MODELLING OF SOME PROBLEMS ON PRODUCTION PLANNING AND INVENTORY MANAGEMENT

Thesis submitted to the
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DOCTOR OF PHILOSOPHY IN SCIENCE

By

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Declaration

I hereby declare that the thesis entitled **"Modelling of Some Problems on Production Planning and Inventory Management"** submitted for the degree of Doctor of Philosophy in Science is my original work carried out under the supervision of **Dr. Biswajit Sarkar**, Assistant Professor, Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University and **Prof. Adrijit Goswami**, Department of Mathematics, Indian Institute of Technology. I further declare that the work presented in this thesis has not been submitted previously, as a whole or in part, to any University or Institution for the award of any academic degree or diploma.

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Abstract

The major issues of recent research is the products nature live when it can be deteriorated, when it can be used fully, when deterioration rate is random or when products have fixed lifetime. These problems are solved by this research studies. Along with the product's nature, the optimal buffer inventory, replenishment rate, imperfect production, and inspection errors during the shipment of production system from *in-control-state* to *out-of-control* state are efficiently studied by this research studies. The dissertation consists of six different studies on products produced by production system. The first chapter of the thesis is devoted to discuss the background of different study, basic concepts, and terminology, historical review of the inventory model and objective and organization of the thesis.

In Chapter 2, an inventory model of two-warehouse is considered with quadratically increasing demand and time-varying deterioration. Comparing to the existing literature, the model is derived with finite replenishment rate and unequal length of the time cycle. The associated cost of the system is minimized. A numerical example, the graphical representation and sensitivity analysis are provided to illustrate the model.

Chapter 3 describes a deteriorating inventory model with *ramp-type* demand pattern under stock-dependent consumption rate. The deterioration of the product is considered as probabilistic to make the research a more realistic one. The proposed model assumes partially backorder rate, which follows a negative exponential with the waiting time. The purpose of this study is to develop an optimal replenishment policy such that the total profit is maximized. Here, it is provided a simple solution procedure to obtain the optimal solution.

Chapter 4 illustrates a mathematical model to obtain an optimal replenishment policy for deteriorating items with maximum lifetime, *ramp-type* demand, and shortages. Holding cost and deterioration function both are linear function of time, which are treated as

constants in most of the deteriorating inventory model. Numerical examples along with graphical representations are provided to illustrate the model.

In Chapter 5, a two-echelon supply chain model with variable setup cost and deterioration cost are analyzed. It is assumed that the setup cost is directly proportional and the deterioration rate is inversely proportional to reliability. Basic algebraical procedure is used to obtain the optimal closed-form solution of this model. The objective is to minimize the total cost of the entire system by considering reliability as a decision variable.

In Chapter 6, an effort has been made to obtain an optimal buffer inventory and inspection policy with preventive maintenance and two types of inspection errors. This model considerers a production system that is subject to a random deterioration from an *in-control* state to an *out-of-control* state with a specific distribution. An on-line inspection is started after some time to inspect products. To detect the defective items another human inspection policy is considered at the end of production cycle.

Chapter 7 deals with the problem of determining the optimal production run time for an imperfect production system with inspection policy. This model considerers that product inspection perform at any arbitrary time of the production cycle. Two types of inspection errors (Type I and Type II) are considered to make the model more realistic rather than the existing models. Defective items are salvaged at some cost before being shipped. Non-inspected defective items are passed to customers with free minimal repair warranty.

Lastly, Chapter 8 concludes the thesis and discusses its achievements. It also suggests further studies in this area.

Keywords: Inventory; probabilistic deterioration; time-varying deterioration; time-varying demand; two-warehouse system; Ramp-type demand; stock-dependent consumption rate; partial backorder; inflation; fixed lifetime; time-dependent holding cost; shortages; Supply chain management; reliability; buffer inventory; maintenance scheduling; imperfect production; product inspection policy; inspection errors; warranty.

Published/Accepted Papers

1. A two-warehouse inventory model with increasing demand and time varying deterioration. *Scientia Iranica E*, 19 (6), 1969-1977 (2012).

2. Mitigation of high-tech products with probabilistic deterioration and inflations. *American Journal of Industrial and Business Management*, 5, 73-89 (2015).

3. Flexible setup cost and deterioration of products in a supply chain model. *International Journal of Applied and Computational Mathematics*, DOI 10.1007/s40819-015-0045-7, (2015).

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Communicated Paper

1. Optimal production run time and inspection errors in an imperfect production system with warranty. Under review in *Journal of Industrial and Management Optimization*.

Contents

Notation

H	total planning horizon
n	number of production cycles during the entire horizon H
d	demand rate of product (units per unit time)
p	constant production rate, where $p > d$
TC	total system cost during planning horizon
ψ	probabilistic deterioration rate
$\delta(t)$	backlogging rate
u	discount rate representing the time-value of money
i	inflation rate per unit time
$\rho = u - i$	discount rate minus inflation rate
s	selling-price per unit
C_o	ordering cost per order
C_h	unit inventory holding cost per unit time
C_p	purchasing cost per unit purchase/production cost per unit product
C_b	backorder cost per unit backorder
C_l	lost sell cost per unit
C_d	deterioration cost
S_c	shortage cost

M_c	preventive maintenance cost
C_w	warranty cost
C_s	salvaged cost
C_r	rework cost
C_m	variable cost (labor cost, energy cost)
μ	parameter of the ramp-type demand function (break point)
$I(t)$	on-hand inventory level at time t
t_1	length of time in which the inventory level falls to zero
T	production run time/ length of each ordering cycle
Q	order quantity per cycle (units)/production lot size per batch-cycle (units)
$\alpha(t)$	time-dependent deterioration rate
α	constant deterioration rate
q	delivery lot size (units)
N	number of deliveries per production-batch, $N \geq 1$
R	reliability parameter
S	total shortage amount
S_o	initial setup cost for a production batch
S_1	variable setup cost for production batch
A_b	area under the buyer's inventory level

A_s	area under the supplier's inventory level
K	transportation cost per delivery
HC_s	holding cost for the supplier
HC_b	holding cost for the buyer
V_c	unit variable cost for order handling and receiving
t_p	production time duration for the supplier
t_n	non-production time duration for the supplier
t_d	duration between the two successive deliveries
X	time after which the production process shifts *in-control* to *out-of-control* state (random variable)
m_1	probability of Type I error (random variable)
m_2	probability of Type II error (random variable)
τ	preventive maintenance time (random variable)
$g(x)$	probability density function of X
$G(x)$	distribution function of X
$\overline{G}(x)$	survival function of X, i.e., $\overline{G}(x) = 1 - G(x)$
$g(m_1)$	probability density function of m_1
$g(m_2)$	probability density function of m_2
$g(\tau)$	probability density function of τ

θ_1 percentage of defective items when the production process is in the *in-control* state

θ_2 percentage of defective items when the production process is in the *out-of-control* state.

M_c material cost

r system restoration cost

η system inspection cost

I_c product inspection cost

B buffer inventory

C_A cost of falsely accepted defective item

C_R cost of falsely rejected non-defective item

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Chapter 1

Introduction

1.1 Background

"Sorry, we are out of stock", we often heard these words during shopping time. In many of these cases, we have encountered that the stores aren't doing a good job of managing their inventory (stocks of physical goods that contain economic value and use or sale in a future point of time). They aren't placing orders to the suppliers of the product in time to replenish inventories to avoid shortage. In this situation, lost sale occurred and at the same time goodwill of the shop is lost. Maintenance of inventory is necessary for any company dealing with any physical products.

Operations Research (OR) is something that helps management to achieve its goals using the scientific process. It is the use of advanced analytical techniques to improve decision making. In today's global markets, customers expect high-quality products and services when they need them, where they need them. Any organizations need to provide these products and services as effectively and efficiently as possible. This requires careful planning and analysis of the whole system to make a concrete decision.

One of the most developed field of OR is inventory management. Inventory has been defined as "an asset that is owned by a business that has the express purpose of being sold to a customer". It includes different raw materials, work-in-process goods and completely finished goods, which have not yet been sold to the market. The management of inventory is a key concern of all businesses. The main aim of any production firms/shops is to hold adequate stock of items after meeting up the customer's demand. Stocking of items depends upon the various factors like demand, deterioration, lead time, etc. Study of such type of problems fall under the topics on inventory management.

Due to the unpredictable nature of our society and markets, it is a common problem to all organizations in any sector of economy to control and maintain an inventory. The following are reasons why any organization need to hold inventory:

(1) It gives satisfactory service to customers. If inventories are not maintained properly,

then goods have to be procured frequently in a given system precisely to meet the demands. Which is neither physically possible nor economically sound.

(2) The demands of certain kind of products such as ice-cream, umbrella, air-conditioner, etc are vastly unpredictable. Because their demand revolve around their irregular changeability of the weather. Thus, holding stocks might be the main option to ensure on-time supply to the customer.

(3) Without inventories, customers have to wait until the required items are delivered to the customers. This may cause inconvenience to many customers and sometimes the customer can't wait for it, they can collect their required items from different sources. Hence, lost sale may occur. Inventory can solve this type of problems.

(4) It acts as a buffer stock against the lack of goods in the market.

(5) There could also be the possibility of price-discount for bulk purchases.

(6) The loss due to deterioration, inflation, damages, and different obstacles are minimized with the help of inventory.

(7) There are some instances, when buyer and suppliers are far away from each other. Therefore, there comes the need to possible long lead time (time-lack between placing an order and receiving the items) i.e. it serves as a buffer stock in case of delayed deliveries by the suppliers.

(8) There are some products (precious metals, metallic furniture, fine arts etc.), which increase the valuation over time. Therefore, products are to held in stock to prevent high purchase cost in future.

(9) It reduces the production cost and maximizes the profit. When machine produces defective items, a certain amount of production cost is increased due to re-process the defective items. Designing reliable inventory system helps to control production in such a way that the defective items are not produced or if produced then it is minimum in numbers.

(10) It moves the cash flow by timely shipment of customers order that helps to run the

business smoothly and efficiently.

(11) It improves the man power, equipment and facility utilization by better planing and scheduling.

(12) It reduces the production cost, because of an advantage of batching and long interrupted production runs.

(13) Usually the demand rate is decided by the amount of the stock level. The motivational effect on the people may be caused by presentation of stock. It is a matter of fact that large quantity of goods displayed in markets according to seasons motivate the customer to buy more.

Thus, with a good amount of inventory, a firm is able to make purchases in economic lots, maintain continuity of operations and guarantee prompt delivery of finished products. Although, it provides an alternative to production or purchase in future, but this also means locking-up capital for this time period of the organization. There is some expenses on stores, insurance, equipment, etc to maintain inventory. Hence excess inventories are undesirable. Thus, it is necessary for any organization to maintained inventory properly such that the profit is maximum. The two fundamental questions in controlling these inventories of any physical goods are:

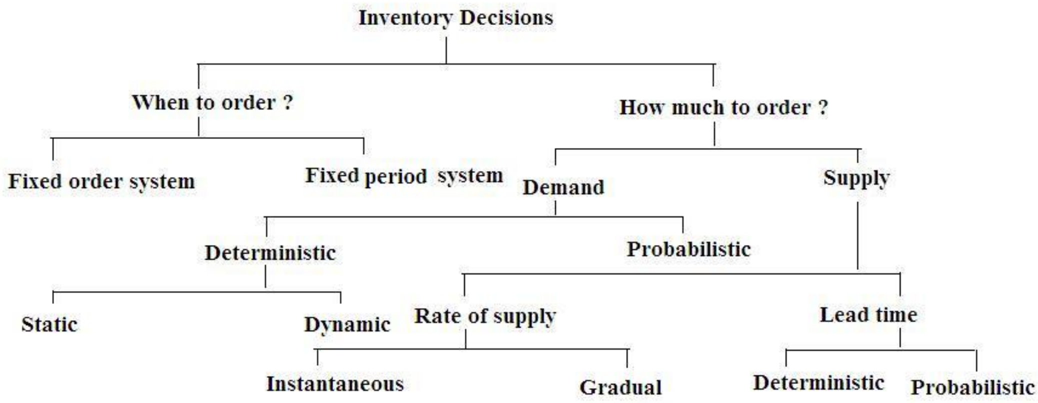
(i) When the industry has to order for buying the raw materials or any other required products?

(ii) How much amount of lot has to ordered/produced/purchased by the industry at the beginning of each time interval?

An inventory problem is a problem of making optimal decision (i.e. to minimize the total system cost or to maximize the overall profit) regarding the above questions.

In today's competitive and continuously changing environment, it is essential for any business organization to make daily decisions on many issues, such as how much and where to produce, when to produce, what prices to set the product, etc. Mathematical models can help to take the best decision, among all the possible alternatives. Here the

Figure 1.1: Classification of inventory decisions



main target is to develop operating rules for controlling inventory system by using mathematical point of view. In this regard, the first task is to express the inventory system in a mathematical framework i.e., a mathematical model is constructed which is based on various assumptions and approximations that fit the model. Then, the set of all specific values of the variables are to be set such that the total average system cost is minimum or the total profit of the system is maximum.

1.2 Basic concepts and terminologies

The following basic concepts and terminologies are used to developed the research.

1.2.1 Demand

Demand is defined as the number of units of an item required per period of time by the customer. There are two types of demand in the market. One is deterministic demand and another is probabilistic demand. In the deterministic case, where quantities of demand needed throughout a period of time are known exactly. The known demand may be constant or variable. These demands are called static or dynamic, respectively. Dynamic demand depends on time, stock level, price, etc.(i.e., time-dependent demand, inventory level-dependent demand, price-dependent demand, etc.). In the probabilistic case, the demand over certain period of time is not known exactly, but its pattern can be described and explained by a known probability distribution. It includes two types of demands. The first type is stationary over time. It is characterized by a known demand distribution. The second type is non-stationary over time. It is characterized by arbitrary demand distribution.

1.2.2 Lead time or delivery lag

When any buyer places an order for a product to a supplier, then the supplier may instantaneously supply the product or may take time to supply the product i.e., there is a time-gap between placing of an order and its actual addition in stock. This time-gap is called lead time. It has an important role for determining inventory levels. The lead time may be negligible or constant, deterministic, variable or probabilistic. Any industry or manufacturing system tries to minimize the lead time. Because if, the lead time is large enough the inventory level have to be large as well. If the lead time is increased, the demand uncertainty during this time interval is increased as well. The size of the safety stock depends on this uncertainty. If the lead time is minimize, the inventory cost also be minimized.

1.2.3 Replenishment rate

The rate at which stock-keeping units are ordered to replace depleted-stocks is known as replenishment rate. A replenishment rate may be fixed and occur within the same time frame every month, quarter, or year, when projected needs are well established or, a replenishment rate may be variable and depend on a number of factors such as seasonal changes in demand and whether shortages are allowed.

1.2.4 Buffer inventory

Buffer inventory is the specific level of excess stock of inventory that is maintained for protection against uncertainties of future demand and lead time. In general, demand and lead time are random variable with known probability distribution. Thus, to avoid unpredictable shortage, additional stock of inventory items is maintained in addition to the regular stock. Such additional stocks of items is known as buffer stock as it provides buffer against future uncertainty.

1.2.5 Deterioration

Deterioration is defined as damage, spoilage, breakage, decay, and dryness vaporization of the items such that those items are not in a condition of being used for their original purpose. Any product has a lifetime after which its utility reaches to zero. The products, like green vegetables, foods, photographic film, etc. having a maximum usable lifetime and its deterioration is started after they produce. The perishable products are those products, which have full utilities within the duration of their lifetimes, but after the end of lifetime, total quantity of product cannot be used any more like blood, medicine, etc. The products, like alcohol, gasoline, radioactive elements, etc. having no self-time at all, are known as decaying products. Any product deteriorates when storage conditions are unsatisfactory or inadequate or poor handling/packing in the warehouse.

1.2.6 Planning horizon

The time period over which a particular inventory level will be maintained is called planning horizon. Depending on the demand pattern, it may be finite or infinite.

1.2.7 Supply chain management

A manufacturer's inventory initially stored at the point of manufacture (first echelon), then at regional warehouses (second echelon), after that a field distributor centers (third echelon), and so on. Each stage, where the inventory is held in progression through a multi-stage inventory system is called echelon of the inventory system. A system with multiple echelons of inventory is referred to as a multi-echelon inventory system. Supply chain is a network of different players that first procures raw materials, transforms them into intermediate goods, and finished products, and finally delivers these finished products to various customers through a distribution system, which comprises a multi-echelon inventory system. Therefore, a supply chain spans procurement, manufacturing and distribution. In all these stages, inventories are needed. Hence, effective inventory management is one key element in managing the supply chain. Key success factor for any leading companies is the integrated management of the supply chain.

1.2.8 Cost

When optimize the inventory levels, one has to determine the different costs associated with carrying inventory.

(a) Ordering cost:

Ordering cost includes all costs which are involved each time an order is placed for procuring items into inventory or stores from the outside suppliers. This cost consists of the following heads:

(i) Purchasing: The clerical and administrative costs associated with the purchasing, placing an order, follow-up, receiving and evaluating quotations, etc. are involved within it.

(ii) Accounting: It covers the cost of checking supplies against each order, auditing, maintaining record of purchases, etc.

(iii) Services: Cost of services includes cost of mailing, telephone calls, transportation, etc.

(b) Setup cost:

It deals with the cost of setting up machines before initial production. The clerical and administrative cost related to purchase the raw material or any items from the market, receiving and evaluating quotations, different types of postage, telephone charges, loading and unloading costs, transportation costs etc are involved in setup costs. These costs are assumed to be independent of quantity produced. The setup cost related to the inventory system may not be fixed always, it may depend on demand, time, etc.

(c) Holding cost:

This cost is associated with the money tide up for holding or carrying the goods in stock. Holding cost is assumed to vary directly with the size of inventory as well as the time for which the item is hold in stock. It includes warehouse rent, storage cost like air-conditioning for cold storage, insurance cost, handling cost, lighting cost, theft, pilferage, interest of capital, etc. It can be derived into four categories:

- capital cost
- inventory service cost
- storage space cost
- inventory risk cost

(d) Shortage cost or Stock-out cost:

Demand of an item is uncertain in real life. Consequently, stock out situation may arise in an inventory cycle. Some of the retailers/customers are unable to keep their patience during this shortage period and meet their demand from other sources, which is considered as shortage cost, i.e. when a customer seeks any product and finds empty at the inventory, then the customer may wait or the customer go to another shop to fulfill the demand. Thus, the demand can either go unfulfilled or be satisfied later, when the product becomes available. The former case is called lost sale and the latter is called a backorder. Shortage cost includes lost sale cost, cost of production stoppage, over time payments, special order at higher price, loss of good will, loss of profitability, etc. The shortage costs are found in two ways: one with backorder which means if the unsatisfied demand may be satisfied in future, these costs usually vary directly to the shortage item and waiting time, and another without backorder means if the unsatisfied demand is lost completely, then this cost becomes proportional to the shortage quantity. In inventory system, the shortage cost depends both on the shortage quantities and on the duration of time over which this shortage exists.

(e) Inspection cost:

Major source of lost sale and poor customer satisfaction for retailers is shelf stock-outs. A recent industry study identifies that shelf execution failures and inventory inaccuracies as two main causes of shelf stock-outs. Shelf execution failures occur when there is an inventory in the back room, but store associates fail to move it to the shelf on time. As a result, the product is unavailable on the shelf and no sales are recorded. The shelf remains empty unless and until someone intervenes. Thus, inaccuracy into the data of inventory record is a major obstruction to achieving excellence in management. In this situation, any employer has to make inventory inspection and replenishment decisions at the beginning of each period. There will be a cost associated with each inspection.

(f) Defective cost/Rework cost:

In a production system, defective items may be produced by the manufacturing system due to long-run of the machine or due to lack of maintenance of the system. When defective items are produced, then manufacturer reworked these items at some cost before being shifted or sold them at reduced price or this items are passed to customers with free minimal repair warranty.

1.3 Historical review of the inventory model

The classical inventory model, named economic order quantity (EOQ) model, and economic production quantity (EPQ) model were first developed by Harris in 1915 and Taft in 1918. After that, Raymond (1931) first wrote a full length book on inventory problems. During the World War II, an urgent need was felt to allocate scarce resources to the various military operations in an effective manner. Therefore, British and then the U.S. military management called upon a large number of scientist including Physicists, Mathematician, Biologists, Statistician and Psychologists to apply a scientific approach to the many strategy and tactical problems. At the end of this War II, the scientist group moved to different sectors, e.g., health, education, industry, defence, etc. They analyze the problems in each sector scientifically and investigated the optimum method for carrying out the operations. The urge to study the inventory problems have increased at that time and a huge number of publications have been devoted exclusively to this study. Till date many research are going on to enrich this field. Several books devoted in the direction of inventory control are written by Whitin (1957), Naddor (1966), Silver and Peterson (1985), Porteus (2002), Piasecki (2009), among others. Recently some of the authors like Muller (2011), Richards (2014), Frazelle (2015), etc., published several inventory related models in their books.

Any industry always tries to minimize his production cost to achieve a suitable goal of profit. Thus, the industry has to watch how many quantities are expected and how many quantities are actually achieved i.e., how many quantities are deteriorated through out the cycle of production. Deterioration is defined as decay of items such that those items are not in a condition of being used for their original purpose. Deteriorating items can be classified into two categories. The first category refers to the items that become expired through time like medicine, vegetables, fruits etc. The other category refers to the items that lose their usable value due to the presence of their alternatives by the introduction of new technology like mobile phone, computer chips, fashionable items etc. Deterioration is a key characteristic and its impact on inventory modelling can not be neglected. Thus, deterioration rate should be considered in the development of inventory strategy. The inventory problem with deteriorating items was first proposed by Whitin (1957), he considered the deterioration of the fashion goods at the end of a prescribed storage period. Ghare and Schrader (1963) were the first authors to consider the effect of exponentially decay of an items on inventory. An EOQ type model with finite production rates was consider by Misra (1975). Shah (1977) derived a model for both exponentially and Weibell deterioration cases, which allowed backorder. Datta and Pal (1988) formulated an order level inventory system with variable rate of deterioration. Heng *et al.* (1991) integrated Misra's (1975) and Shah's (1977) approaches and derived a model for deteriorating items with finite replenishment rate and backorder. Kang and Kim (1983) studied on the price and production level of the deteriorating inventory system. Aggarwal and Jaggi (1989) derived an ordering policy for decaying inventory. Law and Wee (2006) developed an integrated production-inventory model for ameliorating and deteriorating items. Heuristic models for deteriorating items with shortages and time-varying demand was developed by Giri and Chaudhuri (1997). Raafat (1991) made a literature survey on continuously deteriorating inventory model. In 2001, Goyal and Giri wrote a survey on the recent trends in modeling of deteriorating inventory.

The storage capacity of any business organization is limited and some situation arises, when the demand goes up quickly or the order cost is high or when the product is seasonal, the supplier of raw materials provides a price-discount for bulk purchase, etc., then the amount of purchase product or the produced product exceeds the storage capacity of the organization's own-warehouse (OW), the excess quantities have to stored in an additional storage space known as rented-warehouse (RW). As the preservation depends on preservation facilities available in the warehouse and the two-warehouse OW and RW have not same preservation facilities, thus the different warehouse may have different deterioration rate. Different enterprisers have the common problem to decide how much product to keep excess stock in RW such that the total system cost is minimum. Sarma (1987) first studied a two-warehouse inventory model with deterioration. Goswami and Chaudhuri (1992) consider an economic order quantity model for items with two levels of storage for a linear-trend in demand. Lee (2006) developed a two-warehouse inventory model with deterioration under FIFO dispatching policy and he modified Pakkala and Achary's (1992) LIFO model and proved that modified LIFO model always has a lower cost if holding cost in RW was greater than holding cost in OW. Lee and Hsu (2009) established a two-warehouse production model for deteriorating inventory items with time-dependent demands.

Classical inventory model considers the demand rate as either constant, time-dependent, or stock-dependent demand instead of constant demand. Mondal and Phaujdar (1989) proposed a production-inventory model for deteriorating items with the assumption that demand is a linear function of inventory level. Sana et al. (2004) analyzed a production inventory model for a deteriorating item with trended demand and shortages. Hou (2006) developed an inventory model for deteriorating items with stock-dependent consumption rate and shortages under inflation and time discounting. Chang et al. (2010) studied an optimal replenishment policies for non-instantaneous deteriorating items with stock-dependent demand. It is observed that the demand rate of electronic goods (e.g., mobile phone, laptop, RAM, processer, etc.), new brand of consumer goods comes to the market, seasonal products (e.g., mango, orange, etc.,) increase linearly with time at the beginning upto a certain moment and then stabilizes to a constant rate until the end of the inventory cycle. To represent such type of demand pattern the term ramp-type is used. Mandal and Pal (1998) was the first author to introduce ramp-type demand in inventory model.

