profit which is a function of t_1 can be obtained by maximized $E[ITP(t_1;\hat{\eta})]$ and is given by

$$\text{Maximize } E[ITP(t_1;\hat{\eta})] = \begin{cases} E[ITP_1(t_1;\hat{\eta})] = E[\Pi_m(t_1;\hat{\eta})] + \Pi_r^{(1)}(t_1), & \text{if } N^i < M < t_2 < T^i \\ E[ITP_2(t_1;\hat{\eta})] = E[\Pi_m(t_1;\hat{\eta})] + \Pi_r^{(2)}(t_1), & \text{if } N^i < t_2 < M < T^i \\ E[ITP_3(t_1;\hat{\eta})] = E[\Pi_m(t_1;\hat{\eta})] + \Pi_r^{(3)}(t_1), & \text{if } t_2 < N^i < M < T^i \end{cases}$$

11.3.4 Model with Fuzzy Credit Period

If we assume that the manufacturer gives an opportunity to all the retailers' by offering a fuzzy credit period $(\widetilde{M})$. Here, the credit period $\widetilde{M}$ is represented in form of triangular fuzzy number. So due to fuzzy credit period $(\widetilde{M})$, the optimum value of integrated profit function $ITP(t_1)$ will be different for various values of $\widetilde{M}$ with some degree of belonging ness. Therefore in such situation, the profit function will be fuzzy in nature and is denoted by $\widetilde{ITP}(t_1;\widetilde{M},\hat{\eta})$, where

$$E[\widetilde{ITP}(t_1;\widetilde{M},\hat{\eta})] = \begin{cases} E[\widetilde{ITP}^{(1)}(t_1;\widetilde{M},\hat{\eta})] = E[\Pi_m(t_1;\hat{\eta})] + \widetilde{\Pi}_r^{(1)}(t_1;\widetilde{M}), & \text{if } N^i < \widetilde{M} < t_2 < T^i \\ E[\widetilde{ITP}^{(2)}(t_1;\widetilde{M},\hat{\eta})] = E[\Pi_m(t_1;\hat{\eta})] + \widetilde{\Pi}_r^{(2)}(t_1;\widetilde{M}), & \text{if } N^i < t_2 < \widetilde{M} < T^i \\ E[\widetilde{ITP}^{(3)}(t_1;\widetilde{M},\hat{\eta})] = E[\Pi_m(t_1;\hat{\eta})] + \widetilde{\Pi}_r^{(3)}(t_1;\widetilde{M}), & \text{if } t_2 < N^i < \widetilde{M} < T^i \end{cases}$$

The imprecise expression of $E[\widetilde{ITP}^{(1)}(t_1;\widetilde{M},\hat{\eta})]$, $E[\widetilde{ITP}^{(2)}(t_1;\widetilde{M},\hat{\eta})]$ and $E[\widetilde{ITP}^{(3)}(t_1;\widetilde{M},\hat{\eta})]$ are given below:

$$\begin{aligned} E[\widetilde{ITP}^{(1)}(t_1;\widetilde{M},\hat{\eta})] &= s_m \sum_{i=1}^{n} d_r^i t_2 - (r_{cm} + e_c E[\hat{\eta}^\gamma])\delta\Big(\beta + \lambda P + \frac{\xi}{2}t_1\Big)Pt_1 \\ &-(c_p + s_c + e_c E[\hat{\eta}^\varepsilon])Pt_1 - (c_d + e_c E[\hat{\eta}^\rho])(1-\delta)\Big(\beta + \lambda P + \frac{\xi}{2}t_1\Big)Pt_1 \\ &-\frac{h_m}{2}\Big[\{P - \sum_{i=1}^{n} d_r^i - (1-\delta)(\beta+\lambda P)P\}t_1^2 - (1-\delta)\frac{\xi}{3}Pt_1^3 + \sum_{i=1}^{n} d_r^i(t_1-t_2)^2\Big] - A_m \\ &+\sum_{i=1}^{n} s_r^i\Big[d_r^i t_2 + \frac{(d_r^i - d_0^i)}{d_1^i}(e^{-d_1^i t_2} - 1) - \frac{d_0^i}{d_1^i}\{1 - e^{-d_1^i(t_2 - T^i)}\}\Big] - \sum_{i=1}^{n} s_m d_r^i t_2 \\ &-\sum_{i=1}^{n} h_r^i\Big[\frac{(d_r^i - d_0^i)}{d_1^i}\Big\{t_2 + \frac{1}{d_1^i}(e^{-d_1^i t_2} - 1)\Big\} - \frac{d_0^i}{d_1^i}\Big\{(T^i - t_2) + \frac{1}{d_1^i}\{1 - e^{-d_1^i(t_2 - T^i)}\}\Big\}\Big] \\ &-\sum_{i=1}^{n} s_m I_p\Big[\frac{(d_r^i - d_0^i)}{d_1^i}\Big\{(t_2 - \widetilde{M}) + \frac{1}{d_1^i}(e^{-d_1^i t_2} - e^{-d_1^i \widetilde{M}})\Big\} - \frac{d_0^i}{d_1^i}\Big\{(T^i - t_2) + \frac{1}{d_1^i}\{1 \\ &-e^{-d_1^i(t_2 - T^i)}\}\Big\}\Big] + \sum_{i=1}^{n} s_r^i I_e\Big[(T^i - N^i)\Big\{d_0^i N^i + (d_r^i - d_0)\Big\{N + \frac{1}{d_1^i}(e^{-d_1^i N^i} - 1)\Big\}\Big\} \\ &+(T^i - \widetilde{M})\Big\{\frac{d_r^i}{2}(\widetilde{M} - N^i)^2 - (\widetilde{M} - N^i)\frac{e^{-d_1^i N^i}}{d_1^i} + \frac{1}{(d_1^i)^2}(e^{-d_1^i N^i} - e^{-d_1^i \widetilde{M}})\Big\} \\ &+(T^i - t_2)\Big\{\frac{d_r^i}{2}(\widetilde{M} - t_2)^2 - (t_2 - \widetilde{M})\frac{e^{-d_1^i \widetilde{M}}}{d_1^i} + \frac{1}{(d_1^i)^2}(e^{-d_1^i \widetilde{M}} - e^{-d_1^i t_2})\Big\} \\ &+\frac{d_0^i}{d_1^i}(T^i - t_2)e^{-d_1^i(t_2 - T^i)} + \frac{d_0^i}{(d_1^i)^2}\{1 - e^{-d_1^i(t_2 - T^i)}\}\Big] - \sum_{i=1}^{n} A_r^i \end{aligned}$$

$$E[\widetilde{ITP}^{(2)}(t_1;\widetilde{M},\hat{\eta})] = s_m\sum_{i=1}^{n} d_r^i t_2 - (r_{cm}+e_cE[\hat{\eta}^{\gamma}])\delta\Big(\beta+\lambda P+\frac{\xi}{2}t_1\Big)Pt_1$$
$$-(c_p+s_c+e_cE[\hat{\eta}^{\varepsilon}])Pt_1-(c_d+e_cE[\hat{\eta}^{\rho}])(1-\delta)\Big(\beta+\lambda P+\frac{\xi}{2}t_1\Big)Pt_1$$
$$-\frac{h_m}{2}\Big[\{P-\sum_{i=1}^{n}d_r^i-(1-\delta)(\beta+\lambda P)P\}t_1^2-(1-\delta)\frac{\xi}{3}Pt_1^3+\sum_{i=1}^{n}d_r^i(t_1-t_2)^2\Big]-A_m$$
$$+\sum_{i=1}^{n}s_r^i\Big[d_r^it_2+\frac{(d_r^i-d_0^i)}{d_1^i}(e^{-d_1^it_2}-1)-\frac{d_0^i}{d_1^i}\{1-e^{-d_1^i(t_2-T^i)}\}\Big]-\sum_{i=1}^{n}s_md_r^it_2$$
$$-\sum_{i=1}^{n}h_r^i\Big[\frac{(d_r^i-d_0^i)}{d_1^i}\Big\{t_2+\frac{1}{d_1^i}(e^{-d_1^it_2}-1)\Big\}-\frac{d_0^i}{d_1^i}\Big\{(T^i-t_2)+\frac{1}{d_1^i}\{1-e^{-d_1^i(t_2-T^i)}\}\Big\}\Big]$$
$$-\sum_{i=1}^{n}\frac{d_0^is_mI_p}{d_1^i}\Big[\frac{1}{d_1^i}\{e^{-d_1^i(\widetilde{M}-T^i)}-1\}-(T^i-\widetilde{M})\Big]+\sum_{i=1}^{n}s_r^iI_e\Big[(T^i-N^i)\Big\{d_0^iN^i$$
$$+(d_r^i-d_0^i)\Big\{N^i+\frac{1}{d_1^i}(e^{-d_1^iN^i}-1)\Big\}\Big\}+(T^i-t_2^i)\Big\{\frac{d_r^i}{2}(t_2^i-N^i)^2-(t_2-N^i)\frac{e^{-d_1^iN^i}}{d_1^i}$$
$$+\frac{1}{(d_1^i)^2}(e^{-d_1^iN^i}-e^{-d_1^it_2})\Big\}+(T^i-\widetilde{M})\Big\{\frac{d_0^i}{d_1^i}(\widetilde{M}-t_2)e^{-d_1^i(t_2-M)}+\frac{d_0^i}{(d^i)_1^2}\{1$$
$$-e^{-d_1^i(t_2-\widetilde{M})}\}\Big\}+\frac{d_0^i}{d_1^i}(T^i-\widetilde{M})e^{-d_1^i(\widetilde{M}-T^i)}+\frac{d_0}{(d_1^i)^2}\{1-e^{-d_1^i(\widetilde{M}-T^i)}\}\Big]-\sum_{i=1}^{n}A_r^i$$

and $$E[\widetilde{ITP}^{(3)}(t_1;\widetilde{M},\hat{\eta})] = s_m\sum_{i=1}^{n} d_r^i t_2 - (r_{cm}+e_cE[\hat{\eta}^{\gamma}])\delta\Big(\beta+\lambda P+\frac{\xi}{2}t_1\Big)Pt_1$$
$$-(c_p+s_c+e_cE[\hat{\eta}^{\varepsilon}])Pt_1-(c_d+e_cE[\hat{\eta}^{\rho}])(1-\delta)\Big(\beta+\lambda P+\frac{\xi}{2}t_1\Big)Pt_1$$
$$-\frac{h_m}{2}\Big[\{P-\sum_{i=1}^{n}d_r^i-(1-\delta)(\beta+\lambda P)P\}t_1^2-(1-\delta)\frac{\xi}{3}Pt_1^3+\sum_{i=1}^{n}d_r^i(t_1-t_2)^2\Big]-A_m$$
$$+\sum_{i=1}^{n}s_r^i\Big[d_r^it_2+\frac{(d_r^i-d_0^i)}{d_1^i}(e^{-d_1^it_2}-1)-\frac{d_0^i}{d_1^i}\{1-e^{-d_1^i(t_2-T^i)}\}\Big]-\sum_{i=1}^{n}s_md_r^it_2$$
$$-\sum_{i=1}^{n}h_r^i\Big[\frac{(d_r^i-d_0^i)}{d_1^i}\Big\{t_2+\frac{1}{d_1^i}(e^{-d_1^it_2}-1)\Big\}-\frac{d_0^i}{d_1^i}\Big\{(T^i-t_2)+\frac{1}{d_1^i}\{1-e^{-d_1^i(t_2-T^i)}\}\Big\}\Big]$$
$$-\sum_{i=1}^{n}\frac{d_0s_mI_p}{d_1^i}\Big[\frac{1}{d_1^i}\{e^{-d_1^i(\widetilde{M}-T^i)}-1\}-(T^i-\widetilde{M})\Big]+\sum_{i=1}^{n}s_r^iI_e\Big[(T^i-N^i)\Big\{d_0^iN^i$$
$$+(d_r^i-d_0^i)\{t_2+\frac{1}{d_1^i}(e^{-d_1^it_2}-1)\}-d_0^i\Big\{(N^i-t_2)+\frac{1}{d_1^i}\{1-e^{-d_1^i(t_2-N^i)}\}\Big\}\Big\}$$
$$+\frac{d_0^i}{d_1^i}(T^i-N^i)e^{-d_1^i(N^i-T^i)}+\frac{d_0^i}{(d_1^i)^2}\{1-e^{-d_1^i(N^i-T^i)}\}\Big]-\sum_{i=1}^{n}A_r^i$$

11.4 Solution Procedure

The optimum values production time (t_1) and expected total profit for the fuzzy stochastic model are obtained through algorithm.

$\underline{Step 1}$: For the random variable '$\hat{\eta}$' with p.d.f '$g(\eta)$' evaluate the expected value of integrated total profit $\widetilde{ITP}^{(1)}(t_1;\widetilde{M},\hat{\eta})$, $\widetilde{ITP}^{(2)}(t_1;\widetilde{M},\hat{\eta})$ and $\widetilde{ITP}^{(3)}(t_1;\widetilde{M},\hat{\eta})$ using the definition $E[\widetilde{ITP}^{(1)}(t_1;\widetilde{M},\hat{\eta})] = \int_{-\infty}^{\infty} \widetilde{ITP}^{(1)}(t_1;\widetilde{M},\eta)g(\eta)\,d(\eta)$, Similarly obtained $E[\widetilde{ITP}^{(2)}(t_1;\widetilde{M},\hat{\eta})]$ and $E[\widetilde{ITP}^{(3)}(t_1;\widetilde{M},\hat{\eta})]$.