In real life situation, due to long-run process, the manufacturing system moves from *in-control* to *out-of-control* state, and then the manufacturing system may produce perfect as well as imperfect (defective) quality items. This defective items may be reworked at a fixed cost or sold at reduced price. In this direction, Porteus (1986) developed an inventory model assuming imperfect quality of products, where he introduced the quality improvement and setup cost reduction by some logarithmic investment functions. Jamal *e*t al. (2004) developed an inventory model with optimum batch quantity in a single-stage system in which rework was done by addressing two different operational policies to minimize the total system cost. C*á*rdenas-Barr*ó*n (2007) corrected an error appearing in the paper of Jamal *e*t al. (2004), although the main idea and contribution of the paper were not affected. C*á*rdenas-Barr*ó*n (2008) developed a simple derivation based on an algebraic derivation of the two inventory policies proposed by Jammal *e*t al. (2004). C*á*rdenas-Barr*ó*n (2009a) developed an economic production quantity model with planned backorders to determine the economic production quantity for a single type of products manufactured in a single-stage production system that generated imperfect quality products, and all defective products were reworked in the same cycle. In both of the above model, the author established the range of real values of the proportion of defective products for which there was an optimal solution, and the close-form for the total cost of inventory system. Sarkar (2012b) derived an EOQ model to investigate the retailer's optimal replenishment policy under permissible delay-in-payment with stock-dependent demand in the presence of imperfect production. Sarkar and Moon (2011) extend the classical EPQ model with imperfect production and stochastic demand under the effect of inflation. Most recently, Sarkar (2012c) considered an economic manufacturing quantity (EMQ) model with price and advertising-demand pattern in an imperfect production process under the effect of inflation. A production-inventory model to determine the optimal product reliability and production rate that achieves the biggest total integrated profit for an imperfect manufacturing process was developed by Sana (2010c). Sana (2010d) presented an EPL (Economic Production Lotsize) model in an imperfect production system, in which the manufacturing process may shift from an *in-control* state to an *out-of-control* state at any random time. Sana and Chaudhuri (2010) discussed a framework of production policy (resumption and non-resumption) in order to find out optimal safety stock, optimal production rate and production lot size. An integrated production-inventory model for a three-layer supply chain, assuming perfect and imperfect quality products was developed by Sana (2011).

Generally, each and every product of a production system deteriorates with time due to age or usage, and exhaustion. Thus, without any maintenance action, the system must converted from *in-control* state to *out-of-control* state (Yeh *et al.*, 2000). For maintenance *in-control* state, preventative maintenance is planned such that any required resources are available. Preventative maintenance also helps to prevent the break down of a machine such that the production system goes through a long-run process and can produce more products. Considering those, Mercier and Pham (2012) proposed a preventive maintenance policy for a continuously monitored system with correlated wear indicators. Suliman and Jawad (2012) optimized preventive maintenance schedule and production lot size. Recently, Optimal preventive maintenance strategy for repairable items under two-dimensional warranty was presented by Wang *et al.* (2015). Zhou *et al.* (2015) arranged a preventive maintenance modeling for multi-component systems with considering stochastic failures and disassembly sequence. Optimal buffer inventory and opportunistic preventive maintenance under random production capacity availability was discussed by Zequeira *et al.* (2008). More recently, Gan *et al.* (2015) focused on the interaction between maintenance, buffer inventory, and spare parts inventory to achieve the minimization of the long-term expected cost rate for a production system. A deteriorating installation model was developed by Kyriakidis and Dimitrakos (2006) in optimal preventive maintenance of a production system with an intermediate buffer. For performing in the perfect way, the preventative maintenance regularly will enhance the condition of the production unit to an acceptable level. Due to this interruption, there may be a breakdown of machine, thus, prevent this situation a buffer inventory is needed such that normal operations will not be interrupted. Regarding this, Salameh and Ghattas (2001) presented an optimal just-in-time buffer inventory for regular preventive maintenance. For multi-stage production systems with failures, the optimal buffer inventories were calculated by Jensen *et al.* (1991). Recently, Massim *et al.* (2010) presented an efficient combined immune decomposition algorithm for optimal buffer allocation in production lines for throughput and profit maximization. The optimal buffer inventory to satisfy the demand during the interruption period due to a maintenance action is discussed by Zequeira *et al.* (2008). Shi and Gershwin (2009) designed an efficient buffer algorithm for production line profit maximization.

1.4 Objective and organization of the thesis

The objective of the thesis is to formulate several types of inventory models based on some realistic physical problems and solve them with the help of different branches of mathematics.

The thesis consists of eight chapters.

Chapter 1: Introduction

In this chapter, some basic concepts and terminologies are discussed.

Chapter 2: A two-warehouse inventory model with increasing demand and time varying deterioration.

Chapter 3: Mitigation of high-tech products with probabilistic deterioration and inflations.

Chapter 4: Optimal replenishment policy with variable deterioration for fixed lifetime products.

Chapter 5: Flexible setup cost and deterioration of products in a supply chain model.

Chapter 6: Optimal buffer inventory and inspection errors in an imperfect production system with preventive maintenance.

Chapter 7: Optimal production run time and inspection errors in an imperfect production system with warranty.

Chapter 8: General conclusions and scope of future research.

Brief descriptions of the problems considered in different chapters of this thesis:

In **Chapter 2**, a two-warehouse production inventory model is formulated. Generally, inventory models are developed for a single warehouse under a constant or linearly time-dependant demand rate and constant deterioration rate. Here, the produced items are deteriorated with respect to time. This model deals with such type of products whose demand increases quadratically with time. The associated cost of this model is obtained by analytical method. A numerical example is given to illustrate the model. The sensitive analysis is allowed to check the deviation of the parameters.

In **Chapter 3**, an inventory model for deteriorating items with ramp-type demand under stock-dependent demand is developed. Different types of probabilistic deteriorations are considered in this model. This model assumes partially backorder rate which, follows a negative exponential with the waiting time. The effects of inflation and time value of money are consider in this model. Here, four different types of continuous probabilities deterioration function are considered. Some numerical examples, graphical representations, special cases, and sensitive analysis are given to illustrate the model.

It is very essential to know the maximum lifetime of products such that deterioration of products can be controlled. **Chapter 4** deals with maximum lifetime of a deteriorating items. Holding cost and deterioration function both are treated as constants in most of the deteriorating inventory model. But in this model they are taken as linear function of time. Here, demand is a ramp-type demand. A simple solution procedure is given and existence and uniqueness of the optimal solutions are obtained analytically. Finally, the sensitivity analysis on the optimal solution with respect to key parameters are studied to illustrate the model and some managerial insights are provided.

In **Chapter 5**, a two-echelon supply chain model (SCM) with variable setup cost and deterioration cost are analyzed. Supply chain management is the coordination of production, inventory, location and transportation among the participants in a supply chain to achieve the best mix of responsiveness and efficiency for the market being served. Manufacturers procures raw material from different sources and processes them into finished goods, and sells those finished goods to different distributors, retailers and finally customers. A multi-echelon inventory system is a system when an item moves through more than one stage before reaching the final customer. A large amount of researches on multi-echelon/two-echelon inventory control has studied during the last decades. Product reliability is of significant importance in today's technological world. People rely more and more upon the sustained functioning of machinery and complex equipments for purposes such as health, economic welfare, safety, to name just a few. Thus in a business arena it is critical to assess the reliability of new product. This SCM model illustrates the effect of reliability on setup cost and deterioration rate. An algebraical procedure is used to minimize the cost for the entire SCM model and obtained a closed-form solution. Minimum cost of the system with integer number of deliveries, optimal lot size, and reliability are obtained by using algebraical procedure. The proposed procedure can be easily used for the computation of the total cost of any SCM model.

Chapter 6 is concerned with an imperfect production system with preventive maintenance to obtain the optimal buffer inventory and inspection policy for sold products with free minimal repair warranty. In most of the inventory model, it is considered so far that the production process is perfectly reliable and the production facility is always in good condition. Furthermore, produced product are of perfect quality. But in reality, due to long-run process i.e. ageing of the operating system or some environmental factors, like the workload of the production system, temperature and humidity of operating place, etc., the production system continuously deteriorates over time and random failures are likely to occur. Thus, the production system is subject to a random movement from an *in-control* to an *out-of-control* state, where some proportion of defective items are produced by the production system during both the *in-control* and *out-of-control* states. An online inspection is continuing after a time variable during the process. Another offline human-based inspection policy is considered at the end of the production cycle to identify the defective items. Defective items found by the inspector are salvaged at some fixed cost before being shipped and the non-inspected items are passed to the customer with free minimal repair warranty. During human-based inspection, some misclassifications may arise from the inspector's side. Thus, two types of inspection errors (Type I and Type II) are considered to make the model more realistic rather than the existing models. A numerical example along with graphical representations are provided to illustrate the proposed model. Sensitivity analysis of the optimal solution with respect to major parameters of the system has been carried out and the implications are discussed.

In **Chapter 7**, an imperfect production system is considered to obtain the optimal production run time and inspection policy. Contrary to the existing literature this model considerers that product inspection performs at any arbitrary time of the production cycle and after the inspection, all defective products produced until the end of the production run are fully reworked. Due to some misclassification during inspection, from the inspector's side two types of inspection errors as Type I and Type II are considered to make the model more realistic rather than existing models. Defective items, found by the inspector, are salvaged at some cost before being shipped. Non-inspected defective items are passed to customers with free minimal repair warranty. The model gives three special cases, where it is found that this model converges over the exiting literature. Some numerical examples along with graphical representations are provided to illustrate the proposed model with comparison with the existing models. Sensitivity analysis of the optimal solution with respect to key parameters of the model has been carried out and the implications are discussed.

In **Chapter** 8, general conclusions are drawn on the basis of the models studied in this dissertation. Further future scopes of works, on the basis of these works, have also been discussed here.

Chapter 2

A two-warehouse inventory model with increasing demand and time varying deterioration

Scientia Iranica E, $19(6), 1969 - 1977, (2012)$.

2.1 Introduction

To decide where to stock goods produced by a production system plays an important role in the inventory management. Every company, in general, has its own warehouse (OW) with a fixed capacity. If the quantity exceeds the capacity of OW, then these quantities should be stored in another rented warehouse (RW) and also the customers are served from RW first then from OW. The basic two-warehouse inventory model was first introduced by Hartley (1976). In this direction, Sarma (1983) developed the inventory model with two levels of storage and optimum release rule. Murdeshwar and Sathe (1985) extended the model of Sarma (1983) by considering a production of non-perishable item. Sarma (1987) generalized the study in his previous model (1983) by including constant demand and deterioration.

An EOQ model with two levels of storage was discussed by Dave (1988). By considering different stage production system, many researchers considered inventory models for deteriorating items. The term, deterioration is defined as damage, spoilage, dryness of an item. Deteriorating items with linear trend in demand was formulated by Chakraborty and Chaudhuri (1997). Many researchers like Giri and Chaudhuri (1997), Harriga (1996), Khanra and Choudhuri (2003), Ghosh and Chaudhuri (2006) and others discussed about deterioration and different types of demand. Furthermore, Cárdenas Barrón (2009a) investigated the economic production quantity (EPQ) model with rework process at a single-stage manufacturing system with planned backorders. Sarkar *et al.* (2011) established an economic manufacturing quantity (EMQ) model for an imperfect production process with time varying demand and inflation. Widyadana *et al.*(2011) addressed an economic order quantity (EOQ) model for deteriorating items and planned backorder level. Sarkar (2012a) investigated an EOQ model with delay-in-payments and time-varying deterioration rate. Taleizadeh *et al.* (2012) studied multi products single machine EPQ model with immediate rework process.

Pakkala and Acharya (1992) formulated a two-warehouse inventory model for deteriorating items with finite replenishment rate and shortage. Goswami and Chaudhuri (1992) found out a two-warehouse inventory model with linear trend in demand and shortage. Benkherouf (1997) proposed two-warehouse model with deterioration and continuously release pattern. Lee and Ma (2000) discussed an optimal inventory policy for deteriorating items with two-warehouse and time-dependent demand. Two-warehouse inventory model with time-varying demand was proposed by Zhou (2003). Two-warehouse inventory models for deteriorating items were studied by the researchers like Yang (2004), Niu and Xie (2008), Hsieh *et al.* (2008) and others. Cárdenas Barrón (2009b) analyzed a model on optimal batch sizing in a multi-stage production system with rework consideration. By considering constant deteriorating items with time dependent demand, a two-warehouse model was developed by Lee and Hsu (2009). Singh and Malik (2010) formulated an optimal ordering policy with linear deterioration, exponential demand and two storage capacity. Liang and Zhou (2011) developed an optimization inventory policy for a deteriorating item with conditionally permissible delay in payment under two-warehouse system.

In this chapter, an effort has been made to develop a two-warehouse inventory model with quadratically increasing demand and time varying deterioration. Since, the deterioration depends on preserving facility available in a warehouse, so the different warehouse may have different deterioration rate. The model is formulated by considering time dependent deterioration rate for different warehouse. It is assumed that the inventory cost (including holding cost and deteriorating cost) in RW is higher than that in OW. The cost of the whole system is derived analytically. The cost function is highly non-linear, thus it can not be solved analytically. Therefore, the total cost of the whole system is minimized by a proposed solution algorithm. A numerical example, graphical illustration and sensitivity analysis are used to illustrate the model. This model has a new managerial insight that helps a manufacturing system/industry to reduce the total system cost at the optimum level.

2.2 Notation and assumptions

The following notation and assumptions are considered to develop this model.

Notation

$f(t)$ $a + bt + ct^2$ $(a, b, c > 0)$ with $0 < f(t) < p$, here a is initial rate of demand, b is the rate with which the demand rate increases. The rate of change in the demand itself increases at a rate c

$\alpha(t)$ $\frac{1}{1+R_1-t}$(deteriorating rate of inventory items in OW, where R_1 is the maximum life time of an item in OW i.e., R_1 is always greater than or equal to t, thus, $\alpha(t) > 0$)

$\beta(t)$ $\frac{1}{1+R_2-t}$(deteriorating rate of inventory items in RW, where R_2 is the maximum life time of an item in RW i.e., R_2 is always greater than or equal to t, thus, $\beta(t) > 0$)

ω storage capacity of OW

P_1 class of production cycle when only OW is used

P_2 class of production cycle when both OW and RW are used

t_{i0} time at the beginning of the *ith* production cycle belonging to P_2

t_{i1} time at which the inventory in OW first reaches ω units

t_{i2} time at the end of production of the *ith* production cycle

t_{i3} time at which all inventory units in RW are exhausted within the *ith* production cycle

I_{i1} inventory level in OW at time t, $t \in [t_{i0}, t_{i1}]$

I_{i2}	inventory level in RW at time t, $t \in [t_{i1}, t_{i2}]$
I_{i3}	inventory level in RW at time t, $t \in [t_{i2}, t_{i3}]$
I_{i4}	inventory level in OW at time t, $t \in [t_{i3}, t_{i+1,0}]$
I_{i5}	inventory level in OW at time t, $t \in [t_{i1}, t_{i3}]$
t_{j0}	time at the beginning of the *jth* production cycle belonging to P_1
t_{j1}	time at the end of production for the *jth* production cycle
I_{j1}	inventory level in OW at time t, $t \in [t_{j0}, t_{j1}]$
I_{j2}	inventory level in OW at time t, $t \in [t_{j1}, t_{j+1,0}]$
D_i	quantity of deteriorated items during the *ith* production cycle
C_{OW}	carrying cost per inventory unit per unit time in OW
C_{RW}	carrying cost per inventory unit per unit time in RW

Assumptions

(1) Demand is increasing quadratically with respect to time as $f(t) = a + bt + ct^2$; $a, b, c > 0$ and the production rate (p) is greater than the demand (d). Hence, there is no shortage.

(2) The OW has limited capacity of ω units and the RW has unlimited capacity.

(3) The inventory cost (including holding cost and deteriorating cost) in RW is higher than that of OW.

(4) Inventory decreases due to demand and deterioration.

(5) Deterioration rate is considered as time-dependent and the deteriorated units can not be repaired or replaced.

(6) The RW is located near the OW such that the transportation cost between them is negligible.

(7) Maximum life time (R_1) of an item in OW is greater than the maximum life time (R_2) of an item in RW i.e. after R_1 time, the items in OW are deteriorated and after R_2 time, the items in RW are deteriorated.

(8) The lead time is considered as negligible.

2.3 Mathematical formulation of the problem

The inventory level in a production system with quadratic demand for deteriorating items is depicted in Fig.2.1 in which Fig.2.1a shows the inventory level during a production cycle when both OW and RW are used and Fig.2.1b shows when only OW is used. Any arbitrary production cycle i belonging to P_2 starts from t_{i0} and ends at $t_{i+1,0}$. Over the period $[t_{i0}, t_{i+1,0}]$, one can identify the points t_{i0}, t_{i1}, t_{i2}, t_{i3} and $t_{i+1,0}$. Production, demand and deterioration starts simultaneously at t_{i0}. During the period $[t_{i0}, t_{i1}]$ produced items accumulate from 0 up to ω units in OW. RW is used after time t_{i1} when production quantity exceeds ω units. The inventory level in RW begins to decrease at t_{i2} and finally reaches at 0 unit at t_{i3} due to demand and deterioration. The inventory level in OW comes to decrease at t_{i1} and falls below ω units up to time t_{i3} only for deterioration and the remaining quantity in OW is fully exhausted at $t_{i+1,0}$.
Any arbitrary production cycle j belonging to P_1, starts from t_{j0} and ends at $t_{j+1,0}$. Here, one can identify a point t_{j1}, the time at the end of production. During $[t_{j0}, t_{j1}]$ the inventory level in OW gradually decreases but it is always less than ω units. During $[t_{j1}, t_{j+1,0}]$, the stocks in OW gradually decreases due to demand and deterioration as well as it is exhausted at $t_{j+1,0}$.
The governing differential equations stating the inventory levels within the *ith* cycle are

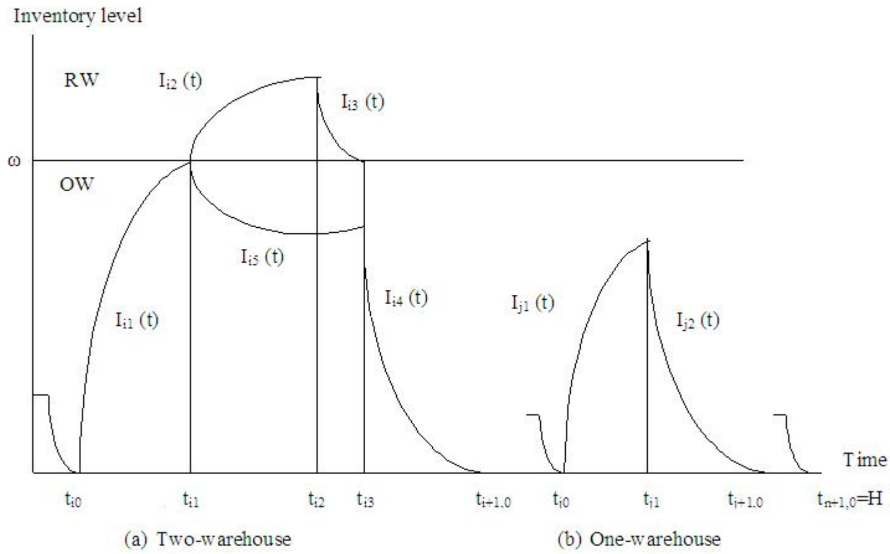


Figure 2.1: Inventory level versus time horizon for two warehouse and one warehouse system.

given as follows:

$$\begin{aligned}
\frac{dI_{i1}(t)}{dt} + \alpha(t)I_{i1}(t) &= p - f(t); \quad t_{i0} \le t \le t_{i1}, && (2.1)\\
\frac{dI_{i2}(t)}{dt} + \beta(t)I_{i2}(t) &= p - f(t); \quad t_{i1} \le t \le t_{i2}, && (2.2)\\
\frac{dI_{i3}(t)}{dt} + \beta(t)I_{i3}(t) &= -f(t); \quad t_{i2} \le t \le t_{i3}, && (2.3)\\
\frac{dI_{i4}(t)}{dt} + \alpha(t)I_{i4}(t) &= -f(t); \quad t_{i3} \le t \le t_{i+1,0}, && (2.4)\\
\frac{dI_{i5}(t)}{dt} + \alpha(t)I_{i5}(t) &= 0; \quad t_{i1} \le t \le t_{i3}, && (2.5)
\end{aligned}$$

Boundary conditions: With the boundary conditions

$I_{i1}(t_{i0}) = 0,\ I_{i2}(t_{i1}) = 0,\ I_{i3}(t_{i3}) = 0,\ I_{i4}(t_{i+1,0}) = 0,\ I_{i5}(t_{i1}) = \omega$

the above differential equations can be solved as follows:

$$
\begin{aligned}
I_{i1}(t) &= (1+R_1-t)\int_{t_{i0}}^{t}\alpha(u)\{p-f(u)\}du \\
&= (1+R_1-t)[(\frac{ct^2}{2}+A_0t)-(\frac{ct_{i0}^2}{2}+A_0t_{i0})-B_0log\frac{1+R_1-t}{1+R_1-t_{i0}}],\ t_{i0}\le t\le t_{i1} \\
I_{i2}(t) &= (1+R_2-t)[(\frac{ct^2}{2}+A_1t)-(\frac{ct_{i1}^2}{2}+A_1t_{i1})-B_1log\frac{1+R_2-t}{1+R_2-t_{i1}}],\ t_{i1}\le t\le t_{i2} \\
I_{i3}(t) &= (1+R_2-t)[-(\frac{ct^2}{2}+A_2t)+(\frac{ct_{i3}^2}{2}+A_2t_{i3})-B_2log\frac{1+R_2-t_{i3}}{1+R_2-t}],\ t_{i2}\le t\le t_{i3} \\
I_{i4}(t) &= (1+R_1-t)[(\frac{ct_{i+1,0}^2}{2}+A_3t_{i+1,0})-(\frac{ct^2}{2}+A_3t)-B_3log\frac{1+R_1-t_{i+1,0}}{1+R_1-t}], \\
& t_{i3}\le t\le t_{i+1,0} \\
I_{i5}(t) &= \frac{1+R_1-t}{1+R_1-t_{i1}}\omega,\quad t_{i1}\le t\le t_{i3}.
\end{aligned}
$$

where,

$A_0=cR_1+c+b, B_0=p-a-(R_1+1)A_0,\ A_1=cR_2+c+b\ \ , B_1=-p-a-(R_2+1)A_1,$

$A_2=cR_2+c+b=A_1, B_2=-a-(R_2+1)A_2,$

$A_3=cR_1+c+b=A_0$ and $B_3=-a-(R_1+1)A_3$

$$
\begin{aligned}
\int_{t_{i0}}^{t_{i1}}\alpha(t)I_{i1}(t)dt &= \int_{t_{i0}}^{t_{i1}}\frac{I_{i1}}{1+R_1-t}(t)dt \\
&= \frac{c(t_{i1}^3-t_{i0}^3)}{6}+\frac{A_0(t_{i1}^2-t_{i0}^2)}{2}-(\frac{ct_{i0}^2}{2}+A_0t_{i0})(t_{i1}-t_{i0}) \\
&+ B_0((1+R_1-t_{i1})(log\frac{1+R_1-t_{i1}}{1+R_1-t_{i0}}-1)+(1+R_1-t_{i0})) \\
\int_{t_{i3}}^{t_{i+1,0}}\alpha(t)I_{i4}(t)dt &= \int_{t_{i3}}^{t_{i+1,0}}\frac{I_{i1}}{1+R_1-t}(t)dt \\
&= -\frac{c(t_{i+1,0}^3-t_{i3}^3)}{6}-\frac{A_3(t_{i+1,0}^2-t_{i3}^2)}{2}+(\frac{ct_{i+1,0}^2}{2}+A_3t_{i+1,0})(t_{i+1,0}-t_{i3}) \\
&+ B_3((1+R_1-t_{i3})(log\frac{1+R_1-t_{i3}}{1+R_1-t_{i+1,0}}-1)+(1+R_1-t_{i+1,0})) \\
\int_{t_{i1}}^{t_{i3}}\alpha(t)I_{i5}(t)dt &= \int_{t_{i1}}^{t_{i3}}\frac{\omega}{1+R_1-t_{i1}}=\frac{\omega(t_{i3}-t_{i1})}{1+R_1-t_{i1}}
\end{aligned}
$$

$$\begin{aligned}
\int_{t_{i1}}^{t_{i2}} \beta(t)I_{i2}(t)dt &= \int_{t_{i1}}^{t_{i2}} \frac{I_{i2}}{1+R_1-t_{i1}} = \frac{c(t_{i2}^3-t_{i1}^3)}{6} \\
&+ \frac{A_1(t_{i2}^2-t_{i1}^2)}{2} - (\frac{ct_{i1}^2}{2}+A_1t_{i1})(t_{i2}-t_{i1}) \\
&+ B_1((1+R_1-t_{i2})(log\frac{1+R_2-t_{i2}}{1+R_2-t_{i1}}-1)+(1+R_1-t_{i1}))
\end{aligned}$$

$$\begin{aligned}
\int_{t_{i2}}^{t_{i3}} \beta(t)I_{i3}(t)dt &= \int_{t_{i2}}^{t_{i3}} \frac{I_{i3}}{1+R_1-t}(t)dt \\
&= -\frac{c(t_{i3}^3-t_{i2}^3)}{6} - \frac{A_2(t_{i3}^2-t_{i2}^2)}{2} + (\frac{ct_{i3}^2}{2}+A_2t_{i3})(t_{i3}-t_{i2}) \\
&+ B_2((1+R_2-t_{i2})(log\frac{1+R_2-t_{i2}}{1+R_1-t_{i3}}-1)+(1+R_2-t_{i3})) \\
\int_{t_{i1}}^{t_{i3}} \alpha(t)I_{i5}(t)dt &= \int_{t_{i1}}^{t_{i3}} \frac{\omega}{1+R_1-t_{i1}} = \frac{\omega(t_{i3}-t_{i1})}{1+R_1-t_{i1}}
\end{aligned}$$

The different inventory levels are obtained as follows:

The inventory level in RW can be derived as

$$I_{RW,i} = \int_{t_{i1}}^{t_{i2}} I_{i2}(t)dt + \int_{t_{i2}}^{t_{i3}} I_{i3}(t)dt$$

where

$$\begin{aligned}
\int_{t_{i1}}^{t_{i2}} I_{i2}(t)dt &= (1+R_2)(\frac{ct_{i2}^3}{6}+\frac{A_1t_{i2}^2}{2}-\frac{ct_{i1}^3}{6}-\frac{A_1t_{i1}^2}{2}) - (\frac{ct_{i2}^4}{8}+\frac{A_1t_{i2}^3}{3} \\
&- ct_{i1}^4 - \frac{A_1t_{i1}^3}{3}) - (\frac{ct_{i1}^2}{2}+A_1t_{i1})((1+R_2)(t_{i2}-t_{i1}) - \frac{t_{i2}^2}{2}+\frac{t_{i1}^2}{2}) \\
&+ \frac{B_1}{4}((1+R_2-t_{i1})^2(2log\frac{1+R_2-t_{i2}}{1+R_2-t_{i1}}-1)+(1+R_2-t_{i2})^2)
\end{aligned}$$

and

$$\int_{t_{i2}}^{t_{i3}} I_{i3}(t)dt = -(1+R_2)(\frac{ct_{i3}^3}{6}+\frac{A_2t_{i3}^2}{2}-\frac{ct_{i2}^3}{6}-\frac{A_2t_{i2}^2}{2})-(\frac{ct_{i3}^4}{8}+\frac{A_2t_{i3}^3}{3}$$
$$-ct_{i2}^4-\frac{A_2t_{i2}^3}{3})-(\frac{ct_{i3}^2}{2}+A_2t_{i3})((1+R_2)(t_{i3}-t_{i2})-\frac{t_{i3}^2}{2}+\frac{t_{i2}^2}{2})$$
$$+\frac{B_2}{4}((1+R_2-t_{i2})^2(2log\frac{1+R_2-t_{i2}}{1+R_2-t_{i3}}-1)+(1+R_2-t_{i3})^2)$$

In addition, the following relation exists:

$$I_{i2}(t_{i2}) \quad = \quad I_{i3}(t_{i2})$$

which gives

$$\begin{aligned} ct_{i2}^2+2A_1t_{i2} &= (\frac{ct_{i1}^2}{2}+A_1t_{i1})+(\frac{ct_{i3}^2}{2}+A_2t_{i3})+B_1log\frac{1+R_1-t_{i2}}{1+R_2-t_{i1}} \\ &- B_2log\frac{1+R_2-t_{i3}}{1+R_2-t_{i2}} \end{aligned} \tag{2.6}$$

The inventory level in OW can be derived as

$$\begin{aligned} I_{OW,i} &= \int_{t_{i0}}^{t_{i1}} I_{i1}(t)dt+\int_{t_{i3}}^{t_{i+1,0}} I_{i4}(t)dt+\int_{t_{i1}}^{t_{i3}} I_{i5}(t)dt \\ \text{where} \quad \int_{t_{i0}}^{t_{i1}} I_{i1}(t)dt &= (1+R_1)(\frac{ct_{i1}^3}{6}+\frac{A_0t_{i1}^2}{2}-\frac{ct_{i0}^2}{6}-\frac{A_0t_{i0}^2}{2}) \\ &- (\frac{ct_{i1}^4}{8}+\frac{A_0t_{i1}^3}{3}-\frac{ct_{i0}^4}{8}-\frac{A_0t_{i0}^3}{3})-(\frac{ct_{i0}^2}{2}+A_0t_{i0})((1+R_1)(t_{i1}-t_{i0}) \\ &- \frac{t_{i1}^2}{2}+\frac{t_{i0}^2}{2})+\frac{B_0}{4}[(1+R_1-t_{i1})^2(2log\frac{1+R_1-t_{i1}}{1+R_1-t_{i0}}-1) \\ &+ (1+R_1-t_{i0})^2], \end{aligned}$$

$$
\begin{aligned}
\int_{t_{i3}}^{t_{i+1,0}} I_{i4}(t)dt &= -(1+R_1)(\frac{ct_{i+1,0}^3}{6}+\frac{A_3t_{i+1,0}^2}{2}-\frac{ct_{i3}^2}{6}-\frac{A_3t_{i3}^2}{2}) \\
&+ (\frac{ct_{i+1,0}^4}{8}+\frac{A_3t_{i+1,0}^3}{3}-\frac{ct_{i3}^4}{8}-\frac{A_3t_{i3}^3}{3})-(\frac{ct_{i+1,0}^2}{2}+A_3t_{i+1,0}) \\
& ((1+R_1)(t_{i+1,0}-t_{i3})-(\frac{t_{i+1,0}^2}{2}+\frac{t_{i3}^2}{2})+\frac{B_3}{4}[(1+R_1-t_{i3})^2 \\
& (2log\frac{1+R_1-t_{i3}}{1+R_1-t_{i+1,0}}-1)+(1+R_1-t_{i+1,0})^2]
\end{aligned}
$$

and $$\int_{t_{i1}}^{t_{i3}} I_{i5}(t)dt = \frac{\omega}{(1+R_1-t_{i1})}[(1+R_1)(t_{i3}-t_{i1})-\frac{(t_{i3}^2-t_{i1}^2)}{2}].$$

Accordingly, the following relations exist:

$$I_{i4}(t_{i3}) = I_{i5}(t_{i3})$$

which indicates

$$(1+R_1-t_{i1})[(\frac{ct_{i+1,0}^2}{2} + A_3t_{i+1,0})-(\frac{ct_{i3}^2}{2}+A_3t_{i3})-B_3log\frac{1+R_1-t_{i+1,0}}{1+R_1-t_{i3}}] = \omega \quad (2.7)$$

Now,

$$I_{i1}(t_{i1}) = \omega$$

gives

$$(1+R_1-t_{i1})[(\frac{ct_{i1}^2}{2}+A_0t_{i1}) - (\frac{ct_{i0}^2}{2}+A_0t_{i0})-B_0log\frac{1+R_1-t_{i1}}{1+R_1-t_{i0}}] = \omega. \quad (2.8)$$

The quantity of deteriorating items during the production cycle i is

$$
\begin{aligned}
D_i &= \int_{t_{i0}}^{t_{i1}} \alpha(t)I_{i1}(t)dt + \int_{t_{i3}}^{t_{i+1,0}} \alpha(t)I_{i4}(t)dt + \int_{t_{i1}}^{t_{i3}} \alpha(t)I_{i5}(t)dt + \int_{t_{i1}}^{t_{i2}} \beta(t)I_{i2}(t)dt \\
&+ \int_{t_{i2}}^{t_{i3}} \beta(t)I_{i3}(t)dt
\end{aligned}
$$

$$\begin{aligned}
&= \frac{c(t_{i1}^3-t_{i0}^3)}{6}+\frac{A_0(t_{i1}^2-t_{i0}^2)}{2}-(\frac{ct_{i0}^2}{2}+A_0t_{i0})(t_{i1}-t_{i0})\\
&+ B_0((1+R_1-t_{i1})(log\frac{1+R_1-t_{i1}}{1+R_1-t_{i0}}-1)+(1+R_1-t_{i0}))-\frac{c(t_{i+1,0}^3-t_{i3}^3)}{6}\\
&- \frac{A_3(t_{i+1,0}^2-t_{i3}^2)}{2}+(\frac{ct_{i+1,0}^2}{2}+A_3t_{i+1,0})(t_{i+1,0}-t_{i3})-B_3((1+R_1-t_{i+1,0})\\
&+ (1+R_1-t_{i3})(log\frac{1+R_1-t_{i3}}{1+R_1-t_{i+1,0}}-1))+\frac{\omega(t_{i3}-t_{i1})}{1+R_1-t_{i1}}+\frac{c(t_{i2}^3-t_{i1}^3)}{6}\\
&+ \frac{A_1(t_{i2}^2-t_{i1}^2)}{2}-(\frac{ct_{i1}^2}{2}+A_1t_{i1})(t_{i2}-t_{i1})+B_1((1+R_2-t_{i2})(log\frac{1+R_2-t_{i2}}{1+R_2-t_{i1}}-1)\\
&+ (1+R_2-t_{i1}))-\frac{c(t_{i3}^3-t_{i2}^3)}{6}-\frac{A_2(t_{i3}^2-t_{i2}^2)}{2}+(\frac{ct_{i3}^2}{2}+A_2t_{i3})(t_{i3}-t_{i2})\\
&+ B_2((1+R_2-t_{i2})(log\frac{1+R_2-t_{i2}}{1+R_1-t_{i3}}-1)+(1+R_2-t_{i3})).
\end{aligned}$$

As ω is known, a unique solution of t_{i1} can be solved from equation (2.8) for a given t_{i0}. By substituting this value in relations (2.6) and (2.7) it is found the relations among t_{i2}, t_{i3} and t_{i3}, $t_{i+1,0}$, respectively.

The previous results can be simplified to express the inventory level in the one-warehouse model. For any arbitrary production cycle j belonging to P_1, I_{j1} is similar to I_{i1} in behavior, the same is I_{j2} to I_{j4}, hence, one can write:

$$\frac{dI_{j1}(t)}{dt}+\alpha(t)I_{j1}(t)=p-f(t);\quad t_{j0}\le t\le t_{j1}, \tag{2.9}$$

$$\frac{dI_{j2}(t)}{dt}+\beta(t)I_{j2}(t)=p-f(t);\quad t_{j1}\le t\le t_{j+1,0}, \tag{2.10}$$

The solutions are as follows:

$$\begin{aligned}
I_{j1}(t) &= (1+R_1-t)\int_{t_{j0}}^{t}\frac{p-(a+bu+cu^2)}{1+R_1-u}du\\
&= (1+R_1-t)[(\frac{ct^2}{2}+A_0t)-(\frac{ct_{j0}^2}{2}+A_0t_{j0})-B_0log\frac{1+R_1-t}{1+R_1-t_{j0}}],\ t_{j0}\le t\le t_{j1}
\end{aligned}$$

and

$$\begin{aligned} I_{j2}(t) &= (1+R_1-t)[-(\frac{ct^2}{2}+A_3t)+(\frac{ct_{j+1,0}^2}{2}+A_3t_{j+1,0})-B_3log\frac{1+R_2-t_{j+1,0}}{1+R_2-t}], \\ & t_{j1}\le t\le t_{j+1,0} \end{aligned}$$

along with the following relation exists

$$I_{j1}(t_{j1}) = I_{j2}(t_{j1})$$

which implies

$$\begin{aligned} & [(\frac{ct_{j1}^2}{2}+A_0t_{j1})-(\frac{ct_{j0}^2}{2}+A_0t_{j0})-B_0log\frac{1+R_1-t_{j1}}{1+R_1-t_{j0}}] = -(\frac{ct_{j1}^2}{2}+A_3t_{j1}) \\ + & (\frac{ct_{j+1,0}^2}{2}+A_3t_{j+1,0})-B_3log\frac{1+R_2-t_{j+1,0}}{1+R_2-t_{j1}}. \end{aligned} \tag{2.11}$$

The inventory levels over the cycle j can be expressed as

$$I_{OW,j} = \int_{t_{j0}}^{t_{j1}} I_{j1}(t)dt + \int_{t_{j1}}^{t_{j+1,0}} I_{j2}(t)dt$$

where

$$\begin{aligned} \int_{t_{j0}}^{t_{j1}} I_{j1}(t)dt &= (1+R_1)(\frac{ct_{j1}^3}{6}+\frac{A_0t_{j1}^2}{2}-\frac{ct_{j0}^2}{6}-\frac{A_0t_{j0}^2}{2}) \\ &- (\frac{ct_{j1}^4}{8}+\frac{A_0t_{j1}^3}{3}-\frac{ct_{j0}^4}{8}-\frac{A_0t_{j0}^3}{3})-(\frac{ct_{j0}^2}{2}+A_0t_{j0})((1+R_1)(t_{j1}-t_{j0}) \\ &- \frac{t_{j1}^2}{2}+\frac{t_{j0}^2}{2})+\frac{B_0}{4}[(1+R_1-t_{j1})^2(2log\frac{1+R_1-t_{j1}}{1+R_1-t_{j0}}-1) \\ &+ (1+R_1-t_{j0})^2] \end{aligned}$$

and

$$\begin{aligned}
\int_{t_{j1}}^{t_{j+1,0}} I_{j2}(t)dt &= -(1+R_1)(\frac{ct_{j+1,0}^3}{6}+\frac{A_3t_{j+1,0}^2}{2}-\frac{ct_{j1}^2}{6}-\frac{A_3t_{j1}^2}{2}) \\
&+ (\frac{ct_{j+1,0}^4}{8}+\frac{A_3t_{j+1,0}^3}{3}-\frac{ct_{j1}^4}{8}-\frac{A_3t_{j1}^3}{3})-(\frac{ct_{j+1,0}^2}{2}+A_3t_{j+1,0}) \\
& ((1+R_1)(t_{j+1,0}-t_{j1})-(\frac{t_{j+1,0}^2}{2}+\frac{t_{j1}^2}{2})+\frac{B_3}{4}[(1+R_1-t_{j1})^2 \\
& (2log\frac{1+R_1-t_{j1}}{1+R_1-t_{j+1,0}}-1)+(1+R_1-t_{j+1,0})^2]
\end{aligned}$$

The total deteriorated items during the production cycle j are

$$\begin{aligned}
D_j &= \int_{t_{j0}}^{t_{j1}} \alpha(t)I_{j1}(t)dt+\int_{t_{j1}}^{t_{j+1,0}} \alpha(t)I_{j2}(t)dt \\
&= \frac{c(t_{j1}^3-t_{j0}^3)}{6}+\frac{A_0(t_{j1}^2-t_{j0}^2)}{2}-(\frac{ct_{j0}^2}{2}+A_0t_{j0})(t_{j1}-t_{j0}) \\
&+ B_0((1+R_1-t_{j1})(log\frac{1+R_1-t_{j1}}{1+R_1-t_{j0}}-1)+(1+R_1-t_{j0}))-\frac{c(t_{j+1,0}^3-t_{j1}^3)}{6} \\
&- \frac{A_3(t_{j+1,0}^2-t_{j1}^2)}{2}+(\frac{ct_{j+1,0}^2}{2}+A_3t_{j+1,0})(t_{j+1,0}-t_{j1})+B_3((1+R_1-t_{j+1,0}) \\
&+ (1+R_1-t_{j1})(log\frac{1+R_1-t_{j1}}{1+R_1-t_{j+1,0}}-1)).
\end{aligned}$$

The total system cost within the planning horizon H consists of setup cost, carrying cost and deteriorating cost can be expressed as

$$TC = nS_0 + C_{RW}\sum_i I_{RW,i} + C_{OW}\sum_i I_{OW,i} + C_{OW}\sum_j I_{OW,j} + C_d\sum_i D_i + C_d\sum_j D_j$$

Taking logarithmic expansion up to second order the relation (2.8) becomes a cubic equation

$$
\begin{aligned}
P_0 t_{i1}^3 &+ 3P_1 t_{i1}^2 + 3P_2 t_{i1} + P_4 = 0 \\
\text{where } P_0 &= \frac{c+B_0}{2} \\
P_1 &= \frac{1}{3}[A_0 - \frac{(1+R_1)(c+B_0)}{2} - B_0(R_1-1)] \\
P_2 &= \frac{1}{3}[(1+R_1)B_0(R_1-1) + B_0(R_1-1)t_{i0} - (1+R_1)A_0 - \frac{ct_{i0}^2}{2} - A_0 t_{i0} - \frac{B_0 t_{i0}^2}{2}] \\
P_3 &= \omega + \frac{(1+R_1)(B_0+c)t_{i0}^2}{2} + (1+R_1)A_0 t_{i0} - B_0(1+R_1)(R_1-1)t_{i0}
\end{aligned}
$$

Solving by Cardan's solution procedure, t_{i1} can be expressed as

$$t_{i1} = g(t_{i0}) \tag{2.12}$$

Similarly, from equations (2.6) and (2.7) respectively, one can obtain

$$t_{i2} = \psi(t_{i0}, t_{i+1,0}) \tag{2.13}$$

and

$$t_{i3} = \phi(t_{i0}, t_{i+1,0}). \tag{2.14}$$

The problem is to find the optimal value of n and t_{i0} $(i = 1, 2, ..., n-1)$ such that the total system cost TC in equation (2.12) is minimum. Assuming n as any integer number, the total system cost (TC) is differentiated with respect to t_{i0} $(i = 1, 2, ..., n-1)$, and the following necessary condition is obtained.

$\frac{dTC}{dt_{i0}} = 0$ i.e.,

$$C_{RW}\sum_i \frac{\partial I_{RW,i}}{\partial t_{i0}} + C_{OW}\sum_i \frac{\partial I_{OW,i}}{\partial t_{i0}} + C_{OW}\sum_j \frac{\partial I_{OW,j}}{\partial t_{i0}} + C_d\sum_i \frac{\partial D_i}{\partial t_{i0}} + C_d\sum_j \frac{\partial D_j}{\partial t_{i0}} = 0$$

which implies

$$\begin{aligned}
C_{RW}\sum_i((1+R_2)a^1_{i21} - b^1_{i21} &- c^1_{i21} + B_1 d^2_{i21} - (1+R_2)a^2_{i32} - b^2_{i32} - c^2_{i32} + B_2 d^2_{i23}) \\
+C_{OW}\sum_i(((1+R_1)a^0_{i10} - b^0_{i10} &- c^0_{i10} + B_0 d^1_{i10} - (1+R_1)a^3_{ii+13} + b^3_{ii+13} - c^3_{ii+13} + B_3 d^1_{i3i+1}) \\
-\frac{\omega t'_{i1}[(1+R_1)(t_{i3}-t_{i1}) - \frac{t^2_{i3}-t^2_{i1}}{2}]}{(1+R_1-t_{i1})^2}) &+ C_d\sum_i(e^0_{i10} + B_0 f^0_{i10} - e^3_{ii+13} - B_3 f^1_{i3i+1} + \frac{\omega(t'_{i3}-t'_{i1})}{1+R_1-t_{i1}} \\
+\frac{\omega(t_{i3}-t_{i1})t'_{i1}}{(1+R_1-t_{i1})^2} + e^1_{i21} &+ B_1 f^2_{i21} - e^2_{i32} + B_2 f^2_{i23}) = 0
\end{aligned}$$

where

$$\begin{aligned}
a^l_{ijk} &= \frac{ct^2_{ij}t'_{ij}}{2} + A_l t_{ij}t'_{ij} - \frac{ct^2_{ik}t'_{ik}}{2} - A_l t_{ik}t'_{ik} \\
b^l_{ijk} &= \frac{ct^3_{ij}t'_{ij}}{2} + A_l t^2_{ij}t'_{ij} - 4ct^3_{ik}t'_{ik} - A_l t^3_{ik}t'_{ik} \\
c^l_{ijk} &= (ct_{ij}t'_{ij} + A_1 t'_{ij})((1+R_l)(t_{ij}-t_{ik}) - \frac{t^2_{ij}}{2} + \frac{t^2_{ik}}{2}) \\
&- (\frac{ct^2_{ij}}{2} + A_1 t_{ij})((1+R_l)(t'_{ij}-t'_{ik}) - t_{ij}t'_{ij} + t_{ik}t'_{ik}) \\
d^l_{ijk} &= (\frac{1}{4})(-2(1+R_l-t_{ij})t'_{ij}(2log\frac{1+R_l-t_{ij}}{1+R_l-t_{ik}} - 1) \\
&+ (1+R_l-t_{ij})^2(2\frac{-t'_{ij}(1+R_l-t_{ij}) + t'_{ik}(1+R_l-t_{ik})}{(1+R_l-t_{ij})(1+R_l-t_{ik})})) - 2(1+R_l-t_{ik})t'_{ik} \\
e^l_{ijk} &= \frac{c}{2}(t^2_{ij}t'_{ij} - t^2_{ik}) = A_l(t_{ij}t'_{ij} - t_{ik}t'_{ik}) - (ct_{ij}+A_l)(t_{ij}-t_{ik}) - (\frac{ct^2_{ij}}{2} + A_l t_{ij})(t'_{ij}-1) \\
f^l_{ijk} &= (1+R_l-t'_{ij})(log\frac{1+R_l-t_{ij}}{1+R_l-t_{ik}} - 1) + \frac{-t'_{ij}(1+R_l-t_{ik}) - (1+R_l-t_{ij})}{(1+R_l-t_{ik})} - 1
\end{aligned}$$

(here $t'_{ij} = \frac{\partial t_{ij}}{\partial t_{i0}}$).

The values of t_{i1}, t_{i2}, t_{i3} can be obtained from the above relation by giving the values of t_{i0} and $t_{i+1,0}$. Once the values of t_{10}, t_{20}, t_{30}...... $t_{n-1,0}$ are determined, the value of t_{n0} is obtained. Therefore, the decision is to obtain the optimal values of the decision variables so that TC is minimum. Since, n is an integer and the optimization of TC is a desecrate optimization as well as the expression of TC is highly non-linear, an algorithm is proposed to find the optimal solution.

2.3.1 Solution algorithm

(**Step 1**.) input H, p, ω, R_1, R_2, S_0, C_d, C_{RW}, C_{OW} , a, b, c, T (time deviation from H)

(**Step 2**.) Set $n = 1$, t_{i0} and $t_{i+1,0}$

(**Step 3**.) From relations (2.13), (2.14) and (2.15) compute t_{i1}, t_{i2}, t_{i3}, compute $TC(n)$ from relation (2.12)

(**Step 4**.) Set $n = n + 1$, $t_{i0} = t_{i0} + h$, where h is a small increment iteratively solve for t_{i0} for $i = 2, 3, 4, ..., n + 1$

(**Step 5**.) if $|t_{n+1,0-H}| \leq T$, goto **Step 3**. else goto **Step 6**

(**Step 6**.) if $TC(n-1) \leq TC(n)$, set $n^* = n - 1$: stop the calculation.

2.4 Numerical example

Following example is considered to illustrate the above results, the $H = 8$ weeks, $p =$ \$2000 per week, $S_0 =$ \$3000 per setup, $C_d =$ \$50 per unit, $C_{ow} =$ \$5 per unit per week, $C_{Rw} =$ \$8 per unit per week, $\omega = 5000$ units per week, $R_1 = 10$ weeks, $R_2 = 7$ weeks and $f(t) = 200 + 100t + 10t^2$ units. Numerical results are shown in Table 2.1. It is

Table 2.1: The optimal value of TC

n	t_{i0}	t_{i1}	t_{i2}	t_{i3}	$t_{i+1,0}$	TC
1	0.0000	1.2110	1.9220	2.8011	3.0000	96,430
2	3.0000	3.7072	4.5156	5.01102	5.5000	78,097
3	5.5000	5.7101	6.0233	6.2243	6.5000	39,311
4*	6.5000	6.7131	6.8904	7.0150	7.5000	**25,439**
5	7.5000	7.5517	7.7234	7.7544	8.0000	26,508
6	8.0000	8.1119	8.3451	8.4162	8.5000	39,903
7	8.5000	8.7253	8.8937	9.1228	9.5000	50,429

observed that the system cost (TC) is minimum when fourth inventory cycles ($n^* = 4$) are involved.

2.5 Sensitivity analysis

Sensitivity analysis of the parameters present in this model are studied. The optimal values of the total system cost (TC) changes significantly with changes ($-50\%, -25\%,$ $+25\%, +50\%$) of different parameters value in the Table 2.2.

On the basis of the sensitivity analysis of the parameters, the following features are observed.

1. If production rate (p) increases then the number of production increases which implies the increasing value of the total cost (TC).

2. If the setup cost per setup (S_0), cost of the deteriorated items (C_d), carrying cost ($C_{OW},\ C_{RW}$) increase by a small amount then the value of total system increases.

3. If the maximum lifetime ($R_1,\ R_2$) of the produced items increases then it is quite natural that the cost for this purpose decreases. Hence, the total cost of the system decreases gradually.

4. Increasing values of a, b and c imply more demands in the market which ensure more production which indicate the reduction of the total costs of the whole system.

(In Table 2.2 “-” is given since $R_1 > R_2$ therefore if R_1 is reduced -50% & R_2 is increased

Table 2.2: Sensitivity analysis of the numerical example

Parameters	Percentage change(in %)	TC^*	$Deviation(\%)$
p	-50%	24214	-4.82
	-25%	24827	-2.41
	$+50\%$	26052	$+2.43$
	$+25\%$	26664	$+4.80$
S_0	-50%	20439	-19.65
	-25%	22939	-9.83
	$+25\%$	27939	$+9.84$
	$+50\%$	30439	$+19.64$
C_d	-50%	17873	-29.74
	-25%	21656	-14.87
	$+25\%$	29223	$+14.89$
	$+50\%$	33006	$+29.75$
C_{RW}	-50%	25433	-0.02
	-25%	25436	-0.01
	$+25\%$	25443	$+0.02$
	$+50\%$	25446	$+0.03$
C_{OW}	-50%	25293	-0.58
	-25%	25366	-0.29
	$+25\%$	25513	$+0.30$
	$+50\%$	25586	$+0.58$
R_1	-50%	-	-
	-25%	45294	$+78.05$
	$+25\%$	18729	-26.38
	$+50\%$	17263	-32.14
R_2	-50%	25340	-0.39
	-25%	24987	-1.78
	$+25\%$	25432	-0.03
	$+50\%$	-	-
a	-50%	25840	$+1.58$
	-25%	25640	$+0.79$
	$+25\%$	25239	-0.79
	$+50\%$	25039	-1.57
b	-50%	26851	$+5.55$
	-25%	26145	$+2.78$
	$+25\%$	24734	-2.78
	$+50\%$	24028	-5.57
c	-50%	29472	$+15.85$
	-25%	27456	$+7.93$
	$+25\%$	23423	-7.92
	$+50\%$	21406	-15.86

+50% then the assumption ($R_1 > R_2$) does not satisfy.)

2.6 Conclusions

Traditional inventory models are developed mainly for a single warehouse under a constant or linearly time dependant demand rate and constant deterioration rate. Sometimes rented warehouse are necessary to store excess items due to capacity limitation of owned warehouse. Additional warehouse are necessary to hold excess stocks. An attempt have been made to develop and to solve a two-warehouse inventory model with the assumption of quadratic demand which is useful for the items whose demand increases very rapidly such as newly launched products in the market. Present study illustrates a two-warehouse model based on time-dependent deterioration rates. The associated cost of this model was obtained by analytical method. As, n was an integer and the optimization of TC was a desecrate optimization as well as the expression of TC was highly non-linear, thus, it was minimized by using an solution algorithm. A numerical example was given to illustrate the model. The sensitivity analysis was given to check the deviation of key parameters in the range -50% to $+50\%$. This model is applicable in an industry, where the production rate is fixed throughout the production-run, demand increases rapidly, and the item has a variable deterioration rate. A possible future research direction is the study of a multi-item model for a variable production rate. Another good direction of research is to consider the inflation and time value of money.