$\underline{Step 2}$: For a given triangular fuzzy number (TFN) $\widetilde{M} = (M - \Delta_1, M, M + \Delta_2)$, the fuzzy expressions of $\widetilde{EITP}^{(1)}_{\widetilde{M}} = E[\widetilde{ITP^{(1)}}] = \left(EITP^{(1)}_l, EITP^{(1)}_m, EITP^{(1)}_r\right)$, $\widetilde{EITP}^{(2)}_{\widetilde{M}} = E[\widetilde{ITP^{(2)}}] = \left(EITP^{(2)}_l, EITP^{(2)}_m, EITP^{(2)}_r\right)$ and $\widetilde{EITP}^{(3)}_{\widetilde{M}} = E[\widetilde{ITP^{(3)}}] = \left(EITP^{(3)}_l, EITP^{(3)}_m, EITP^{(3)}_r\right)$ are obtained using fuzzy extension principal.

$\underline{Step 3}$: Then from the fuzzy expressions $E[\widetilde{ITP}^{(1)}(t_1;\widetilde{M},\hat{\eta})]$, $E[\widetilde{ITP}^{(2)}(t_1;\widetilde{M},\hat{\eta})]$ and $E[\widetilde{ITP}^{(3)}(t_1;\widetilde{M},\hat{\eta})]$ obtained the centroid values $CEITP^{(1)} = C[\widetilde{EITP}^{(1)}_{\widetilde{M}}]$, $CEITP^{(2)} = C[\widetilde{EITP}^{(2)}_{\widetilde{M}}]$ and $CEITP^{(3)} = C[\widetilde{EITP}^{(3)}_{\widetilde{M}}]$ respectively, using the definition presents in § 4, which is the process of defuzzification.

$\underline{Step 4}$: Finally, maximized each $CEITP^{(1)}(t_1)$, $CEITP^{(2)}(t_1)$ and $CEITP^{(3)}(t_1)$ with respect to the decision variable t_1 by using LINGO Solver 12.0 for particular input data.

11.5 Numerical Illustrations

In this section, we illustrate some numerical examples to study the feasibility of the proposed imperfect production inventory model.

Example 11.1. *we consider a production-inventory supply chain model with the following characteristics:*

$P = 42$ units, $\beta_1 = 0.10$, $\lambda = 0.001$, $\xi = 0.02$, $\Delta_1 = 0.01$, $\Delta_2 = 0.02$, $\delta = 0.70$, c_p=\$32 per unit, $c_{sr} = \$2$ per unit, $r_{cm} = \$10$ per unit, c_d=\$5 per unit, $h_m = \$4$ per unit per unit time, $h^{(1)}_r = \$4.5$ per unit per unit time, $h^{(2)}_r = \$4.8$ per unit per unit time, $A^{(m0)}_r = \$140$, $A^{(m1)}_r = \$130$, $A^{(1)}_r = \$140$, $A^{(2)}_r = \$130$, $s_m = \$140$ per unit, $s^{(1)}_r = \$260$ per unit, $s^{(2)}_r = \$250$ per unit, $e_c = \$2.5$ per unit, $s_{min} = \$220$, $s_{max} = \$280$, $d^{(1)}_r = 17$ unit, $d^{(2)}_r = 18$

unit, $d_{c0}^{(1)} = 9.84$ unit, $d_{c0}^{(2)} = 9.54$ unit, $d_{c1}^{(1)} = 1.8$, $d_{c1}^{(2)} = 3$, $d_1^{(1)} = 7$, $d_1^{(2)} = 6$, $\mu^{(1)} = 7$, $\mu^{(2)} = 6$. The carbon-emission rates η^{ϵ}, η^{γ}, η^{ρ} for production process, rework process and disposal units respective followed a Beta distribution with parameters v, w i.e., the p.d.f. of η is

$$g(\eta) \;=\; \begin{cases} \frac{\eta^{v-1}(1-\eta)^{w-1}}{\beta(v,w)}, & 0 \le \eta \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$E[\hat{\eta}^{\epsilon}] = \frac{v}{v+w}$, $E[\hat{\eta}^{\gamma}] = \frac{v(v+1)}{(v+w)(v+w+1)}$, $E[\hat{\eta}^{\rho}] = \frac{v(v+1)(v+2)}{(v+w)(v+w+1)(v+w+2)}$ for $\epsilon = 1, \gamma = 2$ and $\rho = 3$. For $v = 8$, $w = 3$; $E[\hat{\eta}^{\epsilon}] = 0.727$, $E[\hat{\eta}^{\gamma}] = 0.545$ and $E[\hat{\eta}^{\rho}] = 0.419$ numerical computation, we consider applying the proposed computational algorithm yields the results shown in Table 11.2 for different cases.

Table 11.2: Optimal results of illustrated model when $\theta = \beta + \lambda P + \xi t$

Cases	Retailers' credit period (N^1, N^2)	Manufacturer credit period $(\widetilde{M})$	Production time t_1	Period of Manufacturer	Period of Retailers'	Expected Profit
Case-I	(0.12, 0.10)	(0.19, 0.20, 0.22)	0.528	0.605	(0.734, 0.730)	4178.54
Case-II	(0.12, 0.10)	(0.69, 0.70, 0.72)	0.531	0.610	(0.739, 0.732)	3512.53
Case-III	(0.65, 0.62)	(0.69, 0.70, 0.72)	0.538	0.618	(0.726, 0.720)	4128.05

In manager's point of view, 2nd case gives minimum profit. Since, in that case manufacturer lost maximum opportunity of credit period, where as the customers receive maximum benefit from the management system. The first and last cases are nearly same profitable, since in the first case both the members offer lower credit period and in the last case both the members offer higher credit periods. More over, in third case, as retailer offers higher credit period, so demand of the customers become high but the quantity transferred from manufacturer to retailer is same, therefore, period of consumptions of the retailer is reduced. As the demand of the customers are linked with the credit period offered by the retailer, so high demand of the customers reduced the time period of the retailers.

Example 11.2. *Evaluate the optimal policy of the decision maker when the defective rate of produce item depends on production rate only (i.e., $\xi = 0$) and the remain parameters of the system are unalter.*

Following table shows the optimum policies of the decision maker when the defective rate of the produced item is of the form $\theta = \beta + \lambda P$. In comparison of the cases, Example 11.2 reveal same decisions as Example 11.1. More-over, when defective rate does not depend on time, the defective units are quite less, i.e., fresh units are more than that of Example 11.1. So, business periods are larger than the Example 11.1, which also yield more profit than Example 11.1 for each cases.

Table 11.3: Optimal results of illustrated model when $\theta = \beta + \lambda P$

Cases	Retailers' credit period (N^1, N^2)	Manufacturer credit period $(\widetilde{M})$	Production time (t_1)	Period of Manufacturer(t_2)	Period of Retailers' (T^1, T^2)	Expected Profit
Case-I	(0.10, 0.12)	(0.19, 0.20, 0.22)	0.543	0.622	(0.763, 0.757)	4370.80
Case-II	(0.12, 0.10)	(0.69, 0.70, 0.72)	0.548	0.637	(0.767, 0.761)	3872.79
Case-III	(0.65, 0.62)	(0.69, 0.70, 0.72)	0.558	0.642	(0.755, 0.748)	4327.61

Example 11.3. *Find the optimal policies of the decision maker when the defective rate of produce item depend on time only (i.e., $\lambda = 0$) and the remain parameters of the system are unalter.*

When the defective rate of produce item does not depend on production rate but depend on time only i.e., $\theta = \beta + \xi t$, then the optimum results are shown in the following table.

Table 11.4: Optimal results of illustrated model when $\theta = \beta + \xi t$

Cases	Retailers' credit period (N^1, N^2)	Manufacturer credit period $(\widetilde{M})$	Production time t_1	Period of Manufacturer (t_2)	Period of Retailers' (T^1, T^2)	Expected Profit
Case-I	(0.10, 0.12)	(0.19, 0.20, 0.22)	0.728	0.848	(1.020, 1.010)	6074.09
Case-II	(0.12, 0.10)	(0.69, 0.70, 0.72)	0.731	0.851	(1.023, 1.013)	6833.38
Case-III	(0.65, 0.62)	(0.69, 0.70, 0.72)	0.739	0.858	(0.998, 0.991)	6095.70

If defective rate does not depend on the production rate, then production time is much larger. So manufacturer produced more quantity of items. For this reason, the periods of manufacturer and retailers are more than the scenarios when $\theta = \beta + \lambda P + \xi t$ or $\theta = \beta + \lambda P$. So the optimum profit is more than the other two scenarios, as expected.

Example 11.4. *When the defective rate of produce item is constant (i.e., $\lambda = 0$ and $\xi = 0$) then evaluate the optimal profit of the decision maker (the remain parameter of the system remain unalter).*

Table 11.5: Optimal results of illustrated model when $\theta = \beta$

Cases	Retailers' credit period (N^1, N^2)	Manufacturer credit period $(\widetilde{M})$	Production time t_1	Period of Manufacturer (t_2)	Period of Retailers' (T^1, T^2)	Expected Profit
Case-I	(0.10, 0.12)	(0.19, 0.20, 0.22)	0.783	0.913	(1.080, 1.081)	6593.89
Case-II	(0.12, 0.10)	(0.69, 0.70, 0.72)	0.787	0.918	(1.098, 1.090)	7664.13
Case-III	(0.65, 0.62)	(0.69, 0.70, 0.72)	0.794	0.924	(1.033, 1.045)	6638.49

From Table 11.5, it is decided that when defective rate is fixed then the yields optimum profits are maximum for each cases. This is due to the less amount of defective units. The other conclusions regarding the comparison of the three cases remain same as in Example 11.1.

11.5.1 Sensitivity Analysis

In this section, we examine the effects of changes in the system parameters. A Sensitivity analysis is performed by changing some of the parameters as follows. On the basis of the results calculated the following observations can be made.

Sensitivity analysis 11.1. *In this example, we use the same data as in Example 11.1 except the production rate on the optimal solution. The results in Table 11.6 given below:*

Table 11.6: Sensitivity analysis of expected profit w.r.t. P

Production rate (P)	Cases	Defective rate (θ)	Production time (t_1)	Manufacturer business period (t_2)	Retailers' business period (T^1, T^2)	Expected Profit
	Case-I	0.141	0.561	0.612	(0.747, 0.741)	4237.68
40	Case-II	0.145	0.564	0.616	(0.748, 0.741)	3622.90
	Case-III	0.151	0.572	0.625	(0.735, 0.728)	4189.38
	Case-I	0.153	0.528	0.605	(0.734, 0.730)	4178.54
42	Case-II	0.155	0.531	0.610	(0.739, 0.732)	3512.53
	Case-III	0.159	0.538	0.618	(0.726, 0.720)	4128.05
	Case-I	0.156	0.484	0.597	(0.725, 0.719)	4090.69
45	Case-II	0.159	0.490	0.601	(0.728, 0.723)	3347.35
	Case-III	0.165	0.495	0.607	(0.714, 0.707)	4036.89

Here, we conclude that increasing rate of production (P) increases the rate of defectiveness(θ) as it is a increasing function of P, it also reduced value of production time, manufacturing time as well as expected profit due to the increase of production with fixed demand expand the inventory / holding cost. Again, increasing defective amount reduced the amount of fresh unit that caused lower business period and lower profit.

Sensitivity analysis 11.2. *This example outlines the effects of changes in the different values of trade credit periods $\widetilde{M}$ and N's for case-II. The results given in Table 11.7.*

Table 11.7 shows that, more spread of fuzzy credit period gives lower profit of the system as well as lower interest paid by the retailer (as expected). And increasing of down stream credit period (offered by the retailer to the customers) of the system yields more earn of the retailer due to the increasing of demand which consequently decreases the holding cost also.

Table 11.7: Sensitivity analysis of expected profit w.r.t. $\widetilde{M}$ and N's on case-II

Δ_1	Δ_2	Manufacturer credit period $\widetilde{M} = (M - \Delta_1, M, M + \Delta_2)$	Retailers' credit period (N^1, N^2)	Interest earned by the retailers (IE_2)	Interest paid by the retailers (IP_2)	Expected Profit
			(0.09, 0.07)	294.92	895.84	3488.15
0.008	0.020	(0.688, 0.700, 0.720)	(0.12, 0.10)	315.52	895.84	3508.76
			(0.15, 0.13)	333.76	895.84	3526.99
			(0.09, 0.07)	298.70	895.87	3491.93
0.010	0.020	(0.690, 0.700, 0.720)	(0.12, 0.10)	319.31	895.87	3512.53
			(0.15, 0.13)	337.54	895.87	3530.77
			(0.09, 0.07)	302.49	895.89	3495.70
0.012	0.020	(0.692, 0.700, 0.720)	(0.12, 0.10)	323.09	895.89	3516.31
			(0.15, 0.13)	341.34	895.89	3534.55
			(0.09, 0.07)	302.27	896.48	3496.12
0.010	0.018	(0.690, 0.700, 0.718)	(0.12, 0.10)	322.88	896.48	3516.72
			(0.15, 0.13)	341.12	896.48	3534.96
			(0.09, 0.07)	295.13	897.64	3487.74
0.010	0.022	(0.690, 0.700, 0.722)	(0.12, 0.10)	315.74	897.64	3508.35
			(0.15, 0.13)	333.98	897.64	3526.58

11.6 Conclusion

This chapter develops an integrated production inventory model involving manufacturer, retailer and customers with up-stream and down stream credit periods. It will provide the following decision making:

• If a manufacturer produces an item with defective quality also, then the rate of defective may depend on production rate and/or time. And the effect of these dependencies are shown here numerically with efficient cause.

• The duration of upstream credit period may fluctuate due to different causes, in this regard here, an imprecise nature of upstream credit period is considered and analyzed numerically. More over, its effects are justified by a sensitivity analysis.

• Here, an unavoidable circumstance of carbon-emission is taken into account for a good gesture of society and this emission rate is not fixed through the cycle time, so it is formulated with random nature.

• Furthermore, we discuss some special cases of credit periods to show their effects on the management.

• Finally, we run several numerical examples and sensitivity analysis to illustrate the problem and provide some managerial insights.

Part VI

Summary and Extension of the Thesis

Chapter 12

Summary and Future Research Work

12.1 Summary of the Thesis

In this dissertation, some imperfect production inventory models are formulated and solved in crisp, stochastic, fuzzy and fuzzy stochastic environments. Major emphasis in this thesis has been given on the realistic model formation in crisp, stochastic, fuzzy and fuzzy stochastic environments. Here, nine virgin imperfect production inventory models have been presented. The models are solved applying new/modified methods and these are numerically illustrated with some data.

Part II of the thesis contains Chapter 3 in which an imperfect production inventory model with production rate dependent defective rate and advertisement dependent demand in crisp environments has been developed. In this model we consider different production rate and screening rate.

In Part III, through Chapter 4, 5 & 6 three different types of imperfect production inventory models in stochastic environments have been presented. In Chapter 4, multi item imperfect production inventory model with promotional demand in random planning horizon has been illustrated. In this model, the demand rate depend on both selling price and advertisement. In Chapter 5, a deteriorating manufacturing system is considered with inspection errors. In this model, the demand rate depends on discount and warranty period. In Chapter 6, two layers supply chain in an imperfect production inventory model with two storage facilities under reliability consideration has been described. In this model, the defective rate depends on both production rate and time. Rework of imperfect item has been considered in all models in this part.

In fourth part containing Chapter 7, 8 & 9, there are presented three different types of imperfect production inventory models in fuzzy environments. In Chapter 7, three-layer supply chain in an imperfect production inventory model with two storage facilities under fuzzy rough environment has been developed. Here, the demand of the customers is considered as stock dependent. Chapter 8 discusses an imperfect production inventory model based on fuzzy differential and fuzzy integral method. In Chapter 9, controlling GHG emission from industrial waste in two plant production and reproduction inventory model with interval valued fuzzy pollution parameters has been discussed.

Fifth part contains Chapter 10 & 11 which discuss two different types of imperfect production inventory models in fuzzy stochastic environments. In Chapter 10, multi-item EPQ model with shortages, rework and learning effect on imperfect production over fuzzy-random planning horizon has been developed. In Chapter 11, two layers supply chain imperfect production inventory model with fuzzy credit period, time and production rate dependent imperfectness has been discussed. In this model, the demand rate of the customers is considered as stock dependent and credit period. Rework of imperfect item has been considered all models in this part.

In this thesis, several new techniques or existing techniques in modified forms have been developed and implemented to solve the above mentioned imperfect production inventory models. These methods are: Generalized Reduced Gradient (GRG) technique, Genetic Algorithm (GA), Population Varying Genetic Algorithm (PVGA), Multi-Objective Genetic Algorithm(MOGA), Fuzzy Simulation Based Genetic Algorithm (FSGA), Possibility/ Necessity/ Credibility representation, Solution of Fuzzy Differential Equation (FDE), Fuzzy Programming Technique(FPT).

In this thesis, some statistical tests have been developed and implemented to above mentioned imperfect production inventory models. These test are: ANOVA test for comparison of means in Chapter 4, Fishers ‘t test for Comparison of two means in Chapter 8.

12.2 Future Research Work

There are lot of scopes to improve the production inventory models of this thesis.

In Chapter 3, the proposed model can be extended in several ways such as, First, one can extend this model for stock dependent demand, probabilistic demand. Second, this model can be generalized by considering two level credit policy. Third, this model can be extended to fuzzy demand rate and fuzzy percentage of defective products.

In Chapter 4, the proposed model investigates a multi-item imperfect production inventory model with promotional effort over random planning horizon and the proposed model is solved via Population Varying Genetic Algorithm (PVGA). This model can be extended in fuzzy and/or fuzzy-stochastic environment instead of stochastic (random) planning horizon. Moreover, the randomness of the horizon also can be proposed with other continuous distribution. The items also can be treated as breakable or damageable, etc. Thus, the scope for future work includes rigorous testing of this model with real and simulated data.

In Chapter 5 & 6, the model can be extended further considering the uncertain demand which is the limitation of our model; the stock out situation at each stage of the chain may be incorporated further. Moreover, multi-retailer levels may also be introduced in the model.

In Chapter 8, the proposed work can be extended to the rough, fuzzy-rough, random, fuzzy-random environment taking constant part of screening cost, holding cost, set-up cost, etc.

In Chapter 9, the proposed model investigates GA approach for controlling GHG emission from industrial waste in two plant production and reproduction inventory model with interval valued fuzzy pollution parameters. This model can be extended multi item production inventory model in fuzzy stochastic environment instead of stochastic (random) planning horizon. The items also can be treated as breakable or damageable, etc. Thus, the scope for future work includes rigorous testing of this model with real and simulated data.

In Chapter 10, there are several interesting extensions to research work. First, more general distributions such as normal distribution, standard normal distribution, etc. for random time horizons can be considered. Another direction could be to consider fuzzy dependent demand, probabilistic demand, time-dependent demand, etc. Finally, we can consider the joint optimization of production, maintenance and quality with a two-level credit period policy.

In Chapter 11, we suggest several possible directions for future research. First, one may extend our considered EPQ model to joint optimization of expected total profit and carbon emission (i.e., maximize expected total profit and minimize carbon emission). Second, one immediate possible extension could be allowable shortages, cash discounts, etc. Finally, one can extend the fully trade credit policy to the partial trade credit policy in which a seller requests its credit-risk customers to pay a fraction of the purchase amount at the time of placing an order as a collateral deposit, and then grants a permissible delay on the rest of the purchase amount.

Part VII

Appendices, Bibliography and Index

Appendix A

For Chapter 3

Expression of first order partial derivative of the average profit $AP(P,T)$ w.r.t P,

$$\frac{\partial}{\partial P}\{AP(P,T)\} = \frac{x}{T}\Big\{\frac{s'\theta}{2} - c_p + c_{sr}\Big\}(t_2 + P\frac{\partial t_2}{\partial P}) - \frac{x}{2T}\Big\{(1-x)h_c + \theta h'_m\Big\}$$
$$\times(t_2^2 + 2Pt_2\frac{\partial t_2}{\partial P}) - \frac{h_m}{2T}\Big[(1-\theta)xPt_2 - \frac{D_0}{\eta^2}(\eta - e^{-\eta T}) - \frac{v_0}{\eta^2}(\eta T + e^{-\eta T} - 1)$$
$$-\frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2}(1-e^{-v_2 T}) - \frac{1}{\eta}(1-e^{-\eta T})\Big\}\Big]\frac{\partial t_2}{\partial P} - \frac{h_m}{4T}(1-\theta)xt_2^2$$
$$= F(P,T), \text{ say}$$

The expression of first order partial derivative of $\frac{\partial}{\partial P}\{AP(P,T)\}$ w.r.t T,

$$\frac{\partial^2}{\partial T\partial P}\{AP(P,T)\} = \frac{x}{T}\Big\{\frac{s'\theta}{2} - c_p - c_{sr}\Big\}\Big[(\frac{\partial t_2}{\partial T} + P\frac{\partial t_2}{\partial T\partial P}) - \frac{1}{T}(t_2 + P\frac{\partial t_2}{\partial P})\Big]$$
$$+\frac{x}{2T^2}\{h_c(1-x) + \theta h'_m\}\Big[(t_2^2 + 2Pt_2\frac{\partial t_2}{\partial P}) - 2T\Big\{(t_2 + P\frac{\partial t_2}{\partial P})\frac{\partial t_2}{\partial T} + Pt_2\frac{\partial^2 t_2}{\partial T\partial P}\Big\}\Big]$$
$$-\frac{h_m}{2T^2}\Big[(1-\theta)xPt_2 - \frac{v_1}{v_2-\eta}\Big\{\frac{1}{v^2}(1-e^{-v_2 T}) - \frac{1}{\eta}(1-e^{-\eta T})\Big\} - \frac{v_0}{\eta}T$$
$$-\frac{1}{\eta}(D_0 - \frac{v_0}{\eta})(1-e^{-\eta T})\Big]\Big(T\frac{\partial^2 t_2}{\partial T\partial P} - \frac{\partial t_2}{\partial P}\Big) + \frac{h_m}{4T^2}(1-\theta)xt_2(t_2 - 2T\frac{\partial t_2}{\partial T})$$
$$+\frac{h_m}{2T}\Big[(1-\theta)xP\frac{\partial t_2}{\partial T} + \frac{1}{\eta}(D_0 - v_0)e^{-\eta T}) + \frac{v_0}{\eta} + \frac{v_1}{v_2-\eta}(e^{-v_2 T} - e^{-\eta T})\Big]\frac{\partial t_2}{\partial P}$$

The expression of second order partial derivative of the average profit $AP(P,T)$ w.r.t P,

$$\frac{\partial^2}{\partial P^2}\{AP(P,T)\} = \frac{x}{T}\Big\{\frac{s'\theta}{2} - c_p - c_{sr}\Big\}(2\frac{\partial t_2}{\partial P} + P\frac{\partial^2 t_2}{\partial P^2}) - \frac{h_m}{2T}(1-\theta)x\Big\{P(\frac{\partial t_2}{\partial P})^2$$
$$+2t_2\frac{\partial t_2}{\partial P}\Big\} - \frac{x}{T}\Big\{h_c(1-x) + h'_m\theta\Big\}\Big\{2t_2\frac{\partial t_2}{\partial P} + P(\frac{\partial t_2}{\partial P})^2 + Pt_2\frac{\partial^2 t_2}{\partial P^2}\Big\} - \frac{h_m}{2T}\Big[(1-\theta)xPt_2$$
$$-\frac{1}{\eta}(D_0 - \frac{v_0}{\eta})(1-e^{-\eta T}) - \frac{v_0 T}{\eta} - \frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2^2}(v_2 - v_2 e^{-v_2 T}) - \frac{1}{\eta^2}(\eta - \eta e^{-\eta T})\Big\}\Big]\frac{\partial^2 t_2}{\partial P^2}$$

From Lemma 3.7, we obtained the screening period t_2 is given by

$$t_2 = \frac{1}{(1-\theta)xP}\Big[\frac{1}{\eta}(D_0 - \frac{v_0}{\eta})(1-e^{-\eta T}) + \frac{v_0 T}{\eta} + \frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2}(1-e^{-v_2 T}) - \frac{1}{\eta}(1-e^{-\eta T})\Big\}\Big]$$

The expression of first order partial derivative of the screening time t_2 w.r.t T, we have

$$\frac{\partial t_2}{\partial T} = \frac{1}{(1-\theta)xP}\Big[D_0 e^{-\eta T} + \frac{v_0}{\eta}(1-e^{-\eta T}) + \frac{v_1}{v_2-\eta}\Big\{e^{-v_2 T} - e^{-\eta T}\Big\}\Big]$$

The expression of second order partial derivative of the screening time t_2,

$$\frac{\partial^2 t_2}{\partial T^2} = \frac{1}{(1-\theta)xP}\Big[-\eta D_0 e^{-\eta T} + v_0 e^{-\eta T} - \frac{v_1}{v_2-\eta}\Big\{v_2 e^{-v_2 T} - \eta e^{-\eta T}\Big\}\Big]$$

The expression of first order partial derivative of the screening time t_2 w.r.t P,

$$\frac{\partial t_2}{\partial P} = -\frac{1}{(1-\theta)xP^2}\Big[\frac{1}{\eta}(D_0 - \frac{v_0}{\eta})(1-e^{-\eta T}) + \frac{v_0 T}{\eta} + \frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2}(1-e^{-v_2 T}) - \frac{1}{\eta}(1-e^{-\eta T})\Big\}\Big]$$

The expression of second order partial derivative of the screening time t_2 w.r.t P,

$$\frac{\partial^2 t_2}{\partial P^2} = \frac{2}{(1-\theta)xP^3}\Big[\frac{1}{\eta}(D_0 - \frac{v_0}{\eta})(1-e^{-\eta T}) + \frac{v_0 T}{\eta} + \frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2}(1-e^{-v_2 T}) - \frac{1}{\eta}(1-e^{-\eta T})\Big\}\Big]$$

The expression of first order partial derivative of the average profit $AP(P,T)$ w.r.t T,

$$\begin{aligned}
&\frac{\partial}{\partial T}\{AP(P,T)\} = \frac{s}{T}\Big[D_0 e^{-\eta T} + \frac{v_0}{\eta}(1-e^{-\eta T}) + \frac{v_1}{v_2-\eta}(e^{-v_2 T} - e^{-\eta T})\Big] \\
&-\frac{s}{T^2}\Big[\frac{1}{\eta}(D_0 - \frac{v_0}{\eta})(1-e^{-\eta T}) + \frac{v_0 T}{\eta} + \frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2}(1-e^{-v_2 T}) - \frac{1}{\eta}(1-e^{-\eta T})\Big\}\Big] \\
&+xP\Big\{\frac{s'\theta}{2} - (c_p + c_{sr})\Big\}\Big\{\frac{1}{T}\frac{\partial t_2}{\partial T} - \frac{t_2}{T^2}\Big\} - xP\Big\{(1-x)h_c + \theta h'_m\Big\}\Big\{\frac{t_2}{T}\frac{\partial t_2}{\partial T} - \frac{t_2^2}{2T^2}\Big\} \\
&+\frac{h_m}{2T^2}\Big[Q_0 T + (1-\theta)xP\frac{t_2^2}{2} - \frac{D_0}{\eta^2}\{\eta t_2 - 1 + (1+T-t_2)e^{-\eta T}\} - \frac{v_0}{\eta^3}\Big\{\frac{\eta^2}{2}(2t_2 - T^2) \\
&+1 - \eta t_2 - (1+T-t_2)e^{-\eta T}\Big\} - \frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2^2}\{v_2 t_2 - 1 + (1+v_2 T - v_2 t_2)e^{-v_2 T}\} \\
&-\frac{1}{\eta^2}\{\eta t_2 - 1 + (1+\eta T - \eta t_2)e^{-\eta T}\}\Big\}\Big] - \frac{h_m}{2T}\Big[Q_0 + (1-\theta)xPt_2\frac{\partial t_2}{\partial T} - \frac{D_0}{\eta^2}\Big\{\eta\frac{\partial t_2}{\partial T} \\
&-\eta(1+T-t_2)e^{-\eta T} + (1-\frac{\partial t_2}{\partial T})e^{-\eta T}\Big\} - \frac{v_0}{\eta^3}\Big\{\eta^2(t_2 + T\frac{\partial t_2}{\partial T} - T) - \eta\frac{\partial t_2}{\partial T} \\
&+\eta(1+T-t_2)e^{-\eta T} - (1-\frac{\partial t_2}{\partial T})e^{-\eta T}\Big\} - \frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2}\{\frac{\partial t_2}{\partial T} + (1-\frac{\partial t_2}{\partial T})e^{-v_2 T} \\
&-(1+v_2 T - v_2 t_2)e^{-v_2 T}\} - \frac{1}{\eta}\{\frac{\partial t_2}{\partial T} - (1+\eta T - \eta t_2)e^{-\eta T} + (1-\frac{\partial t_2}{\partial T})e^{-\eta T}\}\Big\}\Big] \\
&+\frac{c_a v_1}{v_2 T^2}(e^{-v_2 T} - 1) + \frac{c_a v_1}{T}e^{-v_2 T} + \frac{A_m}{T^2} \\
= \; & G(P,T), \text{ say}
\end{aligned}$$