Chapter 3

Mitigation of high-tech products with probabilistic deterioration and inflations

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3.1 Introduction

In reality, deterioration of items during storage period is a realistic phenomenon in many inventory sectors. Controlling and regulating the deteriorating items are very difficult in practice. In storage system, fruits, vegetables, foodstuffs, etc., deteriorate during their normal storage period. The deteriorating items cannot be used for its original purpose. The loss of inventory due to deterioration cannot be ignored. Thus, it is very essential to control the deterioration of items. A model with exponentially decaying inventory was initially proposed by Covert and Philip (1973). Dye *et al.* (2006) considered a deteriorating inventory model with time-varying demand and shortage-dependent partial backlogging. Chung and Wee (2008) discussed a deteriorating inventory model for pricing policy with imperfect production, inspection planning, warranty period, and stock-level-dependant demand. Sana (2010b) established an inventory model for both ameliorating and deteriorating items with capacity constraint for storage. Wee *et al.* (2011) proposed an optimal replenishment policy for deteriorating green products. Sett *et al.* (2012) developed a two-warehouse inventory model with increasing demand and time varying deterioration. They considered the maximum lifetime of products. Always all deterioration function are not deterministic type, it may follow probabilistic nature some times. Most recently, Sarkar (2013) developed a production-inventory model for three different types of continuously distributed deterioration functions. Sarkar and Sarkar (2013) explained a control-inventory problem with probabilistic deterioration. They solved the model with the help of Euler-Lagrange method. Sarkar and Sarkar (2013a) developed an inventory model with time varying demand and deterioration. Sarkar *et al.* (2013) considered a deteriorating inventory model with variable demand. Cárdenas-Barrón *et al.* (2013b) developed an improved solution procedure to solve a production model with rework and multiple shipments. Sarkar and Saren (2014) established a partial trade-credit model for retailer with exponentially deterioration. Sarkar *et al.* (2014) considered a deteriorating inventory model with trade-credit policy for fixed lifetime products.

In classical inventory models, it is often assumed that shortages are either completely backlogged or completely lost. But in the real life, when shortages occur, it is observed that some customers may prefer their demands to be backordered, and some may refuse the backorder case. In this direction, Deb and Chaudhuri (1987) were the first to incorporate shortages into inventory model, that model was an extension of Mc.Donald's (1979) model with shortages. Chang and Dye (1999) developed an inventory model in which the backlogging rate is the reciprocal of a linear function of the waiting time. Without using differential calculus, how inventory model with shortages can be solved, Cárdenas-Barrón (2011a) explained it with using basic algebraic procedure. Teng *et al.* (2002) extended the model in which the backlogging rate is any decreasing function of the waiting time up to the next replenishment. Sometimes managers prefers to use planned backorders to reduce the total system cost. In this direction, Cárdenas-Barrón (2009a) presented an inventory model with rework process at a single-stage manufacturing system with planned backorders. Sarkar *et al.* (2010b) described a production policy in order to find out an optimal safety stock, production lotsize, and reliability parameters. Sarkar *et al.* (2014) developed an integrated inventory model with variable lead time, defective units, and delay in payments. Sarkar and Majumder (2013) developed an integrated vendor-buyer supply chain model with vendor's setup cost reduction. Sarkar and Sarkar (2013) presented an improved inventory model with partial backlogging, time varying deterioration, and stock-dependent demand. Most recently, Sarkar *et al.* (2014) extended the inventory model with random defective rate, rework process, and variable backorders. Sarkar *et al.* (2015) developed a continuous review inventory model with backorder price discount under controllable lead time.

Classical inventory model considers constant demand rate. However it is observed that the demand rate of electronic goods (e.g., hard disk, RAM, processer, mobile, etc.), new brand of consumer goods comes to the market, seasonal products (different fruits like mango, orange, etc.,) increases linearly at the beginning up to a certain moment as time increases and then stabilizes to a constant rate until the end of the inventory cycle. To represent such type of demand pattern the term ramp-type is used. Mandal and Pal (1998) was the first author to introduce ramp-type demand in inventory model. Wu (2001) developed an EOQ model with ramp-type demand, Weibull distributed deterioration, and partial backlogging. Giri *et al.* (2003) extended the model of Wu (2001) with more generalized Weibull deterioration distribution. A model with partial backlogging was considered by Skouri *et al.* (2009). Cárdenas-Barrón *et al.* (2013a) developed two easy and improved algorithms to determine jointly both the optimal replenishment lot size and the optimal number of shipments. Sarkar *et al.* (2015) proposed a continuous review manufacturing inventory model with setup cost reduction, quality improvement, and a service level constraint.

The effects of inflation and time-value of money cannot be ignored for the present study. Several researchers have examined the inflationary effect on the inventory policy. Buzacott (1975) was the first researcher to assume inflation in inventory model. Datta and Pal (1991) presented the effect of inflation and time-value of money on an inventory model with linear time dependent demand rate and shortages. Jaggi *et al.* (2006) considered a deteriorating inventory model under inflationary conditions using a discounted cash flow (DCF) approach over a finite planning horizon. Sarkar and Moon (2011) extended an economic production quantity (EPQ) model with inflation in an imperfect production system. Sarkar *et al.* (2014) developed an inventory model for imperfect production with inflation and time value of money.

This model is developed for deteriorating items with ramp-type demand under stock-dependent demand. In addition, different types of probabilistic deteriorations are considered in this model. Shortages are allowed which is backlogged. The effects of inflation and time value of money are incorporated into the model. This chapter develops an optimal replenishment policy which maximized the total profit per unit time. The necessary and sufficient conditions of the existence and uniqueness of the optimal solutions are also provided. Sensitivity analysis of the optimal solution with respect to major parameters and their discursion is carried.

3.2 Assumptions

To derive the model, following assumptions are made:

1. The model is considered for a single type of items.

2. Deterioration rate ψ is probabilistic and there is no replacement or repair of deteriorated units during the period under consideration.

3. The demand rate $d(t)$ is assumed to be a ramp-type function of time, i.e.,

$$d(t) = d_0[t - (t - \mu)H(t - \mu)], \qquad d_0 > 0$$

where $H(t - \mu)$ is the Heaviside's function as follows:

$$H(t - \mu) = \begin{cases} 1 & \text{if} \quad t \geq \mu \\ 0 & \text{if} \quad t < \mu \end{cases}$$

.

4. $S(t)$ is the selling rate at time t, and it is influenced by the demand rate and the on-hand inventory according to relation

$$S(t) = \begin{cases} d(t) + \gamma I(t), & I(t) > 0 \\ d(t), & I(t) \leq 0 \end{cases}$$

where γ is positive constant and $I(t)$ is the on-hand inventory level at time t.

5. Shortages are allowed and partially backlogged at a rate $\delta(t)$, which is a decreasing function of time with $0 \leq \delta(t) \leq 1, \delta(0) = 1$, and $\lim_{t\to\infty} \delta(t) = 0$. The cases with $\delta(t) = 1$ (or) 0 for all t correspond to complete backlogging (or complete lost sales) models.

6. The effects of inflation and time-value of money are considered.

7. Lead time is assumed as negligible.

3.3 Model formulation

The model considers an inventory model for deteriorating items with ramp-type demand and stock-dependent selling rate. Deterioration rate of the items are probabilistic in nature. The replenishment at the beginning of the cycle brings the inventory level up to I_{max}. The inventory level decreases during the time interval $[0, t_1]$ due to demand and deterioration of items, and falls to zero at $t = t_1$. Thereafter shortages occur during the period (t_1, T), which are partially backlogged. The inventory level, $I(t), 0 \leq t \leq T$ satisfies the following differential equations

$$\frac{dI(t)}{dt} + \psi I(t) = -S(t), \quad 0 \leq t \leq t_1, \qquad I(0) = I_{max} \tag{3.1}$$

$$\frac{dI(t)}{dt} = -S(t)\delta(T - t), \quad t_1 \leq t \leq T, \quad I(t_1) = 0 \tag{3.2}$$

The solutions of these differential equations depend on the selling rate. There are two cases considering in this model: (a) $t_1 \leq \mu$, (b) $t_1 \geq \mu$. The fluctuation of the inventory level for the two cases is depicted in Figure 3.1 and Figure 3.2, respectively.

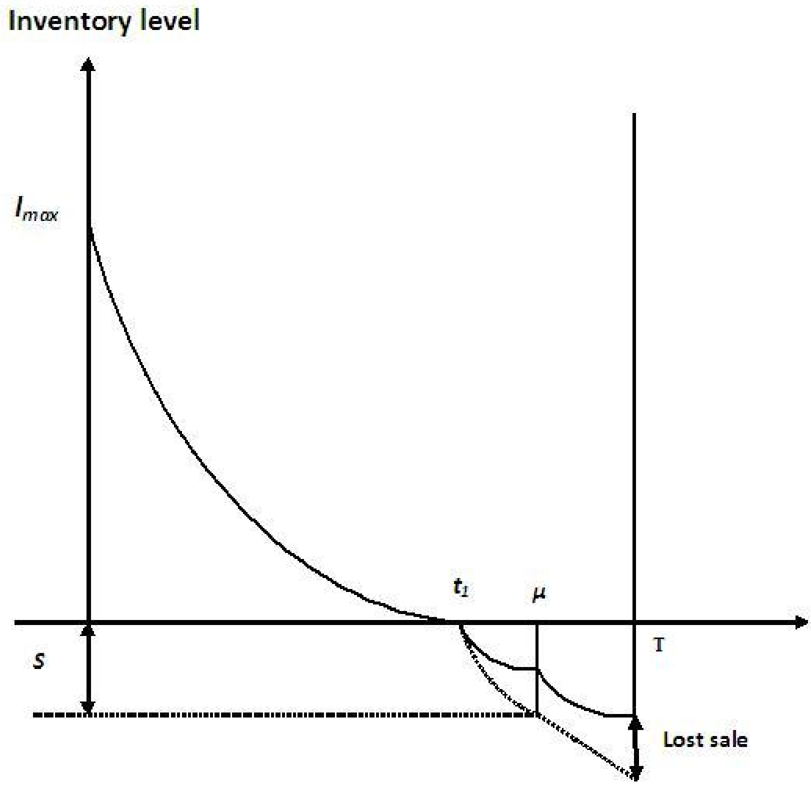

Figure 3.1: Graphical presentation of the inventory system (Case 1 : $t_1 \leq \mu$)

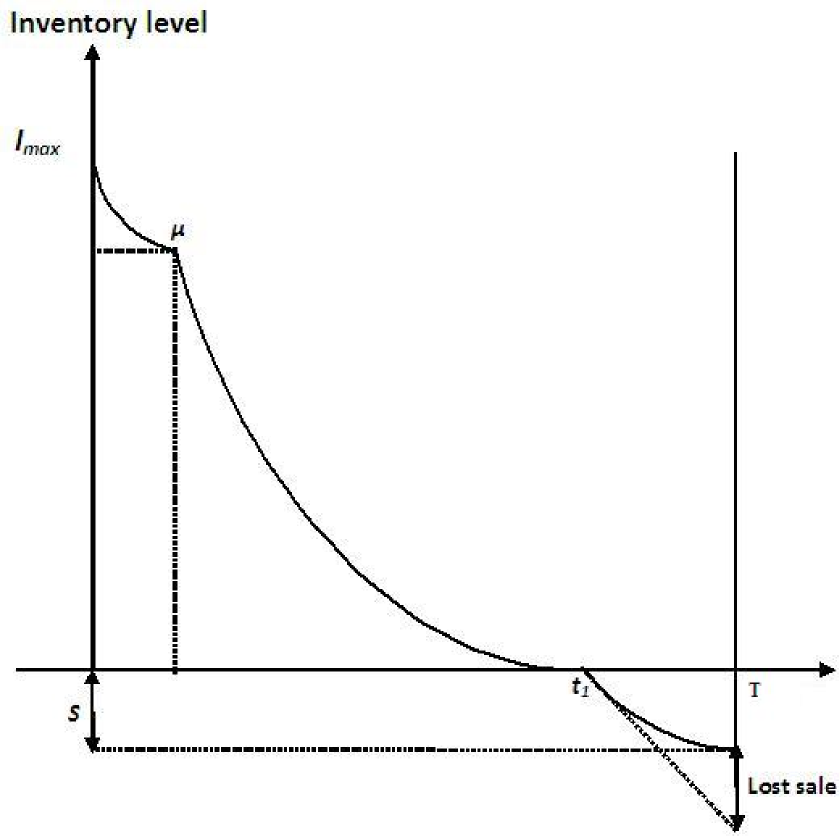

Figure 3.2: Graphical presentation of the inventory system (Case 2 : $t_1 \geq \mu$)

3.3.1 Model 1: $t_1 \leq \mu$

In this case, the selling rate $S(t)$ is

$$S(t) = \begin{cases} d_0 t + \gamma I(t), & 0 \leq t \leq t_1 \\ d_0 t, & t_1 \leq t \leq \mu \\ d_0 \mu, & \mu \leq t \leq T \end{cases}$$

(3.1) and (3.2) are in the form

$$\frac{dI(t)}{dt} + \psi I(t) = -[d_0 t + \gamma I(t)], \quad 0 \leq t \leq t_1 \qquad \text{with } I(0) = I_{max} \tag{3.3}$$

$$\frac{dI(t)}{dt} = -d_0 t \delta(T-t), \quad t_1 \leq t \leq \mu \qquad \text{with } I(t_1) = 0 \tag{3.4}$$

$$\frac{dI(t)}{dt} = -d_0 \mu \delta(T-t), \quad \mu \leq t \leq T \qquad \text{with } -I(T) = S \tag{3.5}$$

Solving (3.3) to (3.5), one can obtained

$$I(t) = \left\{ \begin{array}{lll} I_{max} e^{-(\psi+\gamma)t} - \frac{d_0}{(\psi+\gamma)^2} \left[e^{-(\psi+\gamma)t} + (\psi+\gamma)t - 1 \right], & 0 \leq t \leq t_1 & (3.6.1) \\ \frac{d_0}{\sigma^2 e^{\sigma T}} \left[e^{\sigma t_1}(\sigma t_1 - 1) - e^{\sigma t}(\sigma t - 1) \right], & t_1 \leq t \leq \mu & (3.6.2) \\ \frac{d_0 \mu}{\sigma e^{\sigma T}} \left[e^{\sigma T} - e^{\sigma t} \right] - S, & \mu \leq t \leq T & (3.6.3) \end{array} \right\} (3.6)$$

Using the boundary condition $I(t_1) = 0$ and (3.6.1), the maximum inventory level for each cycle is

$$I_{max} = \frac{d_0}{(\psi+\gamma)^2} \left[(\psi+\gamma) t_1 e^{(\psi+\gamma)t_1} - e^{(\psi+\gamma)t_1} + 1 \right]$$

Considering the continuity of $I(t)$ at $t = \mu$, the maximum amount of demand backlogged per cycle from (3.6.2) and (3.6.3) is

$$S = \frac{d_0 \mu}{\sigma e^{\sigma T}} \left[e^{\sigma T} - e^{\sigma \mu} \right] - \frac{d_0}{\sigma^2 e^{\sigma T}} \left[e^{\sigma t_1}(\sigma t_1 - 1) - e^{\sigma \mu}(\sigma \mu - 1) \right]$$

Now the order quantity Q is

$$\begin{aligned}Q = I_{max} + S &= \frac{d_0}{(\psi+\gamma)^2}\left[(\psi+\gamma)t_1 e^{(\psi+\gamma)t_1} - e^{(\psi+\gamma)t_1} + 1\right] + \frac{d_0\mu}{\sigma e^{\sigma T}}\left[e^{\sigma T} - e^{\sigma\mu}\right] \\ &- \frac{d_0}{\sigma^2 e^{\sigma T}}\left[e^{\sigma t_1}(\sigma t_1 - 1) - e^{\sigma\mu}(\sigma\mu - 1)\right]\end{aligned}$$

The total cost per cycle consists of the following cost as

(a) Ordering cost per cycle

$OC = C_o \int_0^T e^{-\rho t} dt = \frac{C_o}{\rho}\left(1 - e^{-\rho T}\right)$

(b) Purchase cost per cycle

$PC = C_p \int_0^T Q e^{-\rho t} dt = \frac{C_p Q}{\rho}\left(1 - e^{-\rho T}\right)$

(c) Holding cost per cycle

$$\begin{aligned}HC &= C_h \int_0^{t_1} I(t) e^{-\rho t} dt \\ &= \frac{C_h d_0}{(\psi+\gamma)^2}\left[\frac{(\psi+\gamma)t_1 - 1}{(\psi+\gamma+\rho)e^{\rho t_1}}[e^{(\psi+\gamma+\rho)t_1} - 1]\right. \\ &+ \left.\frac{\rho[1-e^{-\rho t_1}] - (\psi+\gamma)\left[1 - (1+\rho t_1)e^{-\rho t_1}\right]}{\rho^2}\right]\end{aligned}$$

(d) Backlogging cost per cycle

$$\begin{aligned}BC &= C_b\left[\int_{t_1}^{\mu}[-I(t)]e^{-\rho t}dt + \int_{\mu}^{T}[-I(t)]e^{-\rho t}dt\right] \\ &= \frac{C_b d_0}{\sigma^2 e^{\sigma T}}\left[\frac{(1-\sigma t_1)[e^{(\sigma-\rho)t_1} - e^{-\rho T}e^{\sigma t_1}] + [e^{-\rho T} - e^{-\rho\mu}]e^{\sigma\mu}}{\rho} + \frac{\sigma[e^{(\sigma-\rho)t_1} - e^{(\sigma-\rho)\mu}]}{(\sigma-\rho)^2}\right. \\ &+ \left.\frac{[\sigma t_1 - 1]e^{(\sigma-\rho)t_1} - \mu\sigma e^{(\sigma-\rho)T} + e^{(\sigma-\rho)\mu}}{(\rho-\sigma)}\right]\end{aligned}$$

(e) Lost sale cost per cycle

$$\begin{aligned}
LC &= C_l\left[\int_{t_1}^{\mu} d_0 t[1-\delta(T-t)]e^{-\rho t}dt + \int_{\mu}^{T} d_0\mu[1-\delta(T-t)]e^{-\rho t}dt\right]\\
&= C_l d_0\left[\frac{(\rho-\sigma)[\mu e^{(\sigma-\rho)T} - t_1 e^{(\sigma-\rho)t_1}] + [e^{(\sigma-\rho)\mu} - e^{(\sigma-\rho)t_1}]}{e^{\sigma T}(\sigma-\rho)^2}\right.\\
&+ \left.\frac{e^{-\rho t_1}[1+\rho t_1] - \rho\mu e^{-\rho T} - e^{-\rho\mu}}{\rho^2}\right]
\end{aligned}$$

(f) Sales revenue per cycle

$$\begin{aligned}
SR &= s\left[\int_0^{t_1} S(t)e^{-\rho t}dt + \int_{t_1}^{T} S(t)\delta(T-t)e^{-\rho t}dt\right]\\
&= sd_0\left[\frac{e^{(\sigma-\rho)t_1} - e^{(\sigma-\rho)\mu} + (\sigma-\rho)[\mu e^{(\sigma-\rho)T} - t_1 e^{(\sigma-\rho)t_1}]}{e^{\sigma T}(\sigma-\rho)^2} + \frac{1-e^{-\rho t_1}[1+\rho t_1]}{\rho^2}\right.\\
&+ \frac{\gamma}{(\psi+\gamma)^2}\left\{\frac{(\psi+\gamma)t_1 - 1}{(\psi+\gamma+\rho)e^{\rho t_1}}[e^{(\psi+\gamma+\rho)t_1} - 1]\right.\\
&+ \left.\left.\frac{\rho[1-e^{-\rho t_1}] - (\psi+\gamma)\left[1-(1+\rho t_1)e^{-\rho t_1}\right]}{\rho^2}\right\}\right]
\end{aligned}$$

Therefore, the total profit per unit time under the effect of inflation and time-value of money is

$$\begin{aligned}
&= Z_1(t_1) = \frac{1}{T}\left[RV - (OC + PC + HC + BC + LC)\right]\\
&= \frac{d_0}{T}\left[\frac{(\gamma s - C_h)}{(\psi+\gamma)^2}\left\{\frac{[(\psi+\gamma)t_1 - 1][e^{(\psi+\gamma+\rho)t_1} - 1]}{(\psi+\gamma+\rho)e^{\rho t_1}} - \frac{(\psi+\gamma)\left[1-(1+\rho t_1)e^{-\rho t_1}\right]}{\rho^2} + \frac{[1-e^{-\rho t_1}]}{\rho}\right\}\right.\\
&+ s\left\{\frac{1-e^{-\rho t_1}[1+\rho t_1]}{\rho^2} + \frac{e^{(\sigma-\rho)t_1} - e^{(\sigma-\rho)\mu} + (\sigma-\rho)[\mu e^{(\sigma-\rho)T} - t_1 e^{(\sigma-\rho)t_1}]}{e^{\sigma T}(\sigma-\rho)^2}\right\} - \frac{C_o[1-e^{-\rho T}]}{d_0\rho}\\
&- C_l\left\{\frac{(\rho-\sigma)[\mu e^{(\sigma-\rho)T} - t_1 e^{(\sigma-\rho)t_1}] + [e^{(\sigma-\rho)\mu} - e^{(\sigma-\rho)t_1}]}{e^{\sigma T}(\sigma-\rho)^2} + \frac{e^{-\rho t_1}[1+\rho t_1] - \rho\mu e^{-\rho T} - e^{-\rho\mu}}{\rho^2}\right\}\\
&- \frac{C_p Q[1-e^{-\rho T}]}{d_0\rho} - \frac{C_b\mu}{\sigma^2 e^{\sigma T}}\left\{\frac{\sigma[e^{\mu(\sigma-\rho)} - e^{T(\sigma-\rho)}]}{\rho-\sigma} + \frac{[\sigma t_1 - 1]e^{(\sigma-\rho)t_1} - [\mu\sigma - 1]e^{(\sigma-\rho)\mu}}{(\rho-\sigma)\mu}\right.\\
&+ \left.\left.\frac{\sigma[e^{(\sigma-\rho)t_1} - e^{(\sigma-\rho)\mu}]}{\mu(\sigma-\rho)^2} + \frac{(1-\sigma t_1)[e^{(\sigma-\rho)t_1} - e^{-\rho T}e^{\sigma t_1}] + [e^{-\rho T} - e^{-\rho\mu}]e^{\sigma\mu}}{\rho\mu}\right\}\right] \qquad (3.7)
\end{aligned}$$

The objective is to obtain the optimal value of t_1 such that the average profit $Z_1(t_1)$ is maximum.

3.3.2 Model 2: $t_1 \geq \mu$

In this case the selling rate $S(t)$ is

$$S(t) = \begin{cases} d_0 t + \gamma I(t), & 0 \leq t \leq \mu, \\ d_0\mu + \gamma I(t), & \mu \leq t \leq t_1, \\ d_0\mu, & t_1 \leq t \leq T. \end{cases}$$

. Hence, (3.1) and (3.2) reduce to the following equations

$$\begin{aligned} \frac{dI(t)}{dt} + \psi I(t) &= -[d_0 t + \gamma I(t)], \quad 0 \leq t \leq \mu, \quad \text{with } I(0) = I_{max} \quad (3.8) \\ \frac{dI(t)}{dt} + \psi I(t) &= -[d_0\mu + \gamma I(t)], \quad \mu \leq t \leq t_1, \quad \text{with } I(t_1) = 0 \quad (3.9) \\ \frac{dI(t)}{dt} &= -d_0\mu\delta(T-t), \quad t_1 \leq t \leq T, \quad \text{with } I(t_1) = 0 \quad (3.10) \end{aligned}$$

Solving (3.8) to (3.10) with the boundary conditions, one can obtained

$$I(t) = \left\{ \begin{array}{lll} I_{max}e^{-(\psi+\gamma)t} - \frac{d_0}{(\psi+\gamma)^2}\left[e^{-(\psi+\gamma)t} + (\psi+\gamma)t - 1\right], & 0 \leq t \leq \mu & (3.11.1) \\ \frac{d_0\mu}{\psi+\gamma}\left[e^{(\psi+\gamma)(t_1-t)} - 1\right], & \mu \leq t \leq t_1 & (3.11.2) \\ \frac{d_0\mu}{\sigma e^{\sigma T}}\left(e^{\sigma t_1} - e^{\sigma t}\right), & t_1 \leq t \leq T & (3.11.3) \end{array} \right\} \quad (3.11)$$

Considering the continuity of $I(t)$ at $t = \mu$, the maximum inventory level I_{max} from equations (3.11.1) and (3.11.2) is

$$I_{max} = \frac{d_0\mu}{\psi+\gamma}e^{(\psi+\gamma)t_1} - \frac{d_0}{(\psi+\gamma)^2}\left[e^{(\psi+\gamma)\mu} - 1\right]$$

Putting $t = T$ in (3.11.3), the maximum amount of demand backlogged per cycle can be obtained as

$$S \equiv -I(T) = \frac{d_0\mu}{\sigma e^{\sigma T}}\left(e^{\sigma T} - e^{\sigma t_1}\right)$$

Now, the order quantity Q is

$$Q = I_{max} + S = \frac{d_0\mu}{\psi+\gamma}e^{(\psi+\gamma)t_1} - \frac{d_0}{(\psi+\gamma)^2}\left[e^{(\psi+\gamma)\mu} - 1\right] + \frac{d_0\mu}{\sigma e^{\sigma T}}\left(e^{\sigma T} - e^{\sigma t_1}\right) \quad (3.12)$$

Next, the total profit per cycle consists of the following cost as

(a) Ordering cost per cycle

$OC = C_o \int_0^T e^{-\rho t}dt = \frac{C_o}{\rho}\left(1 - e^{-\rho T}\right)$

(b) Purchase cost per cycle

$PC = C_p \int_0^T Qe^{-\rho t}dt = \frac{C_p Q}{\rho}\left(1 - e^{-\rho T}\right)$

(c) Holding Cost per cycle

$$\begin{aligned} HC &= C_h\left[\int_0^{\mu} I(t)e^{-\rho t}dt + \int_{\mu}^{t_1} I(t)e^{-\rho t}dt\right] \\ &= \frac{C_h d_0}{(\psi+\gamma)^2}\left[\frac{\rho\mu(\psi+\gamma)e^{-\rho t_1} - [1 - e^{-\rho\mu}](\gamma+\psi-\rho)}{\rho^2}\right. \\ &+ \left.\frac{\mu(\psi+\gamma)[e^{(\psi+\gamma)t_1} - e^{-\rho t_1}] - e^{(\psi+\gamma)\mu} + e^{-\rho\mu}}{(\gamma+\psi+\rho)}\right] \end{aligned}$$

(d) Backlogging cost per cycle

$$\begin{aligned} BC &= C_b\int_{t_1}^{T} -I(t)e^{-\rho t}dt \\ &= \frac{C_b d_0\mu}{\sigma e^{\sigma T}}\left[\frac{e^{\sigma t_1}[e^{-\rho T} - e^{-\rho t_1}]}{\rho} + \frac{e^{(\sigma-\rho)T} - e^{(\sigma-\rho)t_1}}{(\sigma-\rho)}\right] \end{aligned}$$

(e) Lost sell cost per cycle

$$\begin{aligned}
LC &= C_l \int_{t_1}^{T} d_0\mu[1-\delta(T-t)]e^{-\rho t}dt \\
&= C_l d_0 \mu \left[\frac{e^{-\rho t_1}-e^{-\rho T}}{\rho} - \frac{e^{(\sigma-\rho)T}-e^{(\sigma-\rho)t_1}}{e^{\sigma T}(\sigma-\rho)}\right]
\end{aligned}$$

(f) Sale revenue per cycle

$$\begin{aligned}
SR &= s\left[\int_0^{t_1} S(t)e^{-\rho t}dt + \int_{t_1}^{T} S(t)\delta(T-t)e^{-\rho t}dt\right] \\
&= sd_0\left[\frac{1-e^{-\rho\mu}-\rho\mu e^{-\rho t_1}}{\rho^2} + \frac{\mu e^{-\sigma T}[e^{(\sigma-\rho)T}-e^{(\sigma-\rho)t_1}]}{(\sigma-\rho)}\right. \\
&+ \frac{\gamma}{(\psi+\gamma)^2}\left\{\frac{\rho\mu(\psi+\gamma)e^{-\rho t_1}-[1-e^{-\rho\mu}](\gamma+\psi-\rho)}{\rho^2}\right. \\
&+ \left.\left.\frac{\mu(\psi+\gamma)[e^{(\psi+\gamma)t_1}-e^{-\rho t_1}]-e^{(\psi+\gamma)\mu}+e^{-\rho\mu}}{(\gamma+\psi+\rho)}\right\}\right]
\end{aligned}$$

Total profit per unit time under the effect of inflation and time-value of money is

$$\begin{aligned}
&= Z_2(t_1) = \frac{1}{T}[SR-(OC+PC+HC+BC+LC)] \\
&= \frac{d_0}{T}\left[s\left\{\frac{1-e^{-\rho\mu}-\rho\mu e^{-\rho t_1}}{\rho^2} + \frac{\mu e^{-\sigma T}[e^{(\sigma-\rho)T}-e^{(\sigma-\rho)t_1}]}{(\sigma-\rho)}\right\} + C_l\mu\left\{\frac{e^{(\sigma-\rho)T}-e^{(\sigma-\rho)t_1}}{e^{\sigma T}(\sigma-\rho)}\right.\right. \\
&- \left.\frac{e^{-\rho t_1}-e^{-\rho T}}{\rho}\right\} - \frac{C_b\mu}{\sigma e^{\sigma T}}\left\{\frac{e^{\sigma t_1}[e^{-\rho T}-e^{-\rho t_1}]}{\rho} + \frac{e^{(\sigma-\rho)T}-e^{(\sigma-\rho)t_1}}{(\sigma-\rho)}\right\} + \frac{\gamma s - C_h}{(\psi+\gamma)^2} \\
&\quad \left\{\frac{\rho\mu(\psi+\gamma)e^{-\rho t_1}-[1-e^{-\rho\mu}](\gamma+\psi-\rho)}{\rho^2} + \frac{\mu(\psi+\gamma)[e^{(\psi+\gamma)t_1}-e^{-\rho t_1}]-e^{(\psi+\gamma)\mu}+e^{-\rho\mu}}{(\gamma+\psi+\rho)}\right\} \\
&- \left.\frac{[1-e^{-\rho T}]}{d_0\rho}(C_0+C_pQ)\right]
\end{aligned} \tag{3.13}$$

Main objective is to find the optimal value of t_1 such that the average profit $Z_2(t_1)$ is maximum.

The total profit function of the system over $[0, T]$ takes the form

$$Z(t_1) = \begin{cases} Z_1(t_1) & \text{if } t_1 \leq \mu \\ Z_2(t_1) & \text{if } t_1 \geq \mu \end{cases} \tag{3.14}$$

It is easy to check that this function is continuous at μ.

3.4 Solution procedure

This section derives results, which ensure the necessary and sufficient conditions of the existence and uniqueness of the optimal solution to maximize the total profit.
From (3.7), for $t_1 \leq \mu$

$$\frac{dZ_1(t_1)}{dt_1} = \frac{t_1 d_0}{T} F(t_1) \tag{3.15}$$

where

$$\begin{aligned} F(t_1) &= (s + C_l)\left[e^{-\rho t_1} - e^{(\sigma-\rho)t_1 - \sigma T}\right] + \frac{C_b}{\rho e^{\sigma T}}\left[e^{\sigma t_1}(e^{-\rho t_1} - e^{-\rho T})\right] + \frac{(s\gamma - C_h)}{(\gamma + \psi + \rho)} \\ &\quad \left[\frac{e^{(\psi+\gamma+\rho)t_1} - 1}{e^{\rho t_1}}\right] + \frac{C_p(1 - e^{-\rho T})}{\rho}\left[e^{(t_1 - T)\sigma} - e^{(\psi+\gamma)t_1}\right] \end{aligned} \tag{3.16}$$

On the other hand, one can obtain

$$F(0) = \{s + C_l - \frac{C_p}{\rho}[1 - e^{-\rho T}]\}(1 - e^{-\sigma T}) + \frac{C_b}{\rho e^{\sigma T}}(1 - e^{-\rho T}) \tag{3.17}$$

and

$$F(T) = \frac{(s\gamma - C_h)}{(\gamma + \psi + \rho)}\left[e^{(\psi+\gamma)T} - e^{-\rho T}\right] + \frac{C_p}{\rho}(1 - e^{-\rho T})(1 - e^{(\psi+\gamma)T}) \tag{3.18}$$

Taking first order derivative of $F(t_1)$ with respect to t_1, one can find

$$\begin{aligned} F'(t_1) &= \left(\frac{C_b}{\rho} - s - C_l\right)\frac{(\sigma-\rho)e^{(\sigma-\rho)t_1}}{e^{\sigma T}} + \left(\frac{\gamma s - C_h}{\psi+\gamma+\rho} + \frac{C_p}{\rho}\left(e^{-\rho T}-1\right)\right)(\psi+\gamma)e^{(\psi+\gamma)t_1} \\ &+ \left(\frac{s\gamma - C_h}{\psi+\gamma+\rho} - s - C_l\right)\rho e^{-\rho t_1} + \frac{\sigma e^{\sigma t_1}}{\rho e^{(\sigma+\rho)T}}\left((e^{\rho T}-1)C_p - C_b\right) \qquad (3.19) \end{aligned}$$

Now if $F'(t_1) < 0$ and $F(0) > 0, F(T) < 0$, then $F(t_1)$ is a strictly decreasing function of t_1. Therefore, the equation

$$F(t_1) = 0 \qquad (3.20)$$

has a unique root $t_1^* \in (0, T)$ for which

$$\left.\frac{d^2 Z_1(t_1)}{dt_1^2}\right|_{t_1=t_1^*} = \frac{d_0 t_1^*}{T} F'(t_1^*) < 0 \qquad (3.21)$$

From (3.13), for $\mu \le t_1$

$$\frac{dZ_1(t_1)}{dt_1} = \frac{\mu d_0}{T} F(t_1) \qquad (3.22)$$

$$\frac{d^2 Z_1(t_1)}{dt_1^2} = \frac{\mu d_0}{T} F'(t_1) \qquad (3.23)$$

where $F(t_1)$ is given by (3.16).

The above analysis shows that two functions $Z_1(t_1)$ and $Z_2(t_2)$ have the unique and same unstrained maximizing point $t_1^* \in (0, T)$, which is determined by (3.16).

Now if $F'(t_1) < 0$ and $F(0) > 0, F(T) < 0$, then $F(t_1)$ is a strictly decreasing function of t_1. Hence $F(t_1) = 0$ has a unique root $t_1^* \in (0, T)$.

3.5 Numerical examples

Two examples, that consist of the different situations of the ramp-type demand and the deteriorated rates are considered to derive the optimal solution. Let us consider the following parametric values: $d_0 = 400$/units, $C_o = \$50$/order, $s = \$20$/unit, $C_h = \$3$/unit/unit time, $C_p = \$15$/unit, $C_b = \$5$/unit, $C_l = \$8$/unit, $\gamma = 0.1$, $\rho = 0.04$, $\sigma = 0.02, \psi = 0.05$, $T = 1$ week.

Example 3.1

one can assume that $\mu = 0.7$ then solving the equation $F(t_1) = 0$ the optimal replenishment cycle time $t_1^* = 0.6010$ satisfying $t_1^* \leq \mu$ and the maximum total profit per unit of time $Z_2^*(t_1) = \$617.784$/week. The graphical representation of the profit function versus the replenishment time is presented in Figure 3.3.

Now examine whether the optimal solution is unique.

$F'(t) = -8.54 < 0, F(0) = 5.07 > 0$, and $F(T) = -3.44 < 0$.

Hence t_1^* is a unique solution.

Example 3.2

one can assume that $\mu = 0.4$ then solving the equation $F(t_1) = 0$ the optimal replenishment cycle time $t_1^* = 0.6010$ satisfying $t_1^* \geq \mu$ and the maximum total profit per unit of time $Z_1^*(t_1) = \$423.40$/week. The graphical representation of the profit function versus the replenishment time is portrayed in Figure 3.4

Now examine whether the optimal solution is unique.

$F'(t) = -8.54 < 0, F(0) = 5.07 > 0$, and $F(T) = -3.44 < 0$.

Hence t_1^* is a unique solution.

From above numerical examples one can can conclude that the optimal total profit is maximum when $\mu = 0.7$ i.e., for Model I. Now, different continuous probabilistic deterioration functions are considered in these numerical experiments with the same parametric values as in Example 3.1.

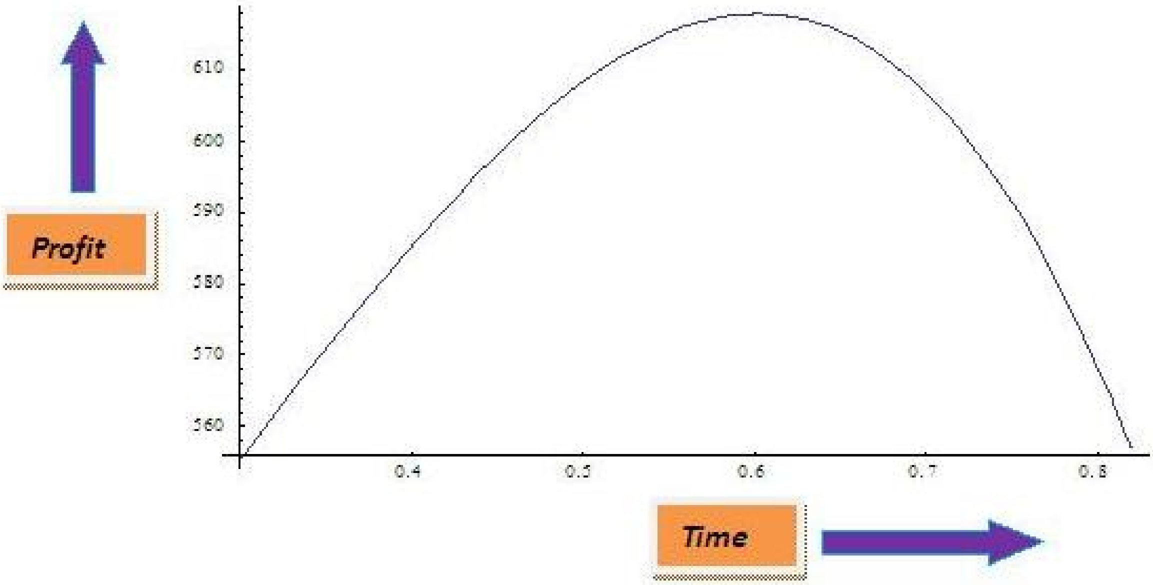

Figure 3.3: Graphical presentation of total profit function verses time (Example 3.1)

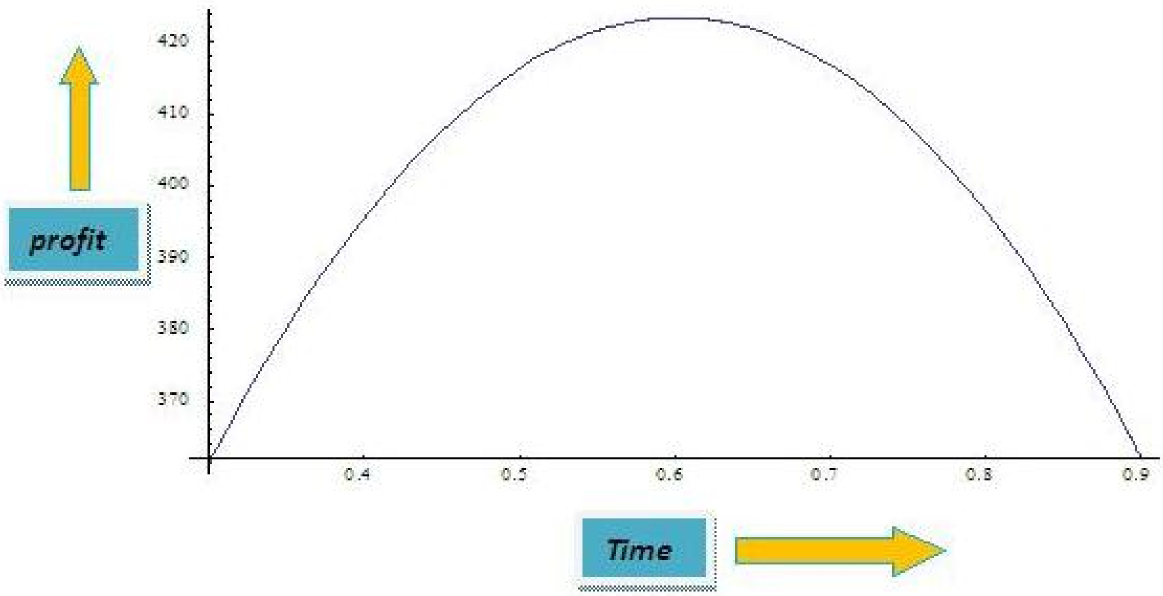

Figure 3.4: Graphical presentation of total profit function verses time (Example 3.2)

Example 3.3

Here, $\psi = E(f(x)) = \frac{a+b}{2}$ (Sarkar and Sarkar [2013]), where $f(x)$ follows a uniform distribution, and $a < b$. Parametric values are $a = 0.05$ and $b = 0.15$. The rest of the values are the same as in Example 3.2. Then, the optimal solution is $t_1^* = 0.5490$ week and $Z^* = \$597.393$/week.

Example 3.4

Here, $\psi = E(f(x)) = \frac{a+b+c}{3}$, where $f(x)$ follows a triangular distribution, and $a \leq c \leq b$. Parametric values are $a = 0.05, b = 0.15$, and $c = 0.13$. The rest of the values are the same as in Example 3.2. Then, the optimal solution is $t_1^* = 0.5394$ week and $Z^* = \$593.91$/week.

Example 3.5

Here, $\psi = E(f(x)) = \frac{a+4m+b}{6}$, where $f(x)$ follows a double triangular distribution, and $a \leq m \leq b$. Parametric values are $a = 0.05, m = 0.13$, and $b = 0.15$. The rest of the values are the same as in Example 3.2. Then, the optimal solution is $t_1^* = 0.5302$ week and $Z^* = \$590.597$/week.

Example 3.6

Here, $\psi = E(f(x)) = \frac{\gamma}{\gamma+\beta}$, where $f(x)$ follows beta distribution, and $\gamma \geq 0$ and $\beta \geq 0$. Parametric values are $\gamma = 0.05$ and $\beta = b = 0.15$. The rest of the values are the same as in Example 3.2. Then, the optimal solution is $t_1^* = 0.4329$ week and $Z^* = \$559.053$/week. The graphical representation of the profit function versus the replenishment time for Example 3.3, Example 3.4, Example 3.5, and Example 3.6 are depicted in Figure 3.5.

3.6 Sensitivity analysis

This section illustrates the effects of changes in parameters such as $C_o, C_h, C_b, C_l, C_p, s, \psi$, and γ on optimal total profit. The sensitivity analysis is performed by changing each of the parameters by $-50\%, -25\%, +25\%$, and $+50\%$ taking one parameter at a time while keeping the remaining parameters unchanged. The results are presented in Table 3.1.

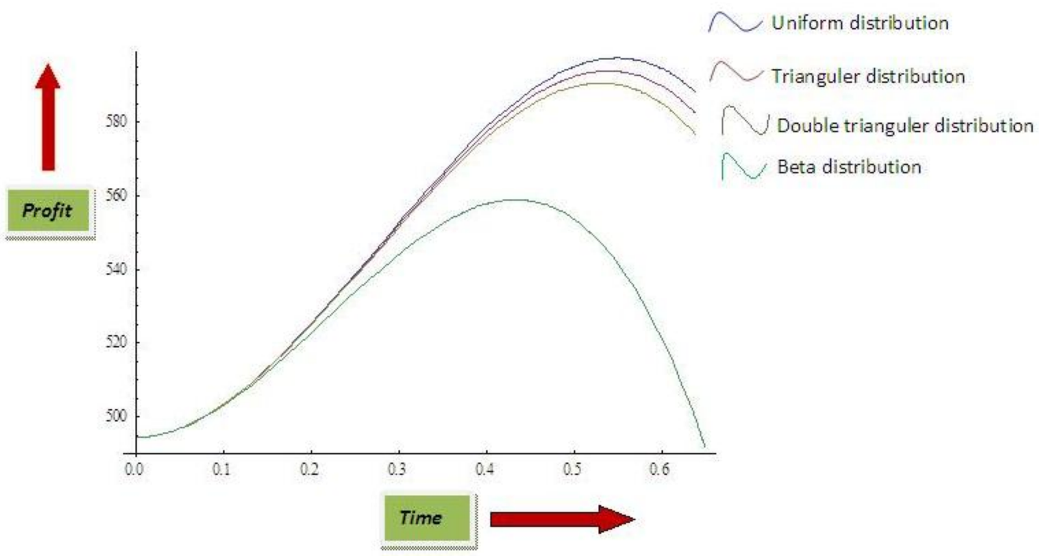

Figure 3.5: Graphical representation of total profit function verses replenishment time for Example 3.3, 3.4, 3.5, and 3.6.

From Table 3.1, the discussion of sensitivity analysis of the key parameters are as follows: From the above table one can conclude that $Z_1(t_1)$ and $Z_2(t_2)$ are highly sensitive towards change in C_p and s whereas slightly sensitive in C_l, ψ and γ. On the other hand, $Z_1(t_1)$ and $Z_2(t_2)$ are moderately sensitive towards change in C_o, C_h and C_b.

3.7 Special cases

This section considers some special cases that influence the total profit.

Table 3.1: Effect of changes in the parameters of Model 1 and Model 2

Parameters	Changes(in %)	$Z_1(t_1)$	$Z_2(t_2)$
	-50%	$+3.97$	$+5.79$
	-25%	$+1.98$	$+2.50$
C_o	$+25\%$	-1.98	-2.89
	$+50\%$	-3.97	-5.79
	-50%	$+9.87$	$+11.34$
	-25%	$+4.19$	$+05.02$
C_h	$+25\%$	-3.18	-04.05
	$+50\%$	-5.65	-07.35
	-50%	$+11.01$	$+10.01$
	-25%	$+04.76$	$+04.18$
C_b	$+25\%$	-03.71	-03.15
	$+50\%$	-06.65	-05.61
	-50%	$+0.27$	$+0.23$
	-25%	$+0.14$	$+0.12$
C_l	$+25\%$	-0.13	-0.12
	$+50\%$	-0.27	-0.23
	-50%	$+223.01$	$+229.91$
	-25%	$+111.08$	$+114.57$
C_p	$+25\%$	-110.43	-113.95
	$+50\%$	-220.35	-227.36
	-50%	-290.31	-299.68
	-25%	-145.38	-150.07
s	$+25\%$	$+145.94$	$+150.62$
	$+50\%$	$+292.59$	$+301.87$
	-50%	$+2.02$	$+2.45$
	-25%	$+0.97$	$+1.19$
ψ	$+25\%$	-0.91	-1.13
	$+50\%$	-1.76	-2.21
	-50%	-1.02	-1.28
	-25%	-0.52	-0.65
γ	$+25\%$	$+0.55$	$+0.68$
	$+50\%$	$+1.12$	$+1.39$

3.7.1 Case 1

$\sigma = 0$ implies a complete backlogging inventory model. In this case, the total profit function is as follows:

$$\begin{aligned}
Z_{11}(t_1) &= \frac{d_0}{T}\Bigg[\frac{(\gamma s - C_h)}{(\psi+\gamma)^2}\Bigg\{\frac{[(\psi+\gamma)t_1 - 1][e^{(\psi+\gamma+\rho)t_1} - 1]}{(\psi+\gamma+\rho)e^{\rho t_1}} - [1-(1+\rho t_1)e^{-\rho t_1}](\psi+\gamma)/\rho^2 \\
&+ [1-e^{-\rho t_1}]/\rho\Bigg\} + s[1-e^{-\rho\mu} - \rho\mu e^{-\rho T}]/\rho^2 - C_b\Bigg\{e^{-\rho t_1}(1+\rho t_1) + \rho^2 e^{-\rho T}(\mu^2+t_1^2)/2 \\
&- \rho\mu e^{-\rho T}(1+\rho T) - e^{-\rho\mu}\Bigg\}/\rho^3 - C_o[1-e^{-\rho T}]/d_0\rho - \frac{C_p}{\rho}(1-e^{-\rho T}) \\
&\quad \left(\left[1+((\psi+\gamma)t_1 - 1)e^{(\psi+\gamma)t_1}\right]/(\psi+\gamma)^2 + T\mu - (\mu^2+t_1^2)/2\right)\Bigg]
\end{aligned}$$

The necessary condition for $Z_{11}(t_1)$ to be maximized is $\frac{dZ_{11}(t_1)}{dt_1} = 0$ which implies

$$\frac{(s\gamma - C_h)}{(\psi+\gamma+\rho)e^{\rho t_1}}(e^{(\psi+\gamma+\rho)t_1} - 1) + C_p(1-e^{-\rho T})(1-e^{(\psi+\gamma)t_1})/\rho = C_b(e^{-\rho T} - e^{-\rho t_1})/\rho$$

3.7.2 Case 2

$\rho = 0$, i.e., the inflationary effect is not considered. For this special case the total profit function is given by

$$\begin{aligned}
Z_{12}(t_1) &= \frac{d_0}{T}\Bigg(\frac{s\gamma - C_h}{(\psi+\gamma)^3}\Bigg\{1-(\psi+\gamma)^2 t_1^2/2 + e^{(\psi+\gamma)t_1}[(\psi+\gamma)t_1 - 1]\Bigg\} + s\Bigg\{e^{\sigma t_1}(1-\sigma t_1) \\
&+ \sigma e^{\sigma T}(\mu + \sigma t_1^2/2) - e^{\sigma\mu}\Bigg\}\Bigg/ e^{\sigma T}\sigma^2 - C_b\Bigg\{\mu e^{\sigma T} + (\mu - T)e^{\sigma\mu} + 2(e^{\sigma t_1} - e^{\sigma\mu})/\sigma \\
&+ e^{\sigma t_1}(\sigma t_1^2 + T(1-\sigma t_1) - 2t_1)\Bigg\}\Bigg/\sigma^2 e^{\sigma T} - C_l\Bigg\{e^{\sigma\mu} + e^{\sigma t_1}(\sigma t_1 - 1) \\
&- \sigma e^{\sigma T}[\sigma(t_1^2+\mu^2)/2 + \mu(1-\sigma T)]\Bigg\}\Bigg/\sigma^2 e^{\sigma T} - \frac{C_o}{d_0} - \frac{QC_p}{d_0}\Bigg)
\end{aligned}$$

Table 3.2: Summary of the optimal solutions under different cases

Special cases	t_1^* (week)	Z^* (\$/week)
Case 1 ($\sigma = 0$)	0.59	622.69
Case 2 ($\rho = 0$)	0.61	649.81
Case 3 ($\sigma = 0$ and $\rho = 0$)	0.60	654.85

The necessary condition for $Z_{12}(t_1)$ to be maximized is $\frac{dZ_{12}(t_1)}{dt_1} = 0$ which implies

$$\begin{aligned} & (s + C_l)(1 - e^{(t_1 - T)\sigma}) + (s\gamma - C_h)(e^{(\psi+\gamma)t_1} - 1)/(\psi + \gamma) \\ = \; & C_b(t_1 - T)e^{(t_1 - T)\sigma} + C_p(e^{(\psi+\gamma)t_1} - e^{(t_1 - T)\sigma}) \end{aligned}$$

3.7.3 Case 3

$\rho = 0$ and $\sigma = 0$ implies the inflationary effect is not considered and the backlogging is complete. For this special case, the total profit function is given by

$$\begin{aligned} Z_{13}(t_1) = \; & \frac{d_0\mu}{T}\Bigg((s - C_p)(T - \frac{\mu}{2}) + \frac{1}{\mu}\left(\frac{\gamma s - C_h}{\psi + \gamma} - C_p\right)\left\{\frac{(\psi + \gamma)t_1 e^{(\psi+\gamma)t_1} - e^{(\psi+\gamma)t_1} + 1}{(\psi + \gamma)^2}\right. \\ & \left. - \frac{t_1^2}{2}\right\} - C_b\left(\frac{\mu^2}{6} + \frac{t_1^3}{3\mu} - \frac{\mu T}{2} - \frac{Tt_1^2}{2\mu} + \frac{T^2}{2}\right) - \frac{C_o}{d_0\mu}\Bigg) \end{aligned}$$

The necessary condition for $Z_{13}(t_1)$ to be maximized is $\frac{dZ_{13}(t_1)}{dt_1} = 0$ which implies

$$\frac{[\gamma(s - C_p) - (C_h + \psi C_p)]}{\psi + \gamma}\left(e^{(\psi+\gamma)t_1} - 1\right) = C_s(t_1 - T)$$

Example 3.7

The results for special cases, which is listed out in Table 3.2 has obtained by using the same parametric values as in Example 3.2. The graphical representation of the profit function versus the replenishment time for special Case 1, Case 2, and Case 3 are presented in Figure. 3.6.