The expression of second order partial derivative of the average profit $AP(P,T)$ w.r.t T,

$$\frac{\partial^2}{\partial T^2}\{AP(P,T)\} = \frac{s}{T}\Big[(v_0-\eta D_0)e^{-\eta T} - \frac{v_1}{v_2-\eta}(v_2 e^{-v_2 T} - \eta e^{-\eta T})\Big] - \frac{2s}{T^2}\Big[D_0 e^{-\eta T}$$
$$+\frac{v_0}{\eta}(1-e^{-\eta T}) + \frac{v_1}{v_2-\eta}(e^{-v_2 T} - e^{-\eta T})\Big] + \frac{2s}{T^3}\Big[\frac{D_0}{\eta}(1-e^{-\eta T}) + \frac{v_0}{\eta^2}(\eta T + e^{-\eta T} - 1)$$
$$+\frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2}(1-e^{-v_2 T}) - \frac{1}{\eta}(1-e^{-\eta T})\Big\}\Big] + xP\Big\{\frac{s'\theta}{2} - (c_p + c_{sr})\Big\}\Big\{\frac{1}{T}\frac{\partial^2 t_2}{\partial T^2}$$
$$-\frac{2}{T^2}\frac{\partial t_2}{\partial T} + \frac{2t_2}{T^3}\Big\} - xP\Big\{(1-x)h_c + \theta h'_m\Big\}\Big\{\frac{t_2}{T}\frac{\partial^2 t_2}{\partial T^2} + \frac{1}{T}(\frac{\partial t_2}{\partial T})^2 - \frac{2t_2}{T^2}\frac{\partial t_2}{\partial T} + \frac{t_2^2}{T^3}\Big\}$$
$$-\frac{h_m}{T^3}\Big[Q_0 T + (1-\theta)xP\frac{t_2^2}{2} - \frac{D_0}{\eta^2}\{\eta t_2 - 1 + (1+T-t_2)e^{-\eta T}\} - \frac{v_0}{\eta^3}\Big\{\frac{\eta^2}{2}(2t_2 - T^2)$$
$$+1 - \eta t_2 - (1+T-t_2)e^{-\eta T}\Big\} - \frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2^2}\{v_2 t_2 - 1 + (1+v_2 T - v_2 t_2)e^{-v_2 T}\}$$
$$-\frac{1}{\eta^2}\{\eta t_2 - 1 + (1+\eta T - \eta t_2)e^{-\eta T}\}\Big\}\Big] + \frac{h_m}{T^2}\Big[Q_0 + (1-\theta)xPt_2\frac{\partial t_2}{\partial T} - \frac{D_0}{\eta^2}\Big\{\eta\frac{\partial t_2}{\partial T}$$
$$-\eta(1+T-t_2)e^{-\eta T} + (1-\frac{\partial t_2}{\partial T})e^{-\eta T}\Big\} - \frac{v_0}{\eta^3}\Big\{\eta^2(t_2 + T\frac{\partial t_2}{\partial T} - T) - \eta\frac{\partial t_2}{\partial T}$$
$$+\eta(1+T-t_2)e^{-\eta T} - (1-\frac{\partial t_2}{\partial T})e^{-\eta T}\Big\} - \frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2}\{\frac{\partial t_2}{\partial T} + (1-\frac{\partial t_2}{\partial T})e^{-v_2 T}$$
$$-(1+v_2 T - v_2 t_2)e^{-v_2 T}\} - \frac{1}{\eta}\{\frac{\partial t_2}{\partial T} - (1+\eta T - \eta t_2)e^{-\eta T} + (1-\frac{\partial t_2}{\partial T})e^{-\eta T}\}\Big\}\Big]$$
$$-\frac{h_m}{2T}\Big[(1-\theta)xP\{(\frac{\partial t_2}{\partial T})^2 + t_2\frac{\partial^2 t_2}{\partial T^2}\} - \frac{D_0}{\eta^2}\Big\{(\eta + e^{-\eta T})\frac{\partial^2 t_2}{\partial T^2} - 2\eta(1-\frac{\partial t_2}{\partial T})e^{-\eta T}$$
$$+\eta^2(1+T-t_2)e^{-\eta T}\Big\} - \frac{v_0}{\eta^3}\Big\{\eta^2(2\frac{\partial t_2}{\partial T} + T\frac{\partial^2 t_2}{\partial T^2} - 1) + (e^{-\eta T} - \eta)\frac{\partial^2 t_2}{\partial T^2}$$
$$+2\eta(1-\frac{\partial t_2}{\partial T})e^{-\eta T} - \eta^2(1+T-t_2)e^{-\eta T}\Big\} - \frac{v_1}{v_2-\eta}\Big\{\frac{1}{v_2}(1-e^{-v_2 T})\frac{\partial^2 t_2}{\partial T^2}$$
$$-2(1-\frac{\partial t_2}{\partial T})e^{-v_2 T} + (1+v_2 T - v_2 t_2)e^{-v_2 T} - \frac{1}{\eta}(1-e^{-\eta T})\frac{\partial^2 t_2}{\partial T^2} + 2(1-\frac{\partial t_2}{\partial T})e^{-\eta T}$$
$$-(1+\eta T - \eta t_2)e^{-\eta T}\Big] - \frac{2c_a v_1}{T^2}\Big[\frac{1}{v_2 T}(e^{-v_2 T} - 1) + e^{-v_2 T} + \frac{T}{2}e^{-v_2 T}\Big] - \frac{2A_m}{T^3}$$

Appendix B

For Chapter 4

(i) $$\sum_{i=1}^{N^j} e^{-R(i-1)T^j} = 1 + e^{-RT^j} + e^{-2RT^j} + ... + e^{-R(N^j-1)T^j} = \frac{1-e^{-N^jRT^j}}{1-e^{-RT^j}}$$

(ii) $$E[TP(T^j)] = \sum_{N^j=0}^{\infty} \int_{N^jT^j}^{(N^j+1)T^j} TP(N^j, T^j) f(h), dh$$

$$= \Big[\frac{s^jD^j}{R}(1-e^{-RT^j}) - \frac{1}{R}(P_0^j + P_1^jD^j)(1-e^{-Rt_1^j})\{c_p^j + c_{sr}^j + r_c^j\delta^j(1-\beta^j)\}$$
$$-\frac{h_c^j}{R^2}\Big\{\{\beta^j + \delta^j(1-\beta^j)\}P_0^j + [\{\beta^j + \delta^j(1-\beta^j)\}P_1^j - 1]D^j\Big\}\{1-(1+Rt_1^j)e^{-Rt_1^j}\}$$
$$-\frac{h_c^jD^j}{R^2}\Big\{e^{-RT^j} - \{R(T^j - t_1^j)+1\}e^{-Rt_1^j}\Big\}\Big] \sum_{N^j=0}^{\infty} \frac{1-e^{-N^jRT^j}}{1-e^{-RT^j}}\Big\{e^{-N^jT^j\lambda} - e^{-(N^j+1)T^j\lambda}\Big\}$$

$$= \Big[\frac{s^jD^j}{R}(1-e^{-RT^j}) - \frac{1}{R}(P_0^j + P_1^jD^j)(1-e^{-Rt_1^j})\{c_p^j + c_{sr}^j + r_c^j\delta^j(1-\beta^j)\}$$
$$-\frac{h_c^j}{R^2}\Big\{\{\beta^j + \delta^j(1-\beta^j)\}P_0^j + [\{\beta^j + \delta^j(1-\beta^j)\}P_1^j - 1]D^j\Big\}\{1-(1+Rt_1^j)e^{-Rt_1^j}\}$$
$$-\frac{h_c^jD^j}{R^2}\Big\{e^{-RT^j} - \{R(T^j - t_1^j)+1\}e^{-Rt_1^j}\Big\}\Big] \frac{e^{-\lambda T^j}}{1-e^{-(\lambda+R)T^j}}$$

(iii) $$\sum_{N^j=0}^{\infty} \int_{N^jT^j}^{(N^j+1)T^j} PC_L^j f(h)\, dh = \sum_{N^j=0}^{\infty} \Big[\int_{N^jT^j}^{N^jT^j+t_1^j} PC_{L_1}^j f(h)\, dh + \int_{N^jT^j+t_1^j}^{(N^j+1)T^j} PC_{L_2}^j f(h)\, dh\Big]$$

$$= \frac{c_p^j}{R}(P_0^j + P_1^jD^j)\Big[\frac{R}{R+\lambda}\{1-e^{-(R+\lambda)t_1^j}\} - (1-e^{-Rt_1^j})e^{-\lambda T^j})\Big] \sum_{N^j=0}^{\infty} e^{-(\lambda+R)N^jT^j}$$

$$= \frac{c_p^j(P_0^j + P_1^jD^j)}{R\{1-e^{-(R+\lambda)T^j}\}}\Big[\frac{R}{R+\lambda}\{1-e^{-(R+\lambda)t_1^j}\} - (1-e^{-Rt_1^j})e^{-\lambda T^j})\Big]$$

(iv) $$\sum_{N^j=0}^{\infty}\int_{N^jT^j}^{(N^j+1)T^j} SC_L^j f(h)\,dh = \sum_{N^j=0}^{\infty}\int_{N^jT^j}^{N^jT^j+t_1^j} SC_{L_1}^j f(h)\,dh + \sum_{N^j=0}^{\infty}\int_{N^jT^j+t_1^j}^{(N^j+1)T^j} SC_{L_2}^j f(h)\,dh$$
$$= \frac{c_{sr}^j(P_0^j + P_1^j D^j)}{R\{1-e^{-(R+\lambda)T^j}\}}\Big[\frac{R}{R+\lambda}\{1-e^{-(R+\lambda)t_1^j}\} - (1-e^{-Rt_1^j})e^{-\lambda T^j})\Big]$$

(v) $$\sum_{N^j=0}^{\infty}\int_{N^jT^j}^{(N^j+1)T^j} RC_L^j f(h)\,dh = \sum_{N^j=0}^{\infty}\int_{N^jT^j}^{N^jT^j+t_1^j} RC_{L_1}^j f(h)\,dh + \sum_{N^j=0}^{\infty}\int_{N^jT^j+t_1^j}^{(N^j+1)T^j} RC_{L_2}^j f(h)\,dh$$
$$= \frac{\delta^j(1-\beta^j)r_c^j}{R\{1-e^{-(R+\lambda)T^j}\}}(P_0^j + P_1^j D^j)\Big[\frac{R}{R+\lambda}\{1-e^{-(R+\lambda)t_1^j}\} - (1-e^{-Rt_1^j})e^{-\lambda T^j})\Big]$$

(vi) $$\sum_{N^j=0}^{\infty}\int_{N^jT^j}^{(N^j+1)T^j} SR_L^j f(h)\,dh = \sum_{N^j=0}^{\infty}\int_{N^jT^j}^{N^jT^j+t_1^j} SR_{L_1}^j f(h)\,dh + \sum_{N^j=0}^{\infty}\int_{N^jT^j+t_1^j}^{(N^j+1)T^j} SR_{L_2}^j f(h)\,dh$$
$$= \frac{D^j s^j}{R\{1-e^{-(R+\lambda)T^j}\}}[(1-e^{-\lambda T^j}) + \frac{\lambda}{R+\lambda}(e^{-(R+\lambda)T^j} - 1)]$$

(vii) In the demand expression $D_1^1(S^1) = \frac{A^1(s_{max}^1 - s^1)}{s^1 - s_{max}^1}$ the input parameter s_{min}^1 is taken per unit total requirement input parameter cost. Other two parameters A^1, s_{max}^1 are computed by the solving following two regression lines for $n = 7$.

$$\frac{1}{n}\sum_{i=1}^{n} D^1(i)s^1(i) - \frac{s_{min}^1}{n}\sum_{i=1}^{n} D^1(i) = A^1\Big[s_{max}^1 - \frac{1}{n}\sum_{i=1}^{n} s^1(i)\Big]$$

and $$\frac{1}{n}\sum_{i=1}^{n}\{D^1(i)\}^2 s^1(i) - \frac{s_{min}^1}{n}\sum_{i=1}^{n}\{D^1(i)\}^2 = A^1\Big[s_{max}^1\frac{1}{n}\sum_{i=1}^{n} D^1(i) - \frac{1}{n}\sum_{i=1}^{n} D^1(i)s^1(i)\Big]$$

$(viii)$ Similarly for the demand expression $D_2^1(\nu^1) = \kappa^1(1 - \frac{1}{\nu^1+1})$ the input parameter κ^1 is estimated by the solving following regression line.

$$k^1 = \frac{1}{n}\sum_{i=1}^{n} D^1(i) - \frac{1}{n}\sum_{i=1}^{n}\frac{D^1(i)}{\nu^1(i)},\ \text{for n=7}$$