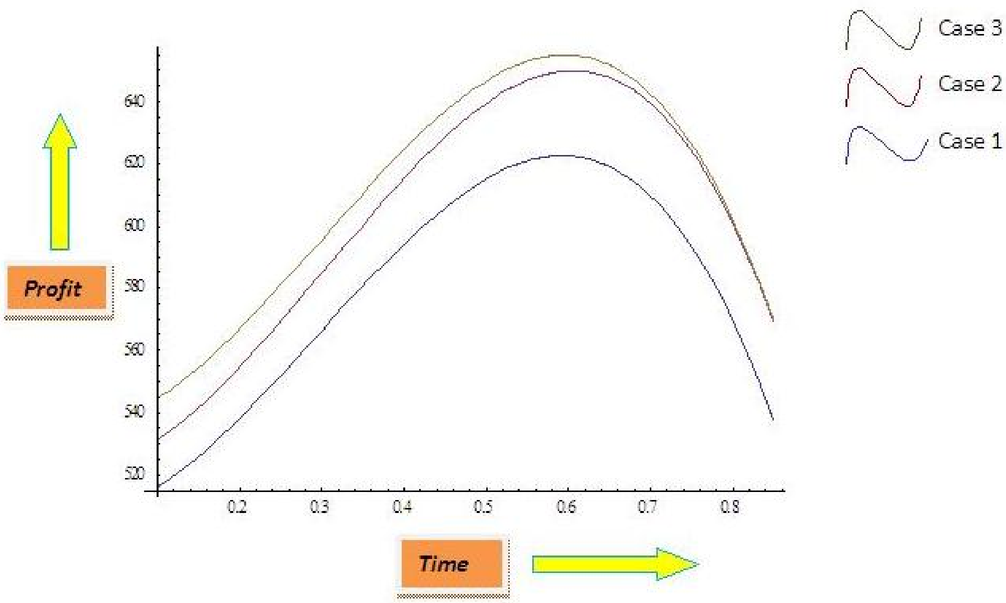

Figure 3.6: Graphical representation of total profit function verses replenishment time for spacial cases

3.8 Conclusions

In this marketing environment, when a new brand of consumer goods are launched, the demand of goods increases quickly to a certain moment and after sometime it stabilizes. Finally, it becomes almost constant. Keeping in mind this type of demand pattern, here, demand is considered as a ramp-type function of time. To make the research a more realistic one, four different types of continuous probabilistic deterioration functions are considered here. The associated profit function was maximized at the optimal values of decision variables. A unique solution procedure was provided as an optimal solution. Some numerical examples, graphical representations, special cases, and sensitivity analysis are given to illustrate the model. There are several extensions of this work that could constitute future research related in this field. This model can be extended in several ways, like multi-item inventory models, and reliability of the items. Another interesting idea is to consider fuzzy demand case.

Chapter 4

Optimal replenishment policy with variable deterioration for fixed-lifetime products

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4.1 Introduction

It is common nature of human behavior that customer would like to buy a single item even though the item is chosen from large amount of stocks, i.e., the demand rate may go up or down if the on-hand inventory level increases or decreases. For this reason, the retailers have to store a large amount of items in their stored places. However, certain types of commodities either deteriorate or become obsolete in course of time. Therefore, they must have an expiry date, i.e., the product will have a maximum lifetime, which is time bounded. For example, goods like fruits, vegetables, meat, foodstuffs, perfumes, etc., where deterioration is usually observed. The deteriorating items cannot be used for its original purpose. The loss of inventory due to deterioration cannot be ignored. Therefore, it is very essential to know the maximum lifetime of products such that deterioration of products can be controlled. Very few of the existing inventory models in the literature assume the fixed lifetime of products.

Wee (1993) considered product's outdate or deterioration as decay, damage, spoilage, evaporation, obsolescence, pilferage, loss of utility or loss of marginal value of a commodity that results in decreasing usefulness from the original one. In this direction, Sana (2010a) developed an optimal selling-price and lot size with time-varying deterioration and partial backlogging. In this paper, he introduced a new time-dependent deterioration function that follows a maximum lifetime of products. By introducing a new algebraic time-varying deterioration function, Sarkar (2012b) developed a deteriorating inventory model with trade-credit policy. Sett *e*t al.(2012) discussed a two-warehouse inventory model with increasing demand and time-varying deterioration. Sarkar *e*t al.(2015) formulated a trade-credit policy with variable deterioration for fixed lifetime products. Wu *e*t al. (2014) discussed an optimal credit-period and lot sizing problem for deteriorating items with expiration dates under two-level trade credit financing. Sarkar *e*t al. (2016) derived retailer's optimal strategy for fixed lifetime products. Sarkar (2016) developed a supply chain coordination with variable backorder, inspections, and discount policy for fixed lifetime products.

Ghare and Schrader (1963) were the first authors to consider deterioration in inventory model. That model is a basic inventory model with constant rate of deterioration. Several authors worked on determining inventory model. Covert and Philip (1973) relaxed the assumption of constant deterioration rate by considering a two-parameter Weibull distribution. Further, Philip (1974) extended the model by assuming the variable deterioration rate. Dave and Patel (1981) invented the deteriorating inventory model with linear increasing demand without shortages. Wee (1997) formulated an optimal replenishment policy for deteriorating items with a linear price-function of demand. Goyal and Giri (2001) discussed a detail review of deteriorating inventory literatures. Yang and Wee (2006) considered an inventory model with finite replenishment rate and price sensitive demand for short life cycle and perishable electronic product. Law and Wee (2006) developed an integrated production-inventory model for ameliorating and deteriorating items. They used the discounted cash flow (DCF) approach and optimization technique to determine the optimal production and replenishment policy. Chung and Wee (2008) discussed a deteriorating inventory model for pricing policy with imperfect production, inspection planning, warranty period, and stock-level-dependant demand. Hsu *e*t al. (2010) addressed a deteriorating inventory policy when the retailer invests on the preservation technology to reduce the rate of product deterioration. Sarkar and Sarkar (2013a) and Sarkar *e*t al. (2013) extended different types of inventory model with deterioration. Sarkar (2013) considered a production-inventory model for three different types of continuously distributed deterioration functions. Sarkar and Sarkar (2013) discussed a control problem with probabilistic deterioration. They solved the model with the help of Euler-Lagrange method. Shah and Cárdenas-Barrón (2015) analyzed the retailer decision for ordering and credit policies, when a supplier offers its retailer either a cash discount or a fixed credit period.

On the issue of shortage case, Deb and Chaudhuri (1987) were the first authors to incorporate shortage into inventory model. Abad (1996) discussed a pricing and lot-size problem for deteriorating product by assuming shortages and partial backlogging. Wee *e*t al. (2008) discussed an inventory model for ameliorating and deteriorating items with partial backordering under inflation. Without using the calculations for optimality from differential calculus, C*á*rdenas-Barr*ó*n (2011a) solved an economic production quantity model with basic algebraic procedure. Taleizadeh *e*t al. (2013) considered an inventory model to determine the optimal order and shortage quantities for a perishable item when the supplier offers a special sale. Sometimes managers prefers to use planned backorders to reduce the total system cost. In this direction, C*á*rdenas-Barr*ó*n (2009a) discussed an inventory model with planned backorders to determine the economic production quantity for a single product. Sarkar and Sarkar (2013a) developed an improved inventory model with partial backlogging, time-varying deterioration, and stock-dependent demand. Most recently, Sarkar *e*t al. (2014) extended the inventory model with random defective rate, rework process, and variable backorders. Vishkaei *e*t al. (2014) used 100% screening process in a economic order quantity model under shortages and delay-in-payment. Sarkar *e*t al. (2015) developed a continuous review inventory model with backorder price-discount under controllable lead time. Recently, Sarkar *e*t al. (2015) discussed a deteriorating inventory model for high-tech products with partial backlogging.

Mandal and Pal (1998) was the first author to introduce ramp-type demand in inventory model. Deterministic and probabilistic demand situations were discussed in that model. Giri *e*t al. (2003) developed a single-item single-period inventory model for deteriorating items with a ramp-type demand and Weibull distributed deterioration. Manna and Chaudhuri (2006) developed an EOQ model with ramp-type demand rate, time-dependent deterioration rate, unit production cost, and shortages. They assumed that the time point at which the demand is stabilized occurs before the production stopping time. Skouri *e*t al. (2009) discussed an inventory model with general ramp-type demand rate, time-dependent deterioration rate, and partial backlogging. The model was studied under two different replenishment policies: (a) starting with no shortages and (b) starting with shortages. Skouri *et* al. (2011) extended Manna and Chaudhuri's (2006) model by considering that (a) for the model with no shortages; the demand rate is stabilized after the production stopping time and (b) for the model with shortages; the demand rate is stabilized after the production stopping time or after the time when the inventory level reaches zero or after the production restarting time. Sarkar *et* al. (2014) developed an inventory models for imperfect production with price and time-dependent demand. Pal *et* al. (2014) developed an inventory model for deteriorating items with ramp-type demand rate. That model was solved under crisp and fuzzy environment to evaluate the optimum solution of in different cases.

This model considers an inventory system for deteriorating items with ramp-type demand. Every deteriorating items have their expiration dates (maximum lifetime), i.e., products may be deteriorate with increasing value of time. An interesting relation between the variable increasing-time and the fixed-lifetime of products is considered in this model. Shortages are allowed, which are fully backlogged. The main purpose of this model is to develop an optimal replenishment policy which minimizes the total cost per unit time. The necessary and sufficient conditions of the existence and uniqueness of the optimal solutions are provided. Finally, several numerical examples are given to illustrate the theoretical results of this model. Sensitivity analysis of the optimal solution with respect to key parameters and their discursion is given. Rest of the paper is designed as follows: In Section 4.2, assumptions are given. In Section 4.3, mathematical model and solution procedure are derived. Numerical experiments and sensitivity analysis are presented to illustrate the model in Sections 4.4 and 4.5 respectively. Finally, conclusions are made in Section 4.6.

4.2 Assumptions

To derive the model, following assumptions are made.

1. The model is considered for a single type of item.

2. The deterioration rate is assumed as $\alpha(t) = \frac{1}{1+\rho-t}$, where $\rho > t$ and ρ is the maximum lifetime of products at which the total on-hand inventory deteriorates. When t increases, $\alpha(t)$ increases and $\text{Lim}_{t\to\rho}\alpha(t) \to 1$.

3. The demand rate $d(t)$ is assumed to be a ramp-type function of time, i.e.,

$$d(t) = d_0[t - (t-\mu)H(t-\mu)], \qquad d_0 > 0$$

   where $H(t-\mu)$ is the Heaviside's function as follows:

$$H(t-\mu) = \begin{cases} 1 & \text{if} \quad t \geq \mu \\ 0 & \text{if} \quad t < \mu \end{cases}$$

   .

4. Holding cost is linear function of time i.e., $C_h(t) = h_0 + h_1 t$, where $h_0, h_1 > 0$.

5. Shortages are allowed and fully backlogged.

6. Lead time is assumed as negligible and the replenishment rate is infinite.

4.3 Model formulation

The model considers an inventory model for deteriorating items with ramp-type demand and stock-dependent selling rate. Deterioration rate of the items are time dependent. The replenishment at the beginning of the cycle brings the inventory level up to I_{max}. The inventory level decreases during the time interval $[0, t_1]$ due to combined effects of the demand, deterioration, and falls to zero at $t = t_1$. Shortages occur during the period (t_1, T), which are fully backlogged.

During the replenishment cycle $[0, T]$, the inventory level, $I(t)$, satisfies the following differential equations

$$\frac{dI(t)}{dt} = -d(t) - \alpha(t)I(t), \quad 0 \le t \le t_1, \qquad \text{with } I(0) = I_{max} \tag{4.1}$$

$$\frac{dI(t)}{dt} = -d(t), \quad t_1 \le t \le T, \quad \text{with } I(t_1) = 0 \tag{4.2}$$

To solve the above differential equations, two cases are considered as (a) $\mu \ge t_1$, (b) $\mu \le t_1$. The fluctuations of the inventory levels for the two cases are depicted in Figure 4.1 and Figure 4.2, respectively.

4.3.1 Model 1

When $\mu \ge t_1$

In this case, the demand rate $d(t)$ is

$$d(t) = \begin{cases} d_0 t, & 0 \le t \le t_1 \\ d_0 t, & t_1 \le t \le \mu \\ d_0 \mu, & \mu \le t \le T \end{cases}$$

Due to deterioration of items and ramp-type demand rate, the inventory level gradually decreases during the period $[0, t_1]$ and ultimately falls to zero at time t_1. Therefore, from (4.1) and (4.2), one obtains

$$\frac{dI(t)}{dt} = -d_0 t - \alpha(t)I(t), \quad 0 \le t \le t_1 \quad \text{with } I(0) = I_{max} \tag{4.3}$$

$$\frac{dI(t)}{dt} = -d_0 t, \qquad t_1 \le t \le \mu \quad \text{with } I(t_1) = 0 \tag{4.4}$$

$$\frac{dI(t)}{dt} = -d_0 \mu, \qquad \mu \le t \le T \quad \text{with } -I(T) = S \tag{4.5}$$

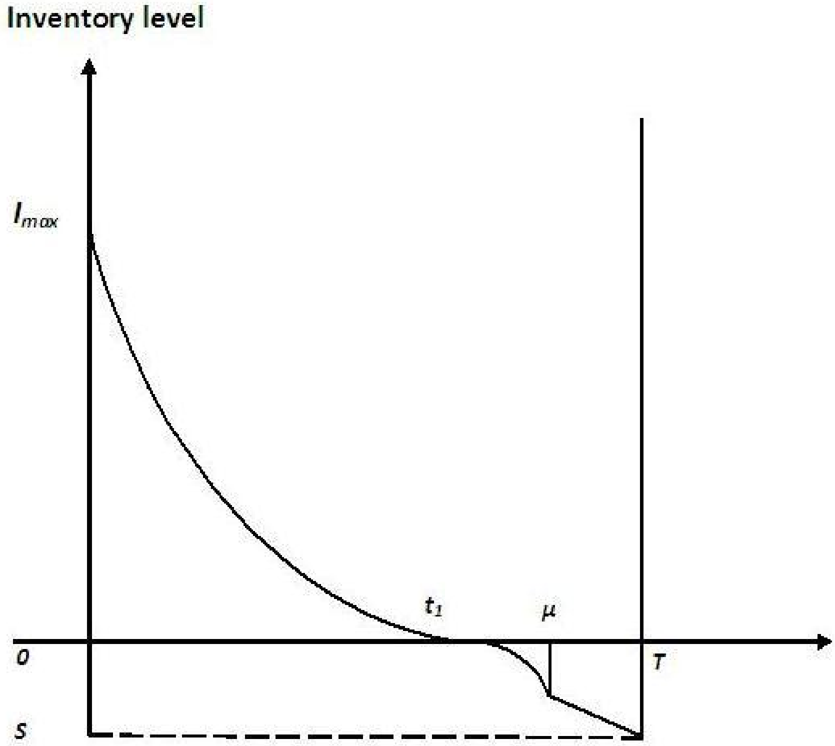

Figure 4.1: Graphical presentation of the inventory system (Case 1 : $t_1 \leq \mu$)

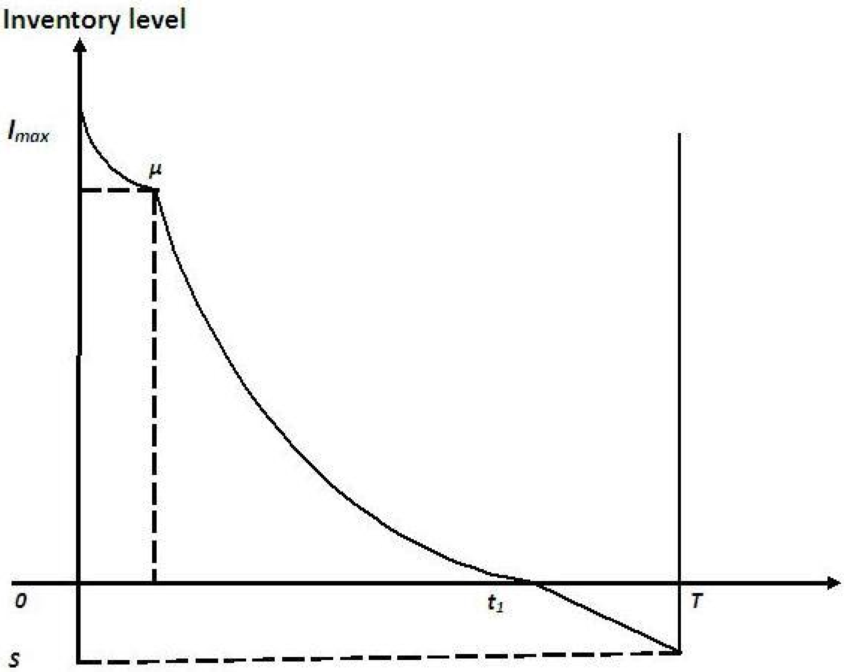

Figure 4.2: Graphical presentation of the inventory system (Case 2 : $t_1 \geq \mu$)

Solution of (4.3) is of the form

$$I(t) \;=\; d_0(1+\rho-t)\left[t+(1+\rho)\ln\left(\frac{1+\rho-t}{1+\rho}\right)+\frac{I_{max}}{d_0(1+\rho)}\right], 0\le t\le t_1 \quad (4.6)$$

Considering the boundary condition $I(t_1)=0$ and (4.6), the maximum inventory level for each cycle is

$$I_{max} \;=\; d_0(1+\rho)\left[(1+\rho)\ln\left(\frac{1+\rho}{1+\rho-t_1}\right)-t_1\right] \quad (4.7)$$

After substituting I_{max} into (4.6), it becomes

$$I(t) \;=\; d_0(1+\rho-t)\left[(1+\rho)\ln\left(\frac{1+\rho-t}{1+\rho-t_1}\right)+(t-t_1)\right], 0\le t\le t_1 \quad (4.8)$$

Solving (4.4) and (4.5) with the boundary conditions, one obtains

$$I(t) \;=\; \frac{d_0}{2}\left(t_1^2-t^2\right), \quad t_1\le t\le \mu \quad (4.9)$$

$$I(t) \;=\; d_0\mu(T-t)-S, \quad \mu\le t\le T \quad (4.10)$$

Considering the continuity of $I(t)$ at $t=\mu$, it follows from (4.9) and (4.10) that the maximum amount of demand backlogged per cycle is

$$S \;=\; d_0\left[\mu T-\frac{1}{2}\left(\mu^2+t_1^2\right)\right] \quad (4.11)$$

Substituting the value of S, (4.10) becomes

$$I(t)=d_0\left[\frac{1}{2}\left(\mu^2+t_1^2\right)-\mu t\right], \quad \mu\le t\le T \quad (4.12)$$

Thus, the order quantity Q is

$$Q = I_{max} + S = d_0 \left\{ (1+\rho) \left[(1+\rho) \ln \left(\frac{1+\rho}{1+\rho-t_1} \right) - t_1 \right] + \mu T - \frac{1}{2} \left(\mu^2 + t_1^2 \right) \right\}$$

Now, the total cost per cycle time consists of the following values

Ordering cost per cycle $OC = C_o$

Purchase cost per cycle

$$\begin{aligned} PC &= C_p Q \\ &= C_p d_0 \left\{ (1+\rho) \left[(1+\rho) \ln \left(\frac{1+\rho}{1+\rho-t_1} \right) - t_1 \right] + \mu T - \frac{1}{2} \left(\mu^2 + t_1^2 \right) \right\} \end{aligned}$$

Holding cost per cycle

$$\begin{aligned} HC &= \int_0^{t_1} C_h(t) I(t) dt = \int_0^{t_1} (h_0 + h_1 t) I(t) dt \\ &= d_0 \left[\left\{ \frac{(1+\rho)^3}{2} \ln \left(\frac{1+\rho}{1+\rho-t_1} \right) + \frac{t_1}{12} \left[2t_1^2 - 3t_1(1+\rho) - 6(1+\rho)^2 \right] \right\} h_0 \right. \\ &+ \left. \left\{ \frac{(1+\rho)^4}{6} \ln \left(\frac{1+\rho}{1+\rho-t_1} \right) + \frac{1}{36} \left[3t_1^4 - 6t_1(1+\rho)^3 - 3t_1^2(1+\rho)^2 - 2t_1^3(1+\rho) \right] \right\} h_1 \right] \end{aligned}$$

Deterioration cost per cycle

$$\begin{aligned} DC &= C_d \left[I_{max} - \int_0^{t_1} d(t) dt \right] \\ &= C_d d_0 \left\{ (1+\rho) \left[(1+\rho) \ln \left(\frac{1+\rho}{1+\rho-t_1} \right) - t_1 \right] - \frac{t_1^2}{2} \right\} \end{aligned}$$

Shortage cost per cycle

$$\begin{aligned} SC &= C_s \int_{t_1}^{T} -I(t) dt \\ &= C_s d_0 \mu \left(\frac{\mu^2}{6} + \frac{t_1^3}{3\mu} - \frac{\mu T}{2} - \frac{T t_1^2}{2\mu} + \frac{T^2}{2} \right) \end{aligned}$$

Therefore, the average total cost per unit time under the condition $\mu \geq t_1$ is

$$
\begin{aligned}
TC_1(t_1) &= \frac{1}{T}\left[OC + PC + HC + DC + SC\right] \\
&= \frac{d_0}{T}\left[\frac{C_o}{d_0} + h_0\left\{\frac{(1+\rho)^3}{2}\ln\left(\frac{1+\rho}{1+\rho-t_1}\right) + \frac{t_1}{12}\left[2t_1^2 - 3t_1(1+\rho) - 6(1+\rho)^2\right]\right\}\right. \\
&+ \left\{\frac{(1+\rho)^4}{6}\ln\left(\frac{1+\rho}{1+\rho-t_1}\right) + \frac{1}{36}\left[3t_1^4 - 6t_1(1+\rho)^3 - 3t_1^2(1+\rho)^2\right.\right. \\
&- \left.\left. 2t_1^3(1+\rho)\right]\right\}h_1 + (C_p + C_d)(1+\rho)\left[(1+\rho)\ln\left(\frac{1+\rho}{1+\rho-t_1}\right) - t_1\right] \\
&+ \left. C_p\left(\mu T - \frac{\mu^2}{2} - \frac{t_1^2}{2}\right) - \frac{C_d t_1^2}{2} + C_s\mu\left(\frac{\mu^2}{6} + \frac{t_1^3}{3\mu} - \frac{\mu T}{2} - \frac{Tt_1^2}{2\mu} + \frac{T^2}{2}\right)\right] \qquad (4.13)
\end{aligned}
$$

4.3.2 Model 2

When $\mu \leq t_1$

In this case, the demand $d(t)$ is

$$
d(t) = \begin{cases} d_0 t, & 0 \leq t \leq \mu, \\ d_0\mu, & \mu \leq t \leq t_1, \\ d_0\mu, & t_1 \leq t \leq T. \end{cases}
$$

Hence, (4.1) and (4.2) reduce to the following equations

$$
\begin{aligned}
\frac{dI(t)}{dt} &= -\alpha(t)I(t) - d_0 t, \quad 0 \leq t \leq \mu, \quad \text{with } I(0) = I_{max} \qquad (4.14) \\
\frac{dI(t)}{dt} &= -\alpha(t)I(t) - d_0\mu, \quad \mu \leq t \leq t_1, \quad \text{with } I(t_1) = 0 \qquad (4.15) \\
\frac{dI(t)}{dt} &= -d_0\mu, \quad t_1 \leq t \leq T, \quad \text{with } I(t_1) = 0 \qquad (4.16)
\end{aligned}
$$

Utilizing the boundary conditions, the solution of (4.14),(4.15), and (4.16) are as follows:

$$I(t) = d_0(1+\rho-t)\left[t+(1+\rho)\ln\left(\frac{1+\rho-t}{1+\rho}\right)+\frac{I_{max}}{d_0(1+\rho)}\right], 0\le t\le \mu \quad (4.17)$$

$$I(t) = d_0\mu(1+\rho-t)\ln\left(\frac{1+\rho-t}{1+\rho-t_1}\right), \quad \mu\le t\le t_1 \quad (4.18)$$

$$I(t) = d_0\mu(t_1-t), \quad t_1\le t\le T \quad (4.19)$$

Considering the continuity of $I(t)$ at $t=\mu$, the maximum inventory level I_{max} is obtained from equations (4.17) and (4.18) as

$$I_{max} = d_0(1+\rho)\left[\mu\left\{\ln\left(\frac{1+\rho-\mu}{1+\rho-t_1}\right)-1\right\}-(1+\rho)\ln\left(\frac{1+\rho-\mu}{1+\rho}\right)\right] \quad (4.20)$$

Using the value of I_{max}, (4.17) reduces to

$$I(t) = d_0(1+\rho-t)\left[(1+\rho)\ln\left(\frac{1+\rho-t}{1+\rho-\mu}\right)+\mu\ln\left(\frac{1+\rho-\mu}{1+\rho-t_1}\right)+(t-\mu)\right] (4.21)$$

Putting $t=T$ in (4.19), the maximum amount of demand backlogged per cycle can be obtained as

$$S \equiv -I(T) = d_0\mu(T-t_1)$$

Thus, the order quantity Q is

$$\begin{aligned} Q &= I_{max}+S \\ &= d_0(1+\rho)\left[\mu\left\{\ln\left(\frac{1+\rho-\mu}{1+\rho-t_1}\right)-1\right\}-(1+\rho)\ln\left(\frac{1+\rho-\mu}{1+\rho}\right)\right]+d_0\mu(T-t_1)\end{aligned}$$

Now, the total cost per cycle time consists of the following:

Ordering cost per cycle $OC=C_o$

Purchase cost per cycle

$$\begin{aligned} PC &= C_pQ \\ &= C_p\left\{d_0(1+\rho)\left[\mu\left\{\ln\left(\frac{1+\rho-\mu}{1+\rho-t_1}\right)-1\right\}-(1+\rho)\ln\left(\frac{1+\rho-\mu}{1+\rho}\right)\right]+d_0\mu(T-t_1)\right\} \end{aligned}$$

Holding Cost per cycle

$$\begin{aligned} HC &= \int_0^{t_1} C_h(t)I(t)dt = \int_0^{\mu}(h_0+h_1t)I(t)dt + \int_{\mu}^{t_1}(h_0+h_1t)I(t)dt \\ &= d_0\Bigg\{h_0\left[\frac{1}{4}(1+\rho)(1+\rho-\mu)^2-\frac{1}{4}(1+\rho)^3+\frac{\mu}{12}(3t_1^2-\mu^2-6t_1(1+\rho))\right] \\ &+ [h_1(1+\rho)+3h_0]\frac{(1+\rho)^2}{6}\left[\mu\ln\left(\frac{1+\rho-\mu}{1+\rho-t_1}\right)+(1+\rho)\ln\left(\frac{1+\rho}{1+\rho-\mu}\right)\right] \\ &+ \frac{h_1\mu}{36}\left[(3\mu-6t_1)(1+\rho)^2+(\mu^2-18\rho-3t_1^2)(1+\rho)-6(1+\rho^3)-\mu^3+4t_1^3\right]\Bigg\} \end{aligned}$$

Deterioration cost per cycle

$$\begin{aligned} DC &= C_d\left[I_{max}-\int_0^{t_1}d(t)dt\right] \\ &= C_dd_0\left\{(1+\rho)\left[\mu\left\{\ln\left(\frac{1+\rho-\mu}{1+\rho-t_1}\right)-1\right\}-(1+\rho)\ln\left(\frac{1+\rho-\mu}{1+\rho}\right)\right]-\left[\frac{\mu^2}{2}+\mu(t_1-\mu)\right]\right\} \end{aligned}$$

Shortage cost per cycle

$$\begin{aligned} SC &= C_s\int_{t_1}^{T} -I(t)dt \\ &= \frac{C_sd_0\mu}{2}(T-t_1)^2 \end{aligned}$$

Therefore, the average total cost per unit time under the condition $\mu \leq t_1$ is

$$\begin{aligned}
TC_2(t_1) &= \frac{1}{T}\left[OC + PC + HC + DC + SC\right] \\
&= \frac{d_0}{T}\left\{\frac{C_o}{d_0} + h_0\left[\frac{1}{4}(1+\rho)(1+\rho-\mu)^2 - \frac{1}{4}(1+\rho)^3 + \frac{\mu}{12}\left[3t_1^2 - \mu^2 - 6t_1(1+\rho)\right]\right]\right. \\
&+ \frac{(1+\rho)^2}{6}\left[h_1(1+\rho) + 3h_0\right]\left[\mu \ln\left(\frac{1+\rho-\mu}{1+\rho-t_1}\right) + (1+\rho)\ln\left(\frac{1+\rho}{1+\rho-\mu}\right)\right] \\
&+ \frac{h_1\mu}{36}\left[(3\mu - 6t_1)(1+\rho)^2 + (\mu^2 - 18\rho - 3t_1^2)(1+\rho) - 6(1+\rho^3) - \mu^3 + 4t_1^3\right] \\
&+ (C_p + C_d)(1+\rho)\left[\mu\left\{\ln\left(\frac{1+\rho-\mu}{1+\rho-t_1}\right) - 1\right\} - (1+\rho)\ln\left(\frac{1+\rho-\mu}{1+\rho}\right)\right] \\
&+ \left. C_p\mu(T-t_1) - C_d\left[\frac{\mu^2}{2} + \mu(t_1-\mu)\right]\right\} + \left.\frac{C_s\mu}{2}(T-t_1)^2\right\} \qquad (4.22)
\end{aligned}$$

From above analysis, one can obtain the total average cost of the system over the interval $[0, T]$ of the form

$$TC(t_1) = \begin{cases} TC_1(t_1) & \text{if } \mu \geq t_1 \\ TC_2(t_1) & \text{if } \mu \leq t_1 \end{cases} \qquad (4.23)$$

It is easy to check that this function is continuous at μ.

4.3.3 Solution procedure

This section provides the results which can ensure the existence of a unique t_1 to minimize the total average cost for the model.

Taking first and second order derivative of $TC_1(t_1)$ with respect to t_1, one has

$$\frac{dTC_1(t_1)}{dt_1} = \xi(t_1)f(t_1) \qquad (4.24)$$

and

$$\frac{d^2TC_1(t_1)}{dt_1^2} = \xi(t_1)\frac{df(t_1)}{dt_1} + \frac{d\xi(t_1)}{dt_1}f(t_1) \tag{4.25}$$

where

$$\begin{aligned} f(t_1) &= \frac{(1+\rho)^2\{3h_0 + (1+\rho)h_1\}}{6(1+\rho-t_1)} - \frac{1}{6}h_1\left\{(1+\rho)(1+\rho-t_1) - 2t_1^2\right\} \\ &- \frac{1}{2}h_0(1+\rho-t_1) + \frac{t_1(C_d+C_p)}{1+\rho-t_1} - C_s(T-t_1) \end{aligned} \tag{4.26}$$

On the other hand

$$f(0) = -C_sT < 0 \tag{4.27}$$

and

$$f(T) = \frac{(1+\rho)}{6}\left(\frac{3h_0T + h_1T^2}{1+\rho-T}\right) + \frac{1}{6}(3h_0T + 2h_1T^2) + \frac{T(C_d+C_p)}{1+\rho-T} > 0 \tag{4.28}$$

Further

$$\begin{aligned} \frac{df(t_1)}{dt_1} &= C_s + \frac{(1+\rho)^2[3h_0 + (1+\rho)h_1]}{6(1+\rho-t_1^2)} + \frac{(1+\rho)(C_d+C_p)}{(1+\rho-t_1)^2} \\ &+ \frac{2}{3}h_1t_1 + \frac{1}{2}h_0 - \frac{1}{6}h_1(1+\rho) > 0 \end{aligned} \tag{4.29}$$

From (4.27),(4.28), and (4.29), one can conclude that $f(t_1)$ is strictly increasing function of t_1. Therefore the equation

$$f(t_1) = 0 \tag{4.30}$$

has a unique root $t_1^* \in (0, T)$ for which

$$\left.\frac{d^2TC_1(t_1)}{dt_1^2}\right|_{t_1=t_1^*} = \xi(t_1^*)\frac{df(t_1^*)}{dt_1^*} + \frac{d\xi(t_1^*)}{dt_1^*}f(t_1^*) = \xi(t_1^*)\frac{df(t_1^*)}{dt_1^*} > 0 \qquad (4.31)$$

so that t_1^* corresponds to the unconstrained global minimum of $TC(t_1)$.

Taking first and second order derivative of $TC_2(t_1)$ with respect to t_1, one has

$$\frac{dTC_1(t_1)}{dt_1} = \xi(\mu)f(t_1) \qquad (4.32)$$

and

$$\frac{d^2TC_1(t_1)}{dt_1^2} = \xi(\mu)\frac{df(t_1)}{dt_1} > 0 \qquad (4.33)$$

where $f(t_1)$ is given by (4.26). From the inequality in (4.33) follows from (4.29), one can ensure the strict convexity of $TC_2(t_1)$.

Based on the properties of $f(t_1)$, it can be concluded that the first order derivative of $TC_2(t_1)$ with respect to t_1 is vanished at the point t_1^* which is a unique solution of $f(t_1) = 0$. This t_1^* corresponds to the unconstrained global minimum of $TC_2(t_1)$.

4.4 Numerical examples

This section considers some numerical examples to check the uniqueness of the solution. Some input parametric values are considered to obtain the optimum results.

Example 4.1

To derive the optimal solution, two examples are given which consist of the different situations of the ramp-type demand and deteriorated rates. Let us consider the following parametric values: $d_0 = 400$, $C_o = \$50$/order, $h_0 = \$0.1$/unit/unit time, $h_1 = \$0.2$/unit/unit

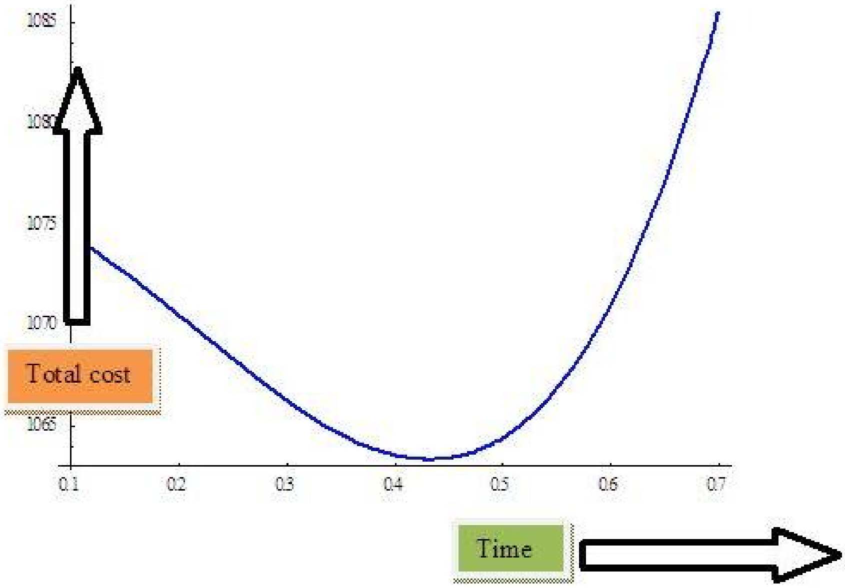

Figure 4.3: Variation of total cost versus time (Example 4.1)

time $C_s = \$1/\text{unit}$, $C_p = \$5/\text{unit}$, $C_d = \$1.5/\text{unit}$, $\mu = 0.8$, $T = 1$ week, $\rho = 5$ months. The optimal value of t_1 is $t_1^* = 0.4318(< \mu)$ week and the minimum cost per unit of time is $Z_1^*(t_1) = \$1063.33/\text{week}$ (See Figure 4.3).

Example 4.2

All the parametric values are identical to Example 4.1 except that $\mu = 0.2$. The optimal value of t_1 is $t_1^* = 0.4318(> \mu)$ week and the minimum cost per unit of time is $Z_2^*(t_1) = \$431.74/\text{week}$ (See Figure 4.4).

4.5 Sensitivity analysis

In this section, the effects of changes in parameters such as $C_o, C_s, C_p, C_d, h_1, h_2$ and L on the total cost are studied. The sensitivity analysis is performed by changing each of the parameters by $-50\%, -25\%, +25\%$, and $+50\%$, taking one parameter at a time while

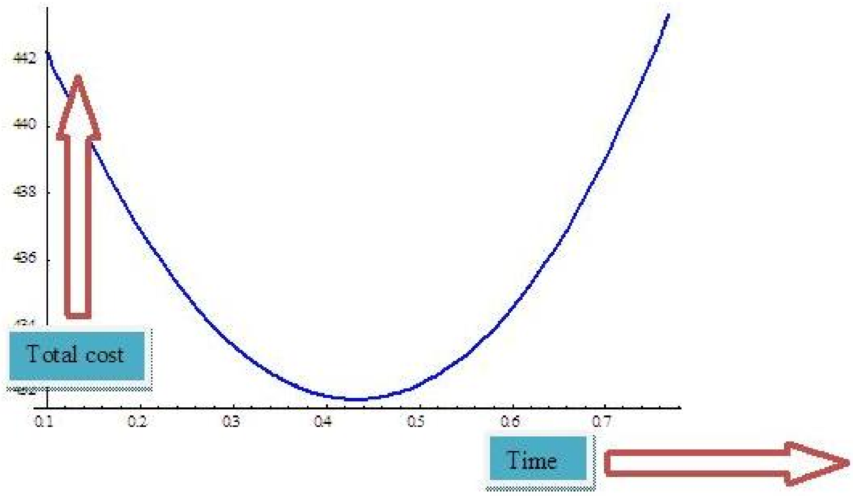

Figure 4.4: Variation of total cost versus time (Example 4.2)

keeping the remaining parameters unchanged. The results are presented in Table 4.1. On the basis of Table 4.1, the following features are observed:

* Increasing value of ordering cost C_o increases the material cost, shipping cost, placing order's cost; as a result the total relevant cost increases. From Table 4.1, it is observed that this parameter is highly sensitive cost parameter for Model II compare to Model I. For the positive change and negative change in total cost for this parameter, it follows symmetrical change.

* Increasing value of shortage cost indicates more shortages in the system that implies increase in total cost. From Table 4.1, one can notice that this parameter is slightly sensitive on total cost for both model.

* From Table 4.1, it is noticed that slight change in purchasing cost results larger change in total cost for both model.

* Increasing value of deterioration cost and holding cost increase the total cost. For both models, these two parameters are slightly sensitive on total cost.

Table 4.1: Effect on total cost for changes in the parameters for Model 1 and Model 2

Parameters	Changes(in %)	$Z_1(t_1)$	$Z_2(t_2)$
	−50%	−02.35	−05.79
	−25%	−01.18	−02.90
C_o	+25%	+01.18	+02.90
	+50%	+02.35	+05.79
	−50%	−02.16	−01.83
	−25%	−01.00	−00.83
C_s	+25%	+00.87	+00.68
	+50%	+01.62	+01.24
	−50%	−45.75	−42.57
	−25%	−22.83	−21.24
C_p	+25%	+22.76	+21.16
	+50%	+45.49	+42.28
	−50%	−00.14	−00.22
	−25%	−00.07	−00.11
C_d	+25%	+00.06	+00.10
	+50%	+00.12	+00.20
	−50%	−00.05	−00.08
	−25%	−00.03	−00.04
h_1	+25%	+00.02	+00.04
	+50%	+00.05	+00.08
	−50%	−00.02	−00.02
	−25%	−00.01	−00.01
h_2	+25%	+00.01	+00.01
	+50%	+00.02	+00.02

4.6 Conclusions

This study considered an inventory model for products with maximum lifetime, time-varying deterioration rate, and ramp-type demand. A simple solution procedure was given and existence and uniqueness of the optimal solutions were obtained analytically. This model minimized the associated cost function at the optimal values of the decision variable. Sensitivity analysis on the optimal solution with respect to key parameters were studied to illustrate the model and some managerial insights were provided. This model used the concept of fixed lifetime of products as in time-varying deterioration rate. The managers of the different industries, where products have fixed lifetime and the deterioration rate is time-dependent, can follow this strategies. This model can be extended for items having linear increasing demand, price, and advertising-dependent demand or power-demand. This study can be extended by assuming multi-item inventory model with inflations.

Chapter 5

Flexible setup cost and deterioration of products in a supply chain model

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5.1 Introduction

Supply chain management (SCM) involves the movement and storage of raw materials and finished goods from point of origin to point of consumption. SCM obtains its importance in global market and network economy as organizations rely increasingly on effective supply chains or networks. Recently, Cárdenas-Barrón and Treviño-Garza (2014) developed an excellent model for an optimal solution to a three-echelon supply chain network. Chung *et al.* (2014) discussed an inventory model with non-instantaneous receipt and exponentially deteriorating items for an integrated three layer supply chain system under two levels of trade credit policy. Taleizadeh and Cárdenas-Barrón (2013) developed a metaheuristic algorithm for supply chain management problems.

In this direction, Goyal (1976) developed a single supplier-single buyer integrated inventory model. Banerjee (1986) derived a joint economic lot size model for the purchaser and the vendor with lot-for-lot policy. Hill (1997) discussed a single-vendor single-buyer integrated production-inventory model as a general policy. Viswanathan and Piplani (2001) explained a coordinating supply chain inventory through common replenishment epochs. Yang and Wee (2002) derived an economic lot size model in an integrated vendor-buyer inventory system without derivatives. Sarkar and Majumder (2013) developed an integrated vendor-buyer supply chain model with vendor's setup cost reduction. Sarkar *et al.* (2015) proposed a continuous review inventory model with setup cost reduction, quality improvement, and a service level constraint. Sarkar *et al.* (2015) discussed an inventory model with quality improvement and setup cost reduction under controllable lead time.

Kim and Ha (2002) proposed a just-in-time (JIT) lot size model to enhance the buyer-supplier linkage. They explained about the single-setup multiple-delivery (SSMD) policy. They proved that SSMD policy is more effective than single-setup single-delivery (SSSD) policy. Khouja (2003) presented an optimizing inventory decisions in a multi-stage multi-customer supply chain model. Cárdenas-Barrón (2007) discussed a note on optimizing inventory decisions in a multi-stage multi-customer supply chain model. Cárdenas-Barrón (2008) developed an optimal manufacturing batch-size with rework in a single-stage production system. Cárdenas-Barrón (2011b) discussed an algebraical procedure to optimize different types of economic order quantity/economic production quantity (EOQ/EPQ) model with the help of basic algebra.

Yan *et al.* (2011) extended Kim and Ha's (2002) model with a constant deterioration rate. Widyadana and Wee (2011) developed an EPQ model for deteriorating items with preventive maintenance policy and random machine breakdown. Teng *et al.* (2011) derived an economic lot size model of the integrated vendor-buyer inventory system without using any derivative. Teng *et al.* (2011) extended an inventory model for buyer-distributor-vendor supply chain with backlogging without derivatives. Chung and Cárdenas-Barrón (2012) found out a complete solution procedure for the EOQ and EPQ inventory models with linear and fixed backorder costs. Sett *et al.* (2012) developed a two-warehouse inventory model with increasing demand and time varying deterioration. They considered the maximum lifetime of products. Sarkar and Saren (2014) established a partial trade-credit model for retailer with exponentially deterioration. Sarkar *et al.* (2014) considered a deteriorating inventory model with trade-credit policy for fixed lifetime products.

The process of degradation of items over time is basically perceived as deterioration. Ghare and Schrader (1963) were the first authors to consider exponential deterioration in an inventory model. Covert and Philip (1973) later discussed an EOQ model for deteriorating items with Weibull distribution. Misra (1975) proposed an optimal production lot size model with deterioration function. Goyal (1987) developed an economic ordering policy for deteriorating items over an infinite time horizon. Datta and Pal (1988) proposed an order-level inventory system with a power demand pattern and variable deterioration rate. Raafat (1991) made a literature survey on continuously deteriorating inventory model. The inventory models with different types of deteriorating rates were extended by Chang and Dye (1999), Skouri and Papachristos (2003), Skouri *et al.* (2009), Sarkar (2012a), Sarkar *et al.* (2013), Sarkar and Sarkar (2013a, 2013b), etc.

Reliability is the ability of a system to perform adequately and maintain its function under routine circumstances. More reliability implies less deteriorating rate of the manufactured items. Thus, the system has to be more reliable to reduce the production of defective items. Sarkar *et al.* (2010b) explained an economic manufacturing quantity (EMQ) model with optimal reliability, production lot size, and safety stock. Sarkar (2012c)explained an inventory model with reliability in an imperfect production process. Sarkar *et al.* (2014) developed an EMQ model with price and time dependent demand under the effect of reliability and inflation.

Recently, Sarkar (2013) developed a SCM model with fixed setup cost and deterioration cost which is an extension of Yan *et al.*'s (2011) model. This study extends Sarkar's (2013) model by considering reliability as a decision variable. Setup cost is directly proportional and the deterioration rate is inversely proportional to the reliability. Therefore, with the increase in reliability the setup cost increases and the deterioration rate decreases. By using algebraical procedure, the total system cost is minimized and obtain a closed-form solution is obtained. There is absolutely no need to use calculus. The orientation of this study is as follows: immediate section contains assumptions. Section 5.3, contains model formulation and solution procedure. In Section 5.4, the model is illustrated by using numerical examples. Sensitivity analysis is studied in Section 5.5. Finally, in Section 5.6, conclusions and the future extensions of the model have been made.

5.2 Assumptions

Following assumptions are made to develop the model.

1. Single type of item is produced by the production-inventory system.

2. Setup cost S_0 and deterioration cost C_d depend on the reliability parameter R.

3. Information regarding the inventory position and demand of the buyer are given to the supplier.

4. Production rate is greater than demand, i.e., $p > d$.

5. Handling and transportation costs are paid by the buyer.

6. Shortage and backlogging are not considered.

5.3 Model formulation

A single-setup-multiple-delivery (SSMD) production is considered in this research. The quantity ordered by the buyer is manufactured at a time and the ready products are delivered after a fixed time interval over multiple deliveries in an equal amount. The splitting of the order quantity into multiple lots is consistent with JIT implementation. The average total cost of the production-inventory model is developed for the buyer's and the supplier's which is then minimized. Without any loss of generality, it is considered that the products arrive at the exact time when the items from the previous delivery has just been depleted. Two inventory versus time graphs for the buyer and the supplier, respectively are shown in Figure 5.1 and Figure 5.2. The total time span T is divided into two components: t_1, the production time duration for the supplier and t_2, the non-production time duration for the supplier. t_3 is considered as the time duration between the two successive deliveries. Calculations are separately done for the buyer's and the supplier's inventory cost.

Decision variables

q delivery lot size (units)

N number of deliveries per production-batch, $N \geq 1$

R reliability parameter

5.3.1 Inventory cost for the buyer

There are three well-known conditions which must prevail for the algebraic method to be used as an appropriate optimization method to minimize a function comprised by several functions and each function with one and more variables. These conditions are that:

(1) functions must be positive functions;

(2) product of the functions must be a constant;

(3) when these functions are equalized; the system of equations can be solved.

Let x be the number of deteriorating items during the time span t_3, then the delivery lot size is given by

$$q = x + dt_3$$

The delivery lot size q is divided into two components: dt_3 and x. dt_3 is for the consumption and x represents the number of deteriorating items. Since the deterioration rate is small, its square and higher powers can be neglected. Hence, during time interval t_3, x can be treated as the deterioration of q units.