(ix) The following formula are used in ANOVA comparison:
For the groups data $X_1, X_2, \ldots, X_k$ of sizes $n_1, n_2, \ldots, n_k$ respectively,

$$N = \sum_{i=1}^{k} n_i\,,\ \overline{X} = \frac{\sum_{i=1}^{k} X_i}{N},\ SS_t = \sum_{i=1}^{k}\sum_{i=1}^{n}(X_i - \overline{X})^2 \text{ with } df_t = N-1 \text{ and } s_t^2 = \frac{SS_t}{df_t}$$

$$\overline{X_1} = \frac{\sum X_1}{n_1};\ \overline{X_2} = \frac{\sum X_2}{n_2};\ldots;\overline{X_k} = \frac{\sum X_k}{n_k},\ SS_b = \sum_{i=1}^{k}[n_i(\overline{X_i} - \overline{X})^2] \text{ with } df_b = k-1$$

$$SS_w = \sum_{i=1}^{k}\sum_{i=1}^{n}(X_i - \overline{X})^2 \text{ with } df_w = N-k \text{ and } s_b^2 = \frac{SS_b}{df_b},\ s_w^2 = \frac{SS_w}{df_w},\ F = \frac{s_b^2}{s_w^2}.$$

Appendix C

For Chapter 5

$$
\begin{aligned}
(i) \int_0^{t_1} f(\tau)\, d\tau &= \lambda \int_0^{t_1} e^{-\lambda\tau}\, d\tau \\
&= 1 - e^{-\lambda t_1} \\
&= \lambda t_1, \ \text{approximating up to the second term of the expansion of } e^{-\lambda t_1}
\end{aligned}
$$

$$
\begin{aligned}
(ii) \int_{t_1}^{\infty} f(\tau)\, d\tau &= \lambda \int_{t_1}^{\infty} e^{-\lambda\tau}\, d\tau \\
&= e^{-\lambda t_1} \\
&= 1 - \lambda t_1, \ \text{approximating up to the second term of the expansion of } e^{-\lambda t_1}
\end{aligned}
$$

$$
\begin{aligned}
(iii) \int_0^{t_1} \tau f(\tau)\, d\tau &= \lambda \int_0^{t_1} \tau e^{-\lambda\tau}\, d\tau \\
&= \frac{1}{\lambda}\{1 - e^{-\lambda t_1}\} - t_1 e^{-\lambda t_1} \\
&= \lambda t_1^2, \ \text{approximating up to the third term of the expansion of } e^{-\lambda t_1}
\end{aligned}
$$

$$
\begin{aligned}
(iv) \int_0^{t_1} \Big(t_1 - \tau\Big) f(\tau)\, d\tau &= \lambda \int_0^{t_1} \Big(t_1 - \tau\Big) e^{-\lambda\tau}\, d\tau \\
&= t_1 - \frac{1}{\lambda}\{1 - e^{-\lambda t_1}\} \\
&= \frac{1}{2}\lambda t_1^2, \ \text{approximating up to the third term of the expansion of } e^{-\lambda t_1}
\end{aligned}
$$

$$
\begin{aligned}
(v)\int_0^{t_1}\tau^2 f(\tau)\,d\tau &= \lambda\int_0^{t_1}\tau^2 e^{-\lambda\tau}\,d\tau\\
&= t_1^2 e^{-\lambda t_1}-\frac{2t_1}{\lambda}e^{-\lambda t_1}+\frac{2}{\lambda^2}\{1-e^{-\lambda t_1}\}\\
&= \frac{1}{2}\lambda^2 t_1^4,\ \text{approximating up to the third term of the expansion of } e^{-\lambda t_1}
\end{aligned}
$$

$$
\begin{aligned}
(vi)\int_0^{t_1}\Big(t_1-\tau\Big)^2 f(\tau)\,d\tau &= \lambda\int_0^{t_1}\Big(t_1-\tau\Big)^2 e^{-\lambda\tau}\,d\tau\\
&= t_1^2-\frac{2t_1}{\lambda}+\frac{2}{\lambda^2}\{1-e^{-\lambda t_1}\}\\
&= \frac{1}{3}\lambda t_1^3,\ \text{approximating up to the fourth term of the expansion of } e^{-\lambda t_1}
\end{aligned}
$$

$$
\begin{aligned}
(vii)\int_0^{t_1}\Big(t_1-\tau\Big)^3 f(\tau)\,d\tau &= \lambda\int_0^{t_1}\Big(t_1-\tau\Big)^3 e^{-\lambda\tau}\,d\tau\\
&= t_1^3-3\int_0^{t_1}\Big(t_1-\tau\Big)^2 e^{-\lambda\tau}\,d\tau\\
&= t_1^3-\frac{3}{\lambda}\int_0^{t_1}\Big(t_1-\tau\Big)^2 f(\tau)\,d\tau\\
&= t_1^3-\frac{3}{\lambda}\Big[t_1^2-\frac{2t_1}{\lambda}+\frac{2}{\{\lambda\}^2}\{1-e^{-\lambda t_1}\}\Big],\ \text{using (iv)}\\
&= \frac{1}{4}\lambda t_1^4,\ \text{approximating up to the fifth term of the expansion of } e^{-\lambda t_1}
\end{aligned}
$$

Appendix D

For Chapter 6

$$(i) \quad \int_0^{t_1} f(\tau)\, d\tau$$
$$= \lambda_1 \phi(P) \int_0^{t_1} e^{-\lambda_1 \phi(P)\tau}\, d\tau$$
$$= 1 - e^{-\lambda_1 \phi(P) t_1}$$
$$= \lambda_1 \phi(P) t_1, \text{ approximating up to the second term of the expansion of } e^{-\lambda_1 \phi(P) t_1}$$

$$(ii) \quad \int_0^{t_1} \Big(t_1 - \tau\Big) \lambda_1 \phi(\tau)\, d\tau$$
$$= \lambda_1 \phi(P) \int_0^{t_1} \Big(t_1 - \tau\Big) e^{-\lambda_1 \phi(P)\tau}\, d\tau$$
$$= t_1 - \frac{1}{\lambda_1 \phi(P)} \{1 - e^{-\lambda_1 \phi(P) t_1}\}$$
$$= \frac{1}{2} \lambda_1 \phi(P) t_1^2, \text{ approximating up to the third term of the expansion of } e^{-\lambda_1 \phi(P) t_1}$$

$$(iii) \quad \int_0^{t_1} \Big(t_2 - \tau\Big) \lambda_1 \phi(\tau)\, d\tau$$
$$= \lambda_1 \phi(P) \int_0^{t_1} \Big(t_2 - \tau\Big) e^{-\lambda_1 \phi(P)\tau}\, d\tau$$
$$= -(t_2 - t_1) e^{-\lambda_1 \phi(P) t_1}) + t_2 - \frac{1}{f(P)} \{1 - e^{-\lambda_1 \phi(P) t_1}\}$$
$$= t_1 (t_2 - t_1) \lambda_1 \phi(P), \text{ approximating up to the second term of the expansion of } e^{-\lambda_1 \phi(P) t_1}$$

$$(iv) \quad \int_0^{t_1} \Big(t_1 - \tau\Big)^2 f(\tau)\, d\tau$$

$$= \lambda_1\phi(P) \int_0^{t_1} \Big(t_1 - \tau\Big)^2 e^{-\lambda_1\phi(P)\tau}\, d\tau$$

$$= t_1^2 - \frac{2t_1}{\lambda_1\phi(P)} + \frac{2}{\{\lambda_1\phi(P)\}^2}\{1 - e^{-\lambda_1\phi(P)t_1}\}$$

$$= \frac{1}{3}\lambda_1\phi(P)t_1^3, \text{ approximating up to the fourth term of the expansion of } e^{-\lambda_1\phi(P)t_1}$$

$$(v) \quad \int_0^{t_1} \Big(t_1 - \tau\Big)^3 f(\tau)\, d\tau$$

$$= \lambda_1\phi(P) \int_0^{t_1} \Big(t_1 - \tau\Big)^3 e^{-\lambda_1\phi(P)\tau}\, d\tau$$

$$= t_1^3 - 3\int_0^{t_1} \Big(t_1 - \tau\Big)^2 e^{-\lambda_1\phi(P)\tau}\, d\tau$$

$$= t_1^3 - \frac{3}{\lambda_1\phi(P)} \int_0^{t_1} \Big(t_1 - \tau\Big)^2 f(\tau)\, d\tau$$

$$= t_1^3 - \frac{3}{\lambda_1\phi(P)}\Big[t_1^2 - \frac{2t_1}{\lambda_1\phi(P)} + \frac{2}{\{\lambda_1\phi(P)\}^2}\{1 - e^{-\lambda_1\phi(P)t_1}\}\Big], \text{using (iv)}$$

$$= \frac{1}{4}\lambda_1\phi(P)t_1^4, \text{ approximating up to the fifth term of the expansion of } e^{-\lambda_1\phi(P)t_1}$$

Appendix E

For Chapter 7

$$\begin{aligned}
T &= t_4 + \frac{1}{\beta_1}\log\Big[(1-\frac{D_r}{\alpha_1})(1-e^{-\beta_1 t_3})\Big] \\
&= t_2 + \frac{S-W_1}{\alpha_1+\beta_1 W_1} + \frac{1}{\beta_1}\log\Big[(1-\frac{D_r}{\alpha_1})(1-e^{-\beta_1 t_3})\Big] \\
&= \frac{P}{D_r}\Big[\frac{1}{\alpha}(1-\beta)(1-e^{-\alpha t_1}) + \beta t_1 + \frac{S-W_1}{\alpha_1+\beta_1 W_1} + \frac{1}{\beta_1}\log\Big[(1-\frac{D_r}{\alpha_1})(1-e^{-\beta_1 t_3})\Big] \\
&= \frac{P}{D_r}t_1 + \frac{S-W_1}{\alpha_1+\beta_1 W_1} + \frac{1}{\beta_1}\log\Big[(1-\frac{D_r}{\alpha_1})(1-e^{-\beta_1 t_3})\Big] \\
&= \frac{1-\theta}{D_r}R + t_0,\ \ (using\, 1-e^{-\alpha t_1} \approx \alpha t_1\Big).
\end{aligned}$$

$$\begin{aligned}
APS(R,P) &= \frac{1}{T}\Big[\{w_s(1-\theta) + w_s'\theta - c_s - s_c\}R - A_s - h_s R[\frac{(1-\theta)t_1}{2} + \frac{R\theta}{x}] - I_{cs}(T-t_1)\Big] \\
&= \frac{1}{T}\Big[-Z_{0s} + Z_{1s}R + Z_{2s}\frac{R}{P} - Z_{3s}\frac{R^2}{2P} - Z_{4s}R^2\Big],\ \text{where } Z_{0s} = (A_s + I_{cs}t_0), \\
& Z_{1s} = \Big[w_s(1-\theta) + w_s' - c_s - s_c - \frac{I_{cs}(1-\theta)}{D_r}\Big], \\
& Z_{2s} = I_{cs}(1-\theta),\ Z_{3s} = h_s(1-\theta)^2,\ Z_{4s} = \frac{h_s\theta}{x}.
\end{aligned}$$

$$\begin{aligned}
& APM(R,P) = \frac{1}{T}\Big[(s_m D_r t_2 + s_m' D_r' t_2') - \Big\{w_s + C(P) + I_{sm}\Big\}Pt_1 \\
& -r_{cm}\Big\{-\frac{P\beta}{\alpha}(1-e^{-\alpha t_1}) + P\beta t_1\Big\} - A_m - I_{cm}(T-t_2) \\
& -h_m\Big\{\frac{P}{\alpha}(1-\beta)t_1 - \frac{P}{\alpha^2}(1-\beta)(1-e^{-\alpha t_1}) + P\beta\frac{t_1^2}{2} - D_r\frac{t_2^2}{2}\Big\} \\
& -h_m'\Big\{\frac{P}{\alpha^2}\gamma(1-\beta)(1-e^{-\alpha t_1}) - \frac{\gamma}{\alpha}(1-\beta)Pt_1 + (\gamma(1-\beta)P - D_r')\frac{t_1^2}{2} + \frac{D_r'}{2}(t_1 - t_2')^2\Big\}\Big]
\end{aligned}$$

$$= \frac{1}{T}\Big[(s_m + \frac{I_{cm}}{D_r})\Big\{\frac{P}{\alpha}(1-\beta)(1-e^{-\alpha t_1}) + P\beta t_1\Big\} - \Big\{w_s + C(P) + I_{sm}\Big\}Pt_1 - A_m$$
$$+ s'_m\Big\{\frac{-P\gamma}{\alpha}(1-\beta)(1-e^{-\alpha t_1}) + P\gamma(1-\beta)t_1\Big\} - r_{cm}[-\frac{P\beta}{\alpha}(1-e^{-\alpha t_1}) + P\beta t_1]$$
$$-I_{cm}T - h_m\Big\{\frac{P}{\alpha}(1-\beta)t_1 - \frac{P}{\alpha^2}(1-\beta)(1-e^{-\alpha t_1}) + P\beta\frac{t_1^2}{2} - D_r\frac{t_2^2}{2}\Big\}$$
$$-h'_m\Big\{\frac{P}{\alpha^2}\gamma(1-\beta)(1-e^{-\alpha t_1}) - \frac{\gamma}{\alpha}(1-\beta)Pt_1 + (\gamma(1-\beta)P - D'_r)\frac{t_1^2}{2} + \frac{D'_r}{2}(t_1 - t'_2)^2\Big\}\Big]$$
$$= \frac{1}{T}\Big[(s_m + \frac{I_{cm}}{D_r} - w_s - I_{sm} - R_1)Pt_1 - I_{cm}T - A_m - Gt_1 - HP^2t_1$$
$$+\Big[s'_m(1-\beta)\alpha\gamma - (s_m + \frac{I_{cm}}{D_r})(1-\beta)\alpha - h_m - r_{cm}\alpha\beta\Big]\frac{Pt_1{}^2}{2}$$
$$+\frac{h_m}{2D_r}P^2t_1{}^2 + h'_mD'_rt_1{}^2\Big], \qquad \Big(Using\, 1 - e^{-\alpha t_1} \approx \alpha t_1\Big)$$
$$= \frac{1}{T}\Big[-Z_{0m} - Z_{1m}\frac{R}{P} + Z_{2m}R + Z_{3m}\frac{R^2}{P^2} + Z_{4m}\frac{R^2}{P} + Z_{5m}R^2 - Z_{6m}PR^2\Big].$$

where $Z_{0m} = I_{cm}t_0 - A_m,\ \ Z_{1m} = G(1-\theta),\ \ Z_{2m} = (s_m - w_s - I_{sm} - R_1)(1-\theta)$
$Z_{4m} = \Big[s'_m(1-\beta)\alpha\gamma - (s_m + \frac{I_{cm}}{D_r})(1-\beta)\alpha - h_m - r_{cm}\alpha\beta\Big]\frac{(1-\theta)^2}{2}$
$Z_{3m} = h'_mD'_r(1-\theta)^2,\ \ Z_{5m} = \frac{h_m}{2D_r}(1-\theta)^2,\ \ Z_{6m} = H(1-\theta).$

$$APR(R,P) = \frac{1}{T}\Big[(s_r - c_{tp})\Big\{(\alpha_1 + \beta_1W_1)(t_4 - t_3) + D_r(t_3 - \frac{1}{\beta_1}) + \frac{D_r - \alpha_1}{\beta_1}e^{-\beta_1 t_3}$$
$$+\frac{\alpha_1}{\beta_1}e^{-\beta_1(t_4 - T)}\Big\} - \frac{h_{rs}}{2}\Big\{(D_r - (\alpha_1 + \beta_1W_1))(t_2 - t_3)^2 + (\alpha_1 + \beta_1W_1)(t_2 - t_4)^2\Big\}$$
$$-h_r\Big\{\frac{D_r - \alpha_1}{\beta_1}(t_3 + \frac{e^{-\beta_1 t_3}}{\beta_1} - \frac{1}{\beta_1}) + W_1(t_4 - t_3) - \frac{\alpha_1}{\beta_1}\Big\{(T - t_4) + \frac{1}{\beta_1} - \frac{e^{-\beta_1(t_4 - T)}}{\beta_1}\Big\}\Big\}$$
$$-\frac{h'_r}{2}\Big\{(D'_r - D'_c)t'_2{}^2 + D'_c(t'_2 - T')^2\Big\} - A_r - A'_r + (s'_r - c'_{tp})D'_rt'_2$$
$$-s_m\Big\{\frac{P}{\alpha}(1-\beta)t_1 - \frac{P}{\alpha^2}(1-\beta)(1-e^{-\alpha t_1}) + (P\beta)\frac{t_1^2}{2} - D'_r\frac{t_2^2}{2}\Big\}$$
$$-s'_m\Big\{\frac{P\gamma}{\alpha^2}(1-\beta)(1-e^{-\alpha t_1}) - \frac{P\gamma}{\alpha}(1-\beta)t_1 + \frac{D'_r}{2}(t_1 - t'_2)^2 + (P\gamma(1-\beta) - D'_r)\frac{t_1^2}{2}\Big\}\Big]$$
$$= \frac{1}{T}\Big[Z_{0r} + Z_{1r}R + Z_{2r}\frac{R^2}{2P} + Z_{3r}R^2 + Z_{4r}\frac{R^3}{P} + Z_{5r}\frac{R^4}{P^2}\Big],\ (using\ 1 - e^{-\alpha t_1} \approx \alpha t_1)$$

where $Z_{0r} = -h_{rs}\Big[\frac{(s-W_1)^2}{2(D_r - (\alpha_1 + \beta_1W_1))} + \frac{(s-W_1)^2}{2(\alpha_1 + \beta_1W_1)}\Big] - A_r - A'_r$
$- h_r\Big[-\frac{\alpha_1}{\beta_1^2}\log(1 + \frac{\beta_1W_1}{\alpha_1}) - \frac{1}{\beta_1}(\frac{D_r - \alpha_1}{\beta_1} - \frac{D_rW_1}{\alpha_1 + \beta_1W_1})\log(1 - \frac{W_1\beta_1}{D_r - \alpha_1})\Big]$
$Z_{1r} = (s_r - c_{tp} - \frac{h_rW_1}{\alpha_1 + \beta_1W_1})(1-\theta)$
$Z_{2r} = \Big[(s'_r - c'_{tp})\gamma\alpha(1-\beta) + (s_r - c_{tp} - \frac{h_rW_1}{\alpha_1 + \beta_1W_1})\alpha(1-\beta) - \frac{W_m}{2}\Big](1-\theta)^2$

$Z_{3r} = \frac{s_m}{2D_r}(1-\theta)^2, \;\; Z_{4r} = \Big[\frac{s'_m\gamma\alpha}{2}(1-\beta) - \frac{s_m\gamma}{2D_r}(1-\beta)\Big](1-\theta)^3$
$Z_{5r} = \Big[s_m\frac{\gamma^2}{4D_r}(1-\beta)^2 - s'_m\frac{\gamma^2\alpha^2}{8D_r}(1-\beta)^2 - \frac{h'_r}{8D'_r}\Big\{(D'_r - D'_c) + \frac{(D'_r-D'_c)^2}{D'^2_c}\Big\}\gamma^2\alpha^2(1-\beta)^2\Big].$

$$IAP(R,P) = \Big[APS + APM + APR\Big]$$
$$= \frac{1}{T}\Big[(Z_{0r} - Z_{0m} - Z_{0s}) + (Z_{2s} - Z_{1m})\frac{R}{P} + (Z_{1s} + Z_{1r} + Z_{2m})R - (Z_{3s} + 2Z_{4m}$$
$$+Z_{2r})\frac{R^2}{2P} + Z_{3m}\frac{R^2}{P^2} - Z_{6m}PR^2 - (Z_{4s} - Z_{3r} - Z_{5m})R^2 + Z_{4r}\frac{R^3}{P} + Z_{5r}\frac{R^4}{P^2}\Big]$$
$$= \frac{1}{T}\Big[Z_0 + Z_1\frac{R}{P} + Z_2R - Z_3\frac{R^2}{2P} + Z_4\frac{R^2}{P^2} - Z_5PR^2 - Z_6R^2 + Z_7\frac{R^3}{P} + Z_8\frac{R^4}{P^2}\Big],$$
where $Z_0 = (Z_{0r} - Z_{0m} - Z_{0s}),\ Z_1 = (Z_{2s} - Z_{1m}),\ Z_2 = (Z_{1s} + Z_{1r} + Z_{2m}),\ Z_5 = Z_{6m},$
$Z_3 = (Z_{3s} + 2Z_{4m} + Z_{2r}),\ Z_4 = Z_{3m},\ \ Z_7 = Z_{4R},\ \ Z_8 = Z_{5R},\ Z_6 = (Z_{4s} - Z_{3r} - Z_{5m}).$

Calculations for optimality test

$$\frac{d}{dP}(APM) = \frac{1}{T}\Big[Z_{1m}\frac{R}{P^2} - 2Z_{3m}\frac{R^2}{P^3} - Z_{4m}\frac{R^2}{P^2} - Z_{6m}R^2\Big].$$
$$\frac{d^2}{dP^2}(APM) = \frac{1}{T}\Big[-Z_{1m}\frac{R}{P^3} + 3Z_{3m}\frac{R^2}{P^4} + Z_{4m}\frac{R^2}{P^3}\Big].$$
$$\frac{d}{dP}(IAP) = \frac{1}{T}\Big[-Z_1\frac{R}{P^2} + Z_3\frac{R^2}{2P^2} - 2Z_4\frac{R^2}{P^3} - Z_5R^2 - Z_7\frac{R^3}{P^2} - 2Z_8\frac{R^4}{P^3}\Big].$$
$$\frac{d^2}{dP^2}(IAP) = \frac{1}{T}\Big[2Z_1\frac{R}{P^3} - Z_3\frac{R^2}{P^3} + 6Z_4\frac{R^2}{P^4} + 2Z_7\frac{R^3}{P^3} + 6Z_8\frac{R^4}{P^4}\Big].$$

$$\frac{\partial}{\partial R}(APM) = \frac{1}{T}\Big[-Z_{1m}\frac{1}{P} + Z_{2m} + 2Z_{3m}\frac{R}{P^2} + 2Z_{4m}\frac{R}{P} + 2Z_{5m}R - 2PRZ_{6m}\Big]$$
$$-\frac{(1-\theta)}{D_rT^2}\Big[-Z_{0m} - Z_{1m}\frac{R}{P} + Z_{2m}R + Z_{3m}\frac{R^2}{P^2} + Z_{4m}\frac{R^2}{P} + Z_{5m}R^2 - Z_{6m}PR^2\Big]$$
$$= \frac{(1-\theta)}{D_rT}Z_{0m} + \Big[\frac{(1-\theta)R}{D_rTP} - \frac{1}{P}]Z_{1m} + [1 - \frac{(1-\theta)R}{D_rT}\Big]Z_{2m} + \Big[\frac{2R}{P^2} - \frac{(1-\theta)R^2}{D_rTP^2}\Big]Z_{3m}$$
$$+\Big[\frac{2R}{P} - \frac{(1-\theta)R^2}{D_rTP}\Big]Z_{4m} + \Big[2R - \frac{(1-\theta)R^2}{D_rT}]Z_{5m} + [\frac{(1-\theta)R^2P}{D_rT} - 2PR\Big]Z_{6m}.$$

$$\frac{\partial}{\partial P}(APM) = \frac{1}{T}\Big[Z_{1m}\frac{R}{P^2} - 2Z_{3m}\frac{R^2}{P^3} - Z_{4m}\frac{R^2}{P^2} - Z_{6m}R^2\Big].$$
$$\frac{\partial^2}{\partial P^2}(APM) = 2\Big[-Z_{1m}\frac{R}{P^3} + 3Z_{3m}\frac{R^2}{P^4} + Z_{4m}\frac{R^2}{P^3}\Big]$$
$$\frac{\partial^2}{\partial R\partial P}(APM) = \frac{1}{T}\Big[Z_{1m}\frac{1}{P^2} - 4Z_{3m}\frac{R}{P^3} - 2Z_{4m}\frac{R}{P^2} - 2Z_{6m}R\Big]$$

$$\frac{\partial^2}{\partial R^2}(APM) = \frac{1}{T}\Big[-\frac{(1-\theta)^2}{{D_r}^2T^2}Z_{0m} + \Big\{\frac{(1-\theta)}{D_rPT} - \frac{(1-\theta)^2R}{{D_r}^2T^2P}\Big\}Z_{1m} + \Big\{\frac{(1-\theta)^2R}{{D_r}^2T^2} - \frac{(1-\theta)}{D_rT}\Big\}Z_{2m}$$
$$+\Big\{\frac{(1-\theta)^2R^2}{{D_r}^2T^2P^2} + \frac{1}{P^2} - \frac{2(1-\theta)R}{D_rTP^2}\Big\}Z_{3m} + \Big\{\frac{(1-\theta)^2}{{D_r}^2T^2P} + \frac{1}{P} - \frac{2(1-\theta)R}{D_rTP}\Big\}Z_{4m}$$
$$+\Big\{1 - \frac{2(1-\theta)R}{D_rPT} + \frac{(1-\theta)^2R^2}{{D_r}^2T^2}\Big\}Z_{5m} - \Big\{P - \frac{2(1-\theta)PR}{D_rT} + \frac{(1-\theta)^2R^2P}{{D_r}^2T^2}\Big\}Z_{6m}\Big].$$

$$\frac{\partial}{\partial R}(IAP) = \frac{1}{T}\Big[Z_1\frac{1}{P} + Z_2 - Z_3\frac{R}{P} + 2Z_4\frac{R}{P^2} - 2Z_5PR - 2Z_6R + 3Z_7\frac{R^2}{P} + 4Z_8\frac{R^3}{P^2}\Big]$$
$$-\frac{(1-\theta)}{D_rT^2}\Big[Z_0 + Z_1\frac{R}{P} + Z_2R - Z_3\frac{R^2}{2P} + Z_4\frac{R^2}{P^2} - Z_5PR^2 - Z_6R^2 + Z_7\frac{R^3}{P} + Z_8\frac{R^4}{P^2}\Big]$$
$$\frac{\partial^2}{\partial R^2}(IAP) = \frac{1}{T}\Big[\frac{2(1-\theta)^2}{{D_r}^2T^2}Z_0 + \Big\{\frac{2(1-\theta)^2R}{{D_r}^2T^2P} - \frac{2(1-\theta)}{D_rTP}\Big\}Z_1 + \{\frac{2(1-\theta)^2R}{{D_r}^2T^2} - \frac{2(1-\theta)}{D_rT}\Big\}Z_2$$
$$-\Big\{1 + \frac{2(1-\theta)^2R^2}{{D_r}^2T^2P} - \frac{2(1-\theta)R}{D_rTP}\Big\}Z_3 + \Big\{\frac{2}{P^2} + \frac{2(1-\theta)^2R^2}{{D_r}^2T^2P^2} - \frac{4(1-\theta)R}{D_rTP}\Big\}Z_4$$
$$-\Big\{2P + \frac{2(1-\theta)^2PR^2}{{D_r}^2T^2} - \frac{4(1-\theta)PR}{D_rT}\Big\}Z_5 - \Big\{2 + \frac{2(1-\theta)^2R^2}{{D_r}^2T^2} - \frac{4(1-\theta)R}{D_rT}\Big\}Z_6$$
$$+\Big\{\frac{6R}{P} + \frac{2(1-\theta)^2R^3}{{D_r}^2T^2P} - \frac{6(1-\theta)R^2}{D_rTP}\Big\}Z_7 + \Big\{\frac{12R^2}{P^2} + \frac{2(1-\theta)^2R^4}{{D_r}^2T^2P^2} - \frac{8(1-\theta)R^3}{D_rTP^2}\Big\}Z_8\Big].$$
$$\frac{\partial}{\partial P}(IAP) = \frac{1}{T}\Big[-Z_1\frac{R}{P^2} + Z_3\frac{R^2}{2P^2} - 2Z_4\frac{R^2}{P^3} - Z_5R^2 - Z_7\frac{R^3}{P^2} - 2Z_8\frac{R^4}{P^3}\Big].$$
$$\frac{\partial^2}{\partial P^2}(IAP) = \frac{1}{T}\Big[2Z_1\frac{R}{P^3} - Z_3\frac{R^2}{P^3} + 6Z_4\frac{R^2}{P^4} + 2Z_7\frac{R^3}{P^3} + 6Z_8\frac{R^4}{P^4}\Big].$$
$$\frac{\partial^2}{\partial R\partial P}(IAP) = \frac{1}{T}\Big[-Z_1\frac{1}{P^2} + Z_3\frac{R}{P^2} - 4Z_4\frac{R}{P^3} - 2Z_5R - 3Z_7\frac{R^2}{P^2} - 8Z_8\frac{R^3}{P^3}\Big]$$
$$-\frac{(1-\theta)}{D_rT^2}\Big[-Z_1\frac{R}{P^2} + Z_3\frac{R^2}{2P^2} - 2Z_4\frac{R^2}{P^3} - Z_5R^2 - Z_7\frac{R^3}{P^2} - 2Z_8\frac{R^4}{P^3}\Big]$$
$$= \frac{1}{T}\Big[\Big\{\frac{(1-\theta)R}{D_rTP^2} - \frac{1}{P^2}\Big\}Z_1 + \Big\{\frac{R}{P^2} - \frac{(1-\theta)R^2}{2D_rTP^2}\Big\}Z_3 + \Big\{\frac{2(1-\theta)R^2}{D_rTP^3} - \frac{4R}{P^3}\Big\}Z_4$$
$$+\Big\{\frac{(1-\theta)R^2}{D_rT} - 2R\Big\}Z_5 + \Big\{\frac{(1-\theta)R^2}{D_rTP^2} - \frac{3R^2}{P^2}\Big\}Z_7 + \Big\{\frac{2(1-\theta)R^2}{D_rTP^3} - \frac{8R^3}{P^3}\Big\}Z_8\Big].$$

Appendix F

For Chapter 9

(i) P_{2i} = Production rate of ith cycle in plant II=$\lambda P_{2(i-1)} = \lambda^{i-1}P_2$, with $P_{21} = P_2$.

$$\sum_{i=1}^{n} P_{2i} = P_2 \sum_{i=1}^{n} \lambda^{i-1} = P_2 \frac{\lambda^n - 1}{\lambda - 1}, \quad \lambda > 1.$$

(ii) D_{2i} = Demand rate of ith cycle in plant II=$\mu D_{2(i-1)} = \mu^{i-1}D_2$ with $D_{21} = D_2$.

$$\sum_{i=1}^{n} D_{2i} = D_2 \sum_{i=1}^{n} \mu^{i-1} = D_2 \frac{\mu^n - 1}{\mu - 1}, \quad \mu > 1.$$

$(iiii)$ α_{ij} = percentage of return of better quality items per unit time collected from the market for ith cycle from jth cycle (j=1,2,...,i) in plant I and we take $\alpha_{ii} = \alpha_1$ and $\alpha_{ij-1} = \alpha_{ij}h$.

Therefore, $\alpha_{i+m,i} \;=\; h\,\alpha_{i+m,i+1} = h^2\,\alpha_{i+m,i+2} = \ldots\ldots\cdots = h^m\,\alpha_{i+m,i+m} = h^m\,\alpha_1,$

$$\begin{aligned}
\sum_{r=i}^{k+i-1} \alpha_{ij} &= \alpha_{i,i} + \alpha_{i+1,i} + \alpha_{i+2,i} + \cdots + \alpha_{i+k-1,i} \\
&= \alpha_{i,i} + h\alpha_{i+1,i+1} + h^2\alpha_{i+2,i+2} + \cdots + h^{k-1}\alpha_{i+k-1,i+k-1} \\
&= \alpha_1 + h\alpha_1 + h^2\alpha_1 + \cdots + h^{k-1}\alpha_1 \\
&= \alpha_1 \frac{h^k - 1}{h - 1}, \quad h > 1,\ i = 1, 2, \ldots, n-k+1.
\end{aligned}$$

$$\begin{aligned}
\sum_{r=i}^{n} \alpha_{ri} &= \alpha_{i,i} + \alpha_{i+1,i} + \alpha_{i+2,i} + \cdots + \alpha_{n,i} \\
&= \alpha_{i,i} + h\alpha_{i+1,i+1} + h^2\alpha_{i+2,i+2} + \cdots + h^{k-1}\alpha_{i+(n-i),i+(n-i)} \\
&= \alpha_1 + h\alpha_1 + h^2\alpha_1 + \cdots + h^{n-i}\alpha_1 \\
&= \alpha_1 \frac{h^{n-i+1} - 1}{h - 1}, \quad h > 1,\ i = n-k+2, \ldots, n-1.
\end{aligned}$$

Similarly, $\sum_{r=i}^{k+i-1}\beta_{ri}=\alpha_2\frac{h'^k-1}{h'-1},\quad h'>1,\, i=1,2,\ldots,n-k+1.$

and $\sum_{r=i}^{n}\beta_{ri}=\alpha_2\frac{h'^{n-i+1}-1}{h'-1},\quad h'>1,\, i=n-k+2,\ldots,n-1.$

(iv) Here $\tilde{\epsilon}_g=[\tilde{\epsilon}_g^L,\tilde{\epsilon}_g^R],\ \tilde{\epsilon}_w=[\tilde{\epsilon_w^L},\tilde{\epsilon_w^R}],\ \widetilde{ER}_g=[\widetilde{ER}_g^L,\widetilde{ER}_g^R]$;
where $\tilde{\epsilon}_g^L=(\epsilon_{g1}^L,\epsilon_{g2}^L,\epsilon_{g3}^L),\ \tilde{\epsilon}_g^R=(\epsilon_{g1}^R,\ \epsilon_{g2}^R,\epsilon_{g3}^R);\ \tilde{\epsilon}_w^L=(\epsilon_{w1}^L,\epsilon_{w2}^L,\epsilon_{w3}^L),\tilde{\epsilon}_w^R=(\epsilon_{w1}^R,\epsilon_{w2}^R,\epsilon_{w3}^R)$;
$\widetilde{ER}_g^L=(ER_{g1}^L,ER_{g2}^L,ER_{g3}^L),\ \widetilde{ER}_g^R=(ER_{g1}^R,ER_{g2}^R,ER_{g3}^R)$ are considered as TFN, so

$$\begin{aligned}\widetilde{TC}_{p1} &= n\Big\{c_p+r_c\delta_1(1-\beta_1)+c_{sr}\Big\}P_1t_1+THC_1+nA_s+nc_w[\tilde{\epsilon}_w^L,\tilde{\epsilon}_w^R]P_1^{\eta_{1w}}t_1+nc_g[\tilde{\epsilon}_g^L,\tilde{\epsilon}_g^R]P_1^{\eta_{1g}}t_1\\ &= [\widetilde{TC}_{p1}^L,\widetilde{TC}_{p1}^R]\\ \widetilde{TC}_{p2} &= TPC_2+RWC_2+TSC_2+THC_2+nA'_s+c_w[\tilde{\epsilon}_w^L,\tilde{\epsilon}_w^R](T-t_1)\sum_{i=1}^n P_{2i}^{\eta_{2g}}\\ &\quad +c_g[\tilde{\epsilon}_g^L,\tilde{\epsilon}_g^R](T-t_1)\sum_{i=1}^n P_{2i}^{\eta_{2g}}=[\widetilde{TC}_{p2}^L,\widetilde{TC}_{p2}^R]\\ \widetilde{Z}_g &= n[\tilde{\epsilon}_g^L,\tilde{\epsilon}_g^R]P_1^{\eta_{1g}}t_1+[\tilde{\epsilon}_g^L,\tilde{\epsilon}_g^R](T-t_1)\sum_{i=1}^n P_{2i}^{\eta_{2g}}+[\widetilde{ER}_g^L,\widetilde{ER}_g^R]Q_4=[\widetilde{Z}_g^L,\widetilde{Z}_g^R]\end{aligned}$$

where $\widetilde{TC}_{p1}^L=n\Big\{c_p+r_c\delta_1(1-\beta_1)+c_{sr}\Big\}P_1t_1+THC_1+nA_s+nc_w\tilde{\epsilon}_w^LP_1^{\eta_{1w}}t_1+nc_g\tilde{\epsilon}_g^LP_1^{\eta_{1g}}t_1,$
$\widetilde{TC}_{p1}^R=n\Big\{c_p+r_c\delta_1(1-\beta_1)+c_{sr}\Big\}P_1t_1+THC_1+nA_s+nc_w\tilde{\epsilon}_w^RP_1^{\eta_1w}t_1+nc_g\tilde{\epsilon}_g^RP_1^{\eta_{1g}}t_1,$

$$\widetilde{TC}_{p2}^L=TPC_2+RWC_2+TSC_2+THC_2+nA'_s+c_w\tilde{\epsilon}_w^L(T-t_1)\sum_{i=1}^n P_{2i}^{\eta_{2w}}+c_g\tilde{\epsilon}_g^L(T-t_1)\sum_{i=1}^n P_{2i}^{\eta_{2g}},$$
$$\widetilde{TC}_{p2}^R=TPC_2+RWC_2+TSC_2+THC_2+nA'_s+c_w\tilde{\epsilon}_w^R(T-t_1)\sum_{i=1}^n P_{2i}^{\eta_{2w}}+c_g\tilde{\epsilon}_g^R(T-t_1)\sum_{i=1}^n P_{2i}^{\eta_{2g}},$$

$\tilde{z}_g^L=n\tilde{\epsilon}_g^LP_1^{\eta_{1g}}t_1+\tilde{\epsilon}_g^L(T-t_1)\sum_{i=1}^n P_{2i}^{\eta_{2g}}+\widetilde{ER}_g^LQ_4,$
$\tilde{z}_g^R=n\tilde{\epsilon}_g^RP_1^{\eta_{1g}}t_1+\tilde{\epsilon}_g^R(T-t_1)\sum_{i=1}^n P_{2i}^{\eta_{2g}}+\widetilde{ER}_g^RQ_4.$

$$\widetilde{TP} = TSR_{p1}+TSR_{p2}-\widetilde{TC}_{p1}-\widetilde{TC}_{p2}-THC_3-c_g\widetilde{ER}_gQ_4-c_tQ_4=[\widetilde{TP}^L,\widetilde{TP}^R]$$

where

$$\begin{aligned}\widetilde{TP}^L &= TSR_{p1}+TSR_{p2}-\widetilde{TC}_{p1}^R-\widetilde{TC}_{p2}^R-THC_3-c_g\widetilde{ER}_g^RQ_4-c_tQ_4\\ \widetilde{TP}^R &= TSR_{p1}+TSR_{p2}-\widetilde{TC}_{p1}^L-\widetilde{TC}_{p2}^L-THC_3-c_g\widetilde{ER}_g^LQ_4-c_tQ_4\end{aligned}$$

Using Lemma 2.3, we have

$E(\tilde{\epsilon}_g^L)=\frac{1}{2}\Big\{(1-\rho)\epsilon_{g1}^L+\epsilon_{g2}^L+\rho\epsilon_{g3}^L\Big\}, E(\tilde{\epsilon}_g^R)=\frac{1}{2}\Big\{(1-\rho)\epsilon_{g1}^R+\epsilon_{g2}^R+\rho\epsilon_{g3}^R\Big\},\ \ 0<\rho<1$

$E(\tilde{\epsilon}_w^L)=\frac{1}{2}\Big\{(1-\rho)\epsilon_{w1}^L+\epsilon_{w2}^L+\rho\epsilon_{w3}^L\Big\}, E(\tilde{\epsilon}_w^R)=\frac{1}{2}\Big\{(1-\rho)\epsilon_{w1}^R+\epsilon_{w2}^R+\rho\epsilon_{w3}^R\Big\},$

$E(\widetilde{ER}_g^L)=\frac{1}{2}\Big\{(1-\rho)ER_{g1}^L+ER_{g2}^L+\rho ER_{g3}^L\Big\}, E(\widetilde{ER}_g^R)=\frac{1}{2}\Big\{(1-\rho)ER_{g1}^R+ER_{g2}^R+\rho ER_{g3}^R\Big\},$

$E(\widetilde{TC}_{p1}^L)=TPC_1+RWC_1+TSC_1+THC_1+nA_s+nc_wE(\tilde{\epsilon}_w^L)P_1^{\eta_{1w}}t_1+nc_gE(\tilde{\epsilon}_g^L)P_1^{\eta_{1g}}t_1,$

$E(\widetilde{TC}_{p1}^R)=TPC_1+RWC_1+TSC_1+THC_1+nA_s+nc_wE(\tilde{\epsilon}_w^R)P_1^{\eta_{1w}}t_1+nc_gE(\tilde{\epsilon}_g^R)P_1^{\eta_{1g}}t_1,$

$$E(\widetilde{TC}_{p2}^L] = TPC_2+RWC_2+TSC_2+THC_2+nA_s'+c_wE(\tilde{\epsilon}_w^L)(T-t_1)\sum_{i=1}^n P_{2i}^{\eta_{2g}}+c_gE(\tilde{\epsilon}_g^L)(T-t_1)\sum_{i=1}^n P_{2i}^{\eta_{2g}},$$

$$E(\widetilde{TC}_{p2}^R] = TPC_2+RWC_2+TSC_2+THC_2+nA_s'+c_wE(\tilde{\epsilon}_w^R)(T-t_1)\sum_{i=1}^n P_{2i}^{\eta_{2g}}+c_gE(\tilde{\epsilon}_g^R)(T-t_1)\sum_{i=1}^n P_{2i}^{\eta_{2g}},$$

$E(\tilde{z}_g^L)=nE(\tilde{\epsilon}_g^L)P_1^{\eta_{1g}}t_1+E(\tilde{\epsilon}_g^L)(T-t_1)\sum_{i=1}^n P_{2i}^{\eta_{2g}}+E(\widetilde{ER}_g^L)Q_4,$

$E(\tilde{z}_g^R)=nE(\tilde{\epsilon}_g^R)P_1^{\eta_{1g}}t_1+E(\tilde{\epsilon}_g^R)(T-t_1)\sum_{i=1}^n P_{2i}^{\eta_{2g}}+E(\widetilde{ER}_g^R)Q_4.$

Appendix G

For Chapter 10

(i) $\displaystyle\sum_{N^j=0}^{\infty}\int_{N^jT^j}^{(N^j+1)T^j} PC_L^j\widetilde{f(h)}\,dh$

$$= \sum_{N^j=0}^{\infty}\int_{N^jT^j}^{N^jT^j+t_1^j} PC_{L_1}^j\widetilde{f(h)}\,dh + \sum_{N^j=0}^{\infty}\int_{N^jT^j+t_1^j}^{N^jT^j+t_2^j} PC_{L_2}^j\widetilde{f(h)}\,dh + \sum_{N^j=0}^{\infty}\int_{N^jT^j+t_2^j}^{(N^j+1)T^j} PC_{L_3}^j\widetilde{f(h)}\,dh$$

$$= \frac{c_p^jP^j}{R}\Big[(1-e^{-\widetilde{\lambda}T^j}) - \frac{\widetilde{\lambda}}{R+\widetilde{\lambda}}\{1-e^{-(R+\widetilde{\lambda})t_1^j}\} - e^{-Rt_1^j}(e^{-\widetilde{\lambda}t_1^j} - e^{-\widetilde{\lambda}T^j})\Big]\sum_{N^j=0}^{\infty} e^{-(RT^j+\widetilde{\lambda}T^j-\beta^j)N^j}$$

$$= \frac{c_p^jP^j}{R\{1-e^{-(RT^j+\widetilde{\lambda}T^j-\beta^j)}\}}\Big[(1-e^{-\widetilde{\lambda}T^j}) - \frac{\widetilde{\lambda}}{R+\widetilde{\lambda}}\{1-e^{-(R+\widetilde{\lambda})t_1^j}\} - e^{-Rt_1^j}(e^{-\widetilde{\lambda}t_1^j} - e^{-\widetilde{\lambda}T^j})\Big]$$

(ii) $\displaystyle\sum_{N^j=0}^{\infty}\int_{N^jT^j}^{(N^j+1)T^j} SC_L^j\widetilde{f(h)}\,dh$

$$= \sum_{N^j=0}^{\infty}\int_{N^jT^j}^{N^jT^j+t_1^j} SC_{L_1}^j\widetilde{f(h)}\,dh + \sum_{N^j=0}^{\infty}\int_{N^jT^j+t_1^j}^{N^jT^j+t_2^j} SC_{L_2}^j\widetilde{f(h)}\,dh + \sum_{N^j=0}^{\infty}\int_{N^jT^j+t_2^j}^{(N^j+1)T^j} SC_{L_3}^j\widetilde{f(h)}\,dh$$

$$= \frac{c_{sr}^jP^j}{R}\Big[(1-e^{-\widetilde{\lambda}T^j}) - \frac{\widetilde{\lambda}}{R+\widetilde{\lambda}}\{1-e^{-(R+\widetilde{\lambda})t_1^j}\} - e^{-Rt_1^j}(e^{-\widetilde{\lambda}t_1^j} - e^{-\widetilde{\lambda}T^j})\Big]\sum_{N^j=0}^{\infty} e^{-(RT^j+\widetilde{\lambda}T^j-\beta^j)N^j}$$

$$= \frac{c_{sr}^jP^j}{R\{1-e^{-(RT^j+\widetilde{\lambda}T^j-\beta^j)}\}}\Big[(1-e^{-\widetilde{\lambda}T^j}) - \frac{\widetilde{\lambda}}{R+\widetilde{\lambda}}\{1-e^{-(R+\widetilde{\lambda})t_1^j}\} - e^{-Rt_1^j}(e^{-\widetilde{\lambda}t_1^j} - e^{-\widetilde{\lambda}T^j})\Big]$$

(iii) $\displaystyle\sum_{N^j=0}^{\infty}\int_{N^jT^j}^{(N^j+1)T^j} RC_L^j \widetilde{f(h)}\,dh$

$$= \sum_{N^j=0}^{\infty}\int_{N^jT^j}^{N^jT^j+t_1^j} RC_{L_1}^j \widetilde{f(h)}\,dh + \sum_{N^j=0}^{\infty}\int_{N^jT^j+t_1^j}^{N^jT^j+t_2^j} RC_{L_2}^j \widetilde{f(h)}\,dh + \sum_{N^j=0}^{\infty}\int_{N^jT^j+t_2^j}^{(N^j+1)T^j} RC_{L_3}^j \widetilde{f(h)}\,dh$$

$$= \frac{r_c^j(1-\delta^j)\theta^j P^j}{R\{1-e^{-\{(R+\widetilde{\lambda})T^j+\alpha^j\}}\}}\Big[(1-e^{-\widetilde{\lambda}T^j}) - \frac{\widetilde{\lambda}}{R+\widetilde{\lambda}}\{1-e^{-(R+\widetilde{\lambda})t_1^j}\} - e^{-Rt_1^j}(e^{-\widetilde{\lambda}t_1^j} - e^{-\widetilde{\lambda}T^j})\Big]$$

(iv) $\displaystyle\sum_{N^j=0}^{\infty}\int_{N^jT^j}^{(N^j+1)T^j} SR_L^j \widetilde{f(h)}\,dh$

$$= \sum_{N^j=0}^{\infty}\int_{N^jT^j}^{N^jT^j+t_1^j} SR_{L_1}^j \widetilde{f(h)}\,dh + \sum_{N^j=0}^{\infty}\int_{N^jT^j+t_1^j}^{N^jT^j+t_2^j} SR_{L_2}^j \widetilde{f(h)}\,dh + \sum_{N^j=0}^{\infty}\int_{N^jT^j+t_2^j}^{(N^j+1)T^j} SR_{L_3}^j \widetilde{f(h)}\,dh$$

$$= \frac{s^jP^j}{R}\Big\{\frac{1}{1-e^{-(R+\widetilde{\lambda})T^j}} - \frac{\delta^j\theta^j}{1-e^{-\{(R+\widetilde{\lambda})T^j+\alpha^j\}}}\Big\}\Big[(1-e^{-\widetilde{\lambda}t_2^j}) - \frac{\widetilde{\lambda}}{\widetilde{\lambda}+R}\{1-e^{-(R+\widetilde{\lambda})t_2^j}\}\Big]$$

$$+s^j\Big\{\frac{(P^j-a^j)}{1-e^{-(R+\widetilde{\lambda})T^j}} - \frac{\delta^j\theta^jP^j}{1-e^{-\{(R+\widetilde{\lambda})T^j+\alpha^j\}}}\Big\}\Big[\frac{1}{R+b^j}(1-e^{-\widetilde{\lambda}t_1^j}) - \frac{\widetilde{\lambda}\{1-e^{-(\widetilde{\lambda}+R+b^j)t_1^j}\}}{(R+b^j)(\widetilde{\lambda}+R+b^j)}$$

$$+\Big\{\frac{1}{R}(1-e^{-Rt_1^j}) - \frac{1}{R+b^j}\{1-e^{-(R+b^j)t_1^j}\}\Big\}(e^{-\widetilde{\lambda}t_1^j} - e^{-\widetilde{\lambda}T^j})\Big]$$

$$-\frac{s^ja^j}{1-e^{-(R+\widetilde{\lambda})T^j}}\Big[\frac{e^{-Rt_1^j}}{R}(e^{-\widetilde{\lambda}t_1^j} - e^{-\widetilde{\lambda}t_2^j}) - \frac{\widetilde{\lambda}}{R(\widetilde{\lambda}+R)}(e^{-(R+\widetilde{\lambda})t_1^j} - e^{-(R+\widetilde{\lambda})t_2^j})$$

$$-\frac{e^{b^jt_2}}{R+b^j}\Big\{e^{-(R+b^j)t_1^j}(e^{-\widetilde{\lambda}t_1^j} - e^{-\widetilde{\lambda}t_2^j}) - \frac{\widetilde{\lambda}}{(\widetilde{\lambda}+R+b^j)}\{e^{-(R+b^j+\widetilde{\lambda})t_1^j} - e^{-(R+b^j+\widetilde{\lambda})t_2^j}\}\Big\}$$

$$+\Big\{\frac{1}{R}(e^{-Rt_1^j} - e^{-Rt_2^j}) - \frac{e^{b^jt_2^j}}{R+b^j}\{e^{-(R+b^j)t_1^j} - e^{-(R+b^j)t_2^j}\}\Big\}(e^{-\widetilde{\lambda}T^j} - e^{-\widetilde{\lambda}t_2^j})\Big]$$

(v) $\displaystyle\sum_{N^j=0}^{\infty}\int_{N^jT^j}^{(N^j+1)T^j} SH_L^j \widetilde{f(h)}\,dh$

$$= \sum_{N^j=0}^{\infty}\int_{N^jT^j}^{N^jT^j+t_1^j} SH_{L_1}^j \widetilde{f(h)}\,dh + \sum_{N^j=0}^{\infty}\int_{N^jT^j+t_1^j}^{N^jT^j+t_2^j} SH_{L_2}^j \widetilde{f(h)}\,dh + \sum_{N^j=0}^{\infty}\int_{N^jT^j+t_2^j}^{(N^j+1)T^j} SH_{L_3}^j \widetilde{f(h)}\,dh$$

$$= \frac{c_{sh}^ja^j}{b^j\{1-e^{-(R+\widetilde{\lambda})T^j}\}}\Big[\frac{b^je^{-Rt_2^j}}{R(R+b^j)}\Big(e^{-\widetilde{\lambda}t_2^j} - e^{-\widetilde{\lambda}T^j}\Big) - \frac{\widetilde{\lambda}}{R(R+\widetilde{\lambda})}\Big\{e^{-(R+\widetilde{\lambda})t_2^j} - e^{-(R+\widetilde{\lambda})T^j}\Big\}$$

$$+\frac{\widetilde{\lambda}e^{bt_2^j}}{(R+b^j)(R+\widetilde{\lambda}+b^j)}\Big\{e^{-(R+\widetilde{\lambda}+b^j)t_2^j} - e^{-(R+\widetilde{\lambda}+b^j)T^j}\Big\}\Big]$$

(vi) $\sum_{N^j=0}^{\infty} \int_{N^jT^j}^{(N^j+1)T^j} HC_L^j \widetilde{f(h)}\, dh$

$$= \sum_{N^j=0}^{\infty} \int_{N^jT^j}^{N^jT^j+t_1^j} HC_{L_1}^j \widetilde{f(h)}\, dh + \sum_{N^j=0}^{\infty} \int_{N^jT^j+t_1^j}^{N^jT^j+t_2^j} HC_{L_2}^j \widetilde{f(h)}\, dh + \sum_{N^j=0}^{\infty} \int_{N^jT^j+t_2^j}^{(N^j+1)T^j} HC_{L_3}^j \widetilde{f(h)}\, dh$$

$$= \frac{h_c^j}{b^j}\Big\{\frac{(P^j-a^j)}{\{1-e^{-(R+\widetilde{\lambda})T^j}\}} - \frac{\delta^j\theta^jP^j}{\{1-e^{-\{(R+\widetilde{\lambda})T^j+\alpha^j\}}\}}\Big\}\Big[\frac{1}{R}(1-e^{-\widetilde{\lambda}t_1^j}) - \frac{\widetilde{\lambda}}{R(\widetilde{\lambda}+R)}\{1-e^{-(R+\widetilde{\lambda})t_1^j}\}$$

$$+\frac{1}{R+b^j}(1-e^{-\widetilde{\lambda}t_1^j}) - \frac{\widetilde{\lambda}}{(R+b^j)(\widetilde{\lambda}+R+b^j)}\{1-e^{-(R+b^j+\widetilde{\lambda})t_1^j}\}$$

$$+\Big\{\frac{1}{R}(1-e^{-Rt_1^j}) - \frac{1}{R+b^j}\{1-e^{-(R+b^j)t_1^j}\}\Big\}(e^{-\widetilde{\lambda}t_1^j}-e^{-\widetilde{\lambda}T^j})\Big]$$

$$-\frac{h_c^ja^j}{b^j\{1-e^{-(R+\widetilde{\lambda})T^j}\}}\Big[\frac{e^{-Rt_1^j}}{R}(e^{-\widetilde{\lambda}t_1^j}-e^{-\widetilde{\lambda}t_2^j}) - \frac{\widetilde{\lambda}}{R(\widetilde{\lambda}+R)}(e^{-(R+\widetilde{\lambda})t_1^j}-e^{-(R+\widetilde{\lambda})t_2^j})$$

$$-\frac{e^{b^jt_2}}{R+b^j}\Big\{e^{-(R+b^j)t_1^j}(e^{-\widetilde{\lambda}t_1^j}-e^{-\widetilde{\lambda}t_2^j}) - \frac{\widetilde{\lambda}}{(\widetilde{\lambda}+R+b^j)}\{e^{-(R+b^j+\widetilde{\lambda})t_1^j}-e^{-(R+b^j+\widetilde{\lambda})t_2^j}\}\Big\}$$

$$+\Big\{\frac{1}{R}(e^{-Rt_1^j}-e^{-Rt_2^j}) - \frac{e^{b^jt_2^j}}{R+b^j}\{e^{-(R+b^j)t_1^j}-e^{-(R+b^j)t_2^j}\}\Big\}(e^{-\widetilde{\lambda}T^j}-e^{-\widetilde{\lambda}t_2^j})\Big]$$

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