Therefore,

$$q = t_3(d + \frac{\alpha q}{2}), \text{ (see for instance Yan et al. (2011))}$$

and

$$q = \frac{T}{N}\left(d + \frac{\alpha q}{2}\right), \quad \text{because } \frac{T}{N} = t_3$$

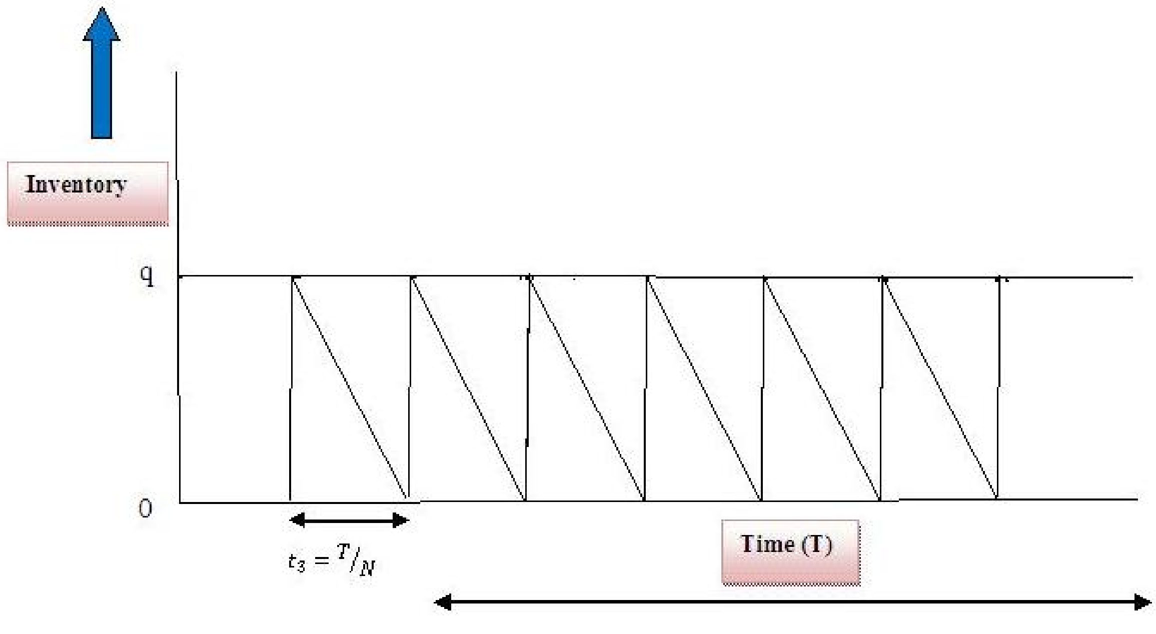

Figure 5.1: Buyer's inventory model, adopted from Sarkar [2013]

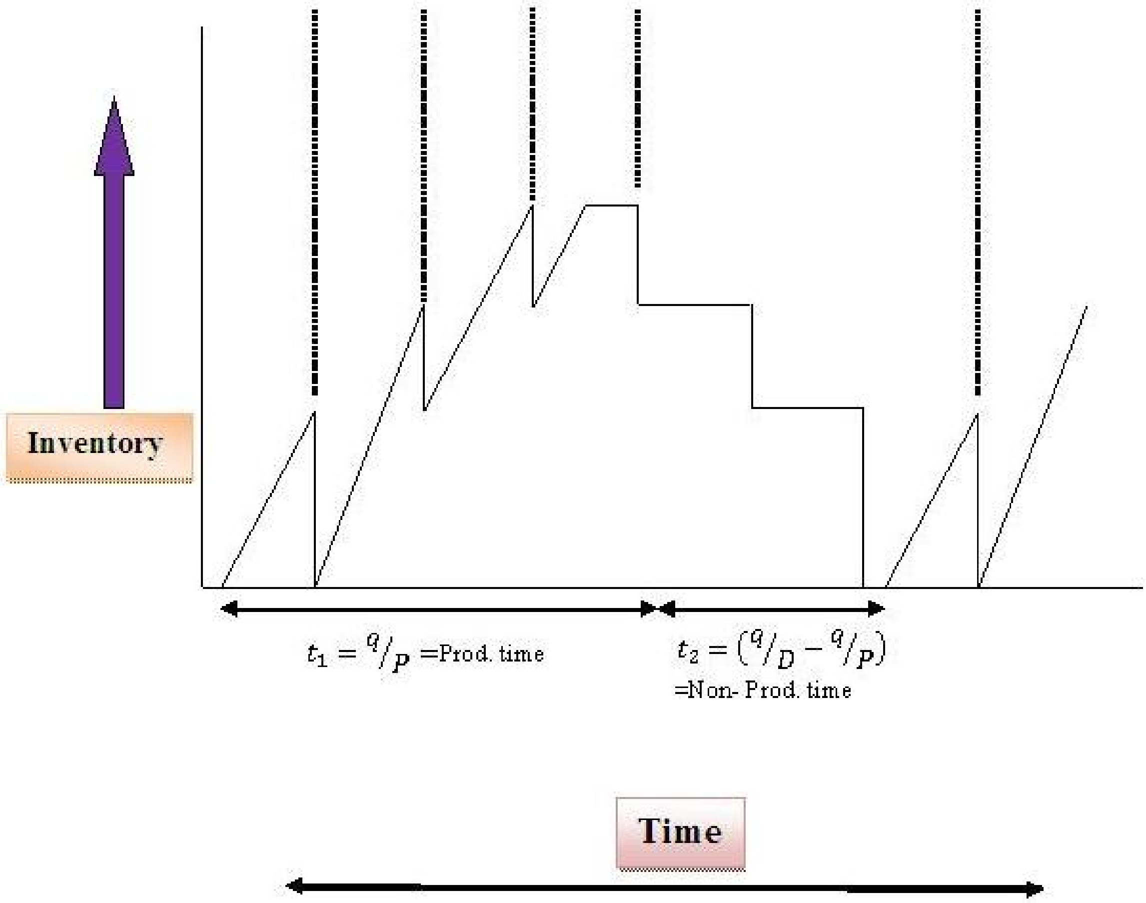

Figure 5.2: Supplier's inventory model, adopted from Sarkar [2013]

Now

$$q = x + dt_3$$

implies

$$\frac{1}{T} = \frac{d}{Nq} + \frac{\alpha}{2N} \tag{5.1}$$

i.e.,

$$\frac{q}{2} = \frac{Nq}{\alpha T} - \frac{d}{\alpha}$$

Again the total deterioration for the buyer is obtained as

$$\alpha A_b = Nq - dT$$

which implies

$$A_b = \frac{(Nq - dT)}{\alpha}$$

i.e.,

$$\frac{A_b}{T} = \frac{q}{2} \tag{5.2}$$

It is considered that the deterioration rate α is inversely proportional to reliability, i.e., $\alpha \propto \frac{1}{R}$ which indicates $\alpha = \frac{\sigma}{R}$, where σ is the proportionality constant. Now the relevant costs for the buyer's are

1. Ordering cost per unit time $= \frac{C_o}{T}$

2. Holding cost per unit per unit time $= \frac{HC_bA_b}{T}$

3. Deterioration cost per unit time $= \frac{\sigma C_dA_b}{RT}$

4. Transportation cost and handling cost per unit time $= \frac{(NK+V_cNq)}{T}$

Therefore, the buyer's total cost function is obtained as

$$TC_b = \frac{1}{T}\left[C_o + HC_bA_b + \frac{\sigma C_d}{R}A_b + NK + V_cNq\right]$$

Using (5.1) and (5.2), the buyer's total cost function becomes

$$TC_b = \left(\frac{d}{Nq} + \frac{\sigma}{2NR}\right)\left(C_o + NK + V_cNq\right) + \frac{q}{2}\left[HC_b + \frac{\sigma C_d}{R}\right]$$

5.3.2 Inventory cost for the supplier

Suppose y represents the number of deteriorating units for the supplier, which can be symbolized as $y = \alpha A_s$. $y + \alpha qT/2$ denotes the total number of deteriorating items for the entire SCM. Here, $Q = Nq + y$ and $t_1 = \frac{Q}{p}$. Considering the initial and the total inventory for the entire SCM, one can obtained

$$y + \frac{\alpha qT}{2} = \frac{\alpha T}{2p}\{2dq + (Nq+y)(p-d)\}$$

Hence,

$$\begin{aligned} A_s &= \frac{y}{\alpha} \\ &= qT\left(\frac{d}{p} + \frac{N-1}{2} - \frac{dN}{2p}\right) \end{aligned} \tag{5.3}$$

If the variable setup cost S_1 is directly proportional to reliability R, i.e., $S_1 \propto R$, i.e., $S_1 = \rho R$, where ρ is the proportionality constant, then, setup cost is the sum of S_o and

ρR. Now the relevant costs for the supplier's are

1. Setup cost per unit time $= \frac{S_o+\rho R}{T}$

2. Holding cost per unit per unit time $= \frac{HC_s A_s}{T}$

3. Deterioration cost per unit time $= \frac{\sigma C_d A_s}{RT}$

Now the equation for the supplier's total cost function can be written as

$$TC_s = \frac{1}{T}\left(S_o + \rho R + HC_s A_s + \frac{\sigma C_d}{R} A_s\right)$$

Using (5.1) and (5.3), the supplier's total cost function is

$$TC_s = \left(\frac{d}{Nq} + \frac{\sigma}{2NR}\right)(S_o + \rho R) + q\left(HC_s + \frac{\sigma C_d}{R}\right)\left(\frac{d}{p} + \frac{N-1}{2} - \frac{dN}{2p}\right)$$

Integrated inventory cost for the entire SCM

The total average cost for the entire SCM is $TC(q, N, R) = TC_b + TC_s$

$$\begin{aligned} TC(q,N,R) &= \left(\frac{d}{Nq} + \frac{\sigma}{2NR}\right)[C_o + (S_o + \rho R) + NK + V_c Nq] + \frac{q}{2}\left(HC_b + \frac{\sigma C_d}{R}\right) \\ &+ \left(HC_s + \frac{\sigma C_d}{R}\right)\left\{\frac{(2-N)d}{p} + N - 1\right\} \end{aligned} \quad (5.4)$$

Minimum order quantity

The required ordered quantity that makes the SSMD policy superior to single-delivery policy is obtained from the savings. Substituting $N = 1$ in (5.4) and subtracting that result from (5.4), one can obtain

$$\begin{aligned} SV(q,N,R) &= \left(\frac{d}{q} + \frac{\sigma}{2R}\right)\{C_o + (S_o + \rho R)\}\left(1 - \frac{1}{N}\right) \\ &+ \frac{q}{2}\left(HC_s + \frac{\sigma C_d}{R}\right)(N-1)\left(\frac{d}{p} - 1\right) \end{aligned} \quad (5.5)$$

When $N = 1$, the saving vanishes. It can be shown that (5.5) is concave and increases at a diminishing rate as the ordered quantity increases which implies that larger ordered quantity indicates more benefit for the supplier and the buyer over a long term contract. The minimum ordered quantity that makes the SSMD policy favorable over the single-delivery policy is obtained by solving $SV(q, N, R) \geq 0$ for q.

Therefore, $SV(q, N, R) \geq 0$ gives

$$q^2 N(HC_s R + \psi C_d)(d - p) + (C_o + S_o + \rho R)p(\sigma q + 2Rd) \geq 0 \tag{5.6}$$

The left hand side (LHS) of the inequality is quadratic in q. Considering the equality and solving for q (suppose, the roots are q_1 and q_2), the LHS of the inequality gives

$$\begin{aligned} q_1 &= \frac{1}{2N(HC_s R + \sigma C_d)(p - d)} \Big\{ p\sigma(C_o + S_o + \rho R) \\ &+ \sqrt{[\sigma(C_o + S_o + \rho R)p]^2 - 8NdpR(HC_s R + \sigma C_d)(d - p)(C_o + S_o + \rho R)} \Big\} \end{aligned}$$

As $(d-p)$ is always less than zero therefore, without any loss of generality, q_1 acquires a positive value.

$$\begin{aligned} q_2 &= \frac{1}{2N(HC_s R + \sigma C_d)(p - d)} \Big\{ -(p\sigma(C_o + S_o + \rho R)) \\ &+ \sqrt{[\sigma(C_o + S_o + \rho R)p]^2 - 8NdpR(HC_s R + \sigma C_d)(d - p)(C_o + S_o + \rho R)} \Big\} \end{aligned}$$

In order to find the nature of the root given by q_2, 'Descartes' rule of signs is considered, which indicates that the equation $q^2 N(HC_s R + \sigma C_d)(d - p) + (C_o + S_o + \rho R)p(\sigma q + 2dR) = 0$ has only one positive real root given by q_1 and hence, q_2 is neglected.

From (5.4), one can find

$$\begin{aligned}
TC(q,N,R) &= \frac{q}{2}\left[\left(HC_b+\frac{\psi C_d}{R}\right)+\left(HC_s+\frac{\sigma C_d}{R}\right)\left(\frac{(2-N)d}{p}+N-1\right)+\frac{V_c\sigma}{R}\right] \\
&+ \frac{1}{q}\left[\frac{d}{N}(C_o+S_o+\rho R+NK)\right]+dV_c+\frac{\sigma}{2NR}(C_o+S_o+\rho R+NK)
\end{aligned}$$

Taking fast order partial derivatives of $TC(q,N,R)$ and equating to zero, it is obtained as

$$\begin{aligned}
\frac{\partial TC}{\partial q} &= \frac{1}{2}\left[\frac{C_d\sigma}{R}+\frac{V_c\sigma}{R}+C_b+\left(\frac{d(2-N)}{p}+N-1\right)\left(\frac{C_d\sigma}{R}+C_s\right)\right]-\frac{d(C_o+KN+R\rho+S_0)}{Nq^2} \\
&= 0 \\
\Rightarrow q &= \sqrt{\frac{2Rdp(C_o+S_o+\rho R+NK)}{N\left[(pRHC_b+\sigma C_d p)+(HC_sR+\sigma C_d)\{(2-N)d+pN-p\}+V_c\sigma p\right]}} \\
\frac{\partial TC}{\partial N} &= \frac{dK}{Nq}+\frac{K\sigma}{2NR}+\frac{1}{2}q\left(1-\frac{d}{p}\right)\left(\frac{C_d\sigma}{R}+C_s\right)-\frac{d(C_o+NK+R\rho+S_0)}{N^2q} \\
&- \frac{\psi}{2NR^2}(C_o+KN+r\rho+S_0)=0 \\
\Rightarrow N &= \sqrt{\frac{p(2Rd+\sigma q)(C_o+S_o+\rho R)}{q^2(p-d)(HC_sR+\sigma C_d)}} \\
\frac{\partial TC}{\partial R} &= \frac{q}{2}\left[-\frac{C_d\sigma}{R^2}-\frac{\sigma C_d}{R^2}\left(\frac{(2-N)d}{p}+N-1\right)-\frac{V_c\psi}{R^2}\right]+\frac{d\rho}{Nq}-\frac{\sigma}{2NR^2}(C_o+S_0+NK)=0 \\
\Rightarrow R &= \sqrt{\frac{\sigma q\left[p\{C_o+S_o+NK+V_cNq\}+C_dNq\{(2-N)d+Np\}\right]}{2pd\rho}}
\end{aligned}$$

When N and R are fixed, TC can be written in the symbolic form as

$$TC(q) = x_1q+\frac{x_2}{q}+x_3=\frac{x_1}{q}\left(q-\sqrt{\frac{x_2}{x_1}}\right)^2+2\sqrt{x_1x_2}+x_3$$

where

$$
\begin{aligned}
x_1 &= \frac{1}{2}\left[\left(HC_b+\frac{\psi C_d}{R}\right)+\left(HC_s+\frac{\sigma C_d}{R}\right)\left(\frac{(2-N)d}{p}+N-1\right)+\frac{V_c\sigma}{R}\right]\\
x_2 &= \left[\frac{d}{N}(C_o+S_o+\rho R+NK)\right]
\end{aligned}
$$

and

$$
x_3 = dV_c+\frac{\sigma}{2NR}(C_o+S_o+\rho R+NK)
$$

Now, the expression $f(q)=x_1q+\frac{x_2}{q}+x_3=(\sqrt{x_1q})^2+\left(\sqrt{\frac{x_2}{q}}\right)^2+x_3=\left(\sqrt{x_1q}-\sqrt{\frac{x_2}{q}}\right)^2+2\sqrt{x_1x_2}+x_3$ attains its minimum when $q=\sqrt{\frac{x_2}{x_1}}$, [See for instance Sarkar (2013)] and the minimum cost is $2\sqrt{x_1x_2}+x_3$.

Therefore, $TC(q)$ is the minimum when

$$
\begin{aligned}
q &= \sqrt{\frac{x_2}{x_1}}\\
&= \sqrt{\frac{2Rdp(C_o+S_o+\rho R+NK)}{N\left[(pRHC_b+\sigma C_d p)+(HC_sR+\sigma C_d)\{(2-N)d+pN-p\}+V_c\sigma p\right]}}
\end{aligned}
$$

and the minimum cost is

$$
\begin{aligned}
TC(q) &= 2\sqrt{x_1x_2}+x_3\\
&= \left[2\left\{\left(HC_b+\frac{\sigma C_d}{R}\right)+\left(HC_s+\frac{\sigma C_d}{R}\right)\left(\frac{(2-N)d}{p}+N-1\right)+\frac{V_c\sigma}{R}\right\}\right.\\
&\quad \left.\left\{\frac{d}{N}(C_o+S_o+\rho R+NK)\right\}\right]^{1/2}+dV_c+\frac{\sigma}{2NR}(C_o+S_o+\rho R+NK) \qquad (5.7)
\end{aligned}
$$

When q and R are fixed

$$\begin{aligned} TC(N) &= N\frac{q}{2}\left(\frac{p-d}{p}\right)\left(HC_s+\frac{\sigma C_d}{R}\right)+\frac{1}{N}\left(\frac{d}{q}+\frac{\sigma}{2R}\right)(C_o+S_o+\rho R) \\ &+ \frac{q}{2}\left[\left(HC_s+\frac{\sigma C_d}{R}\right)\left(\frac{2d-p}{p}\right)+\left(HC_b+\frac{\sigma C_d}{R}\right)\right] \\ &+ \left(\frac{d}{q}+\frac{\sigma}{2R}\right)(K+V_c q) \end{aligned} \tag{5.8}$$

which can be written in the symbolic form as

$$TC(N) = x_4 N + \frac{x_5}{N} + x_6$$

where

$$\begin{aligned} x_4 &= \frac{q}{2}\left(\frac{p-d}{p}\right)\left(HC_s+\frac{\sigma C_d}{R}\right) \\ x_5 &= \left(\frac{d}{q}+\frac{\sigma}{2R}\right)(C_o+S_o+\rho R) \end{aligned}$$

and

$$x_6 = \frac{q}{2}\left[\left(HC_s+\frac{\sigma C_d}{R}\right)\left(\frac{2d-p}{p}\right)+\left(HC_b+\frac{\sigma C_d}{R}\right)\right]+\left(\frac{d}{q}+\frac{\sigma}{2R}\right)(K+V_c q).$$

$TC(N)$ is the minimum when

$$\begin{aligned} N &= \sqrt{\frac{x_5}{x_4}} \\ &= \sqrt{\frac{p(2Rd+\sigma q)(C_o+S_o+\rho R)}{q^2(p-d)(HC_sR+\sigma C_d)}} \end{aligned} \tag{5.9}$$

and the minimum cost is

$$
\begin{aligned}
TC(N) &= 2\sqrt{x_4 x_5} + x_6 \\
&= \sqrt{2q\left(\frac{p-d}{p}\right)\left(HC_s + \frac{\sigma C_d}{R}\right)\left(\frac{d}{q} + \frac{\sigma}{2R}\right)(C_o + S_o + \rho R)} \\
&+ \frac{q}{2}\left[\left(HC_s + \frac{\sigma C_d}{R}\right)\left(\frac{2d-p}{p}\right) + \left(HC_b + \frac{\sigma C_d}{R}\right)\right] \\
&+ \left(\frac{d}{q} + \frac{\sigma}{2R}\right)(K + V_c q)
\end{aligned}
\tag{5.10}
$$

When q and N are fixed

$$
\begin{aligned}
TC(R) &= R\frac{\rho d}{Nq} + \frac{1}{R}\left[\frac{\sigma}{2N}\{C_o + S_o + NK + V_c Nq\} + \frac{\sigma C_d q}{2}\left\{\frac{(2-N)d}{p} + N\right\}\right] \\
&+ \frac{d}{Nq}\{C_o + S_o + NK + V_c Nq\} + \frac{\sigma\rho}{2N} \\
&+ \frac{q}{2}\left[HC_b + HC_s\left\{\frac{(2-N)d}{p} + N - 1\right\}\right]
\end{aligned}
\tag{5.11}
$$

The above equation can be written in the form

$$
TC(R) = x_7 R + \frac{x_8}{R} + x_9
$$

where

$$
\begin{aligned}
x_7 &= \frac{\rho d}{Nq} \\
x_8 &= \left[\frac{\sigma}{2N}\{C_o + S_o + NK + V_c Nq\} + \frac{\sigma C_d q}{2}\left\{\frac{(2-N)d}{p} + N\right\}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
x_9 &= \frac{d}{Nq}\{C_o + S_o + NK + V_c Nq\} + \frac{\sigma\rho}{2N} \\
&+ \frac{q}{2}\left[HC_b + HC_s\left\{\frac{(2-N)d}{p} + N - 1\right\}\right]
\end{aligned}
$$

Therefore, $TC(R)$ is the minimum when

$$\begin{aligned} R &= \sqrt{\frac{x_8}{x_7}} \\ &= \sqrt{\frac{\sigma q\left[p\{C_o + S_o + NK + V_c Nq\} + C_d Nq\{(2-N)d + Np\}\right]}{2pd\rho}} \end{aligned} \tag{5.12}$$

and the minimum cost is

$$\begin{aligned} TC(R) &= 2\sqrt{x_7 x_8} + x_9 \\ &= \sqrt{\frac{2\rho d}{Nq}\left[\frac{\sigma}{N}\{C_o + S_o + NK + V_c Nq\} + \sigma C_d q\left\{\frac{(2-N)d}{p} + N\right\}\right]} \\ &+ \frac{d}{Nq}\{C_o + S_o + NK + V_c Nq\} + \frac{\sigma\rho}{2N} \\ &+ \frac{q}{2}\left[HC_b + HC_s\left\{\frac{(2-N)d}{p} + N - 1\right\}\right] \end{aligned} \tag{5.13}$$

Optimal interval of the lot size

By assumptions of N, the number of deliveries per production batch-cycle, must be greater than or equal to 1 and from the expression for optimum q, N attains its upper bound at $N = 1$, i.e.,

$$q \leq \sqrt{\frac{2pdR(C_o + S_o + \rho R + K)}{R\{pHC_b + dHC_s\} + \sigma\{C_d(d+p) + V_c p\}}}$$

As N, the number of deliveries per production batch-cycle increases, the corresponding lot size value q decreases hence, from the equation of optimum lot size, one can obtained

$$q \geq \sqrt{\frac{2pdR(C_o + S_o + \rho R + NK)}{\{HC_b R + \sigma(C_d + V_c)\}\{2d + N(p-d)\}N}}$$

5.3.3 Solution procedure

The total cost TC, the delivery lot size q, the number of deliveries per production batch N, and the reliability parameter R are obtained in this model. If the value of N, given by (5.9) is not an integer, then one can choose N in such a way, which gives $\min\{TC(N^+), TC(N^-)\}$ for the model where N^+ and N^- represent the closest integers larger or smaller than the optimal N^*. Optimal minimal cost, given by TC, is obtained by substituting the values of N^*, q^* and R^* in $TC(q^*, N^*, R^*)$.

5.4 Numerical examples

In this section, two numerical examples are given to illustrate the proposed model.

Example 5.1

The values of the following parameters are to be taken in appropriate units: $p = 13000$ units/year, $S_o = \$200$/batch, $HC_b = \$7$/unit/year, $HC_s = \$6$/unit/year, $d = 9000$ units/year, $C_o = \$25$/order, $K = \$10$/delivery, $V_c = \$1$/unit, $C_d = \$10$/unit, $\sigma = 0.1$, $\rho = 90$. Then, the optimal solution is $\{TC = \$13873.6$/year, $N = 12$/ production batch-cycle, $q = 126.82$ units, $R = 0.79\}$. Optimality of the cost function are shown in Figures 5.3, 5.4, 5.5.

This model is compared with that of Sarkar (2013) by using the same parametric values.

Example 5.2

The values of the following parameters are to be taken in appropriate units: $p = 10000$ units/year, $S_o = \$800$/batch, $HC_b = \$7$/unit/year, $HC_s = \$6$/unit/year, $d = 4800$ units/year, $C_o = \$25$/order, $K = \$50$/delivery, $V_c = \$1$/unit, $C_d = \$50$/unit, $\sigma = 0.02$, $\rho = 250$. Then, the optimal solution is $\{TC = \$14198.2$/year , $N = 6$/ production batch-cycle, $q = 246.39$ units, $R = 0.86\}$.

[In Sarkar (2013) $S_o = C$, $HC_b = H_B$, $HC_s = H_S$, $K = F$, $V_c = V$, and $C_d = C_d$.]

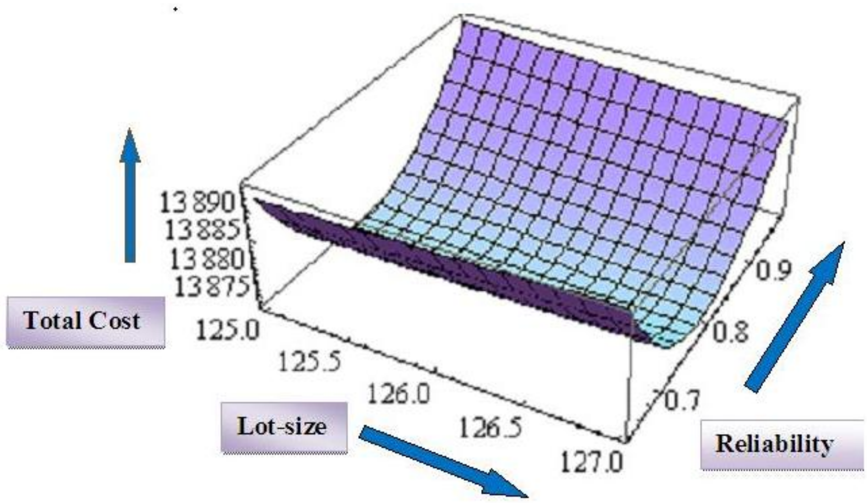

Figure 5.3: Total cost versus lot size and reliability when number of deliveries per production is fixed

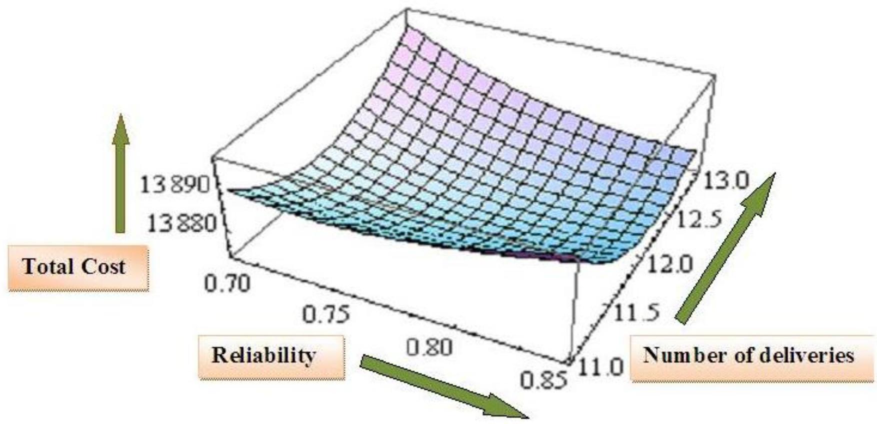

Figure 5.4: Total cost versus number of deliveries per production and reliability when lot size is fixed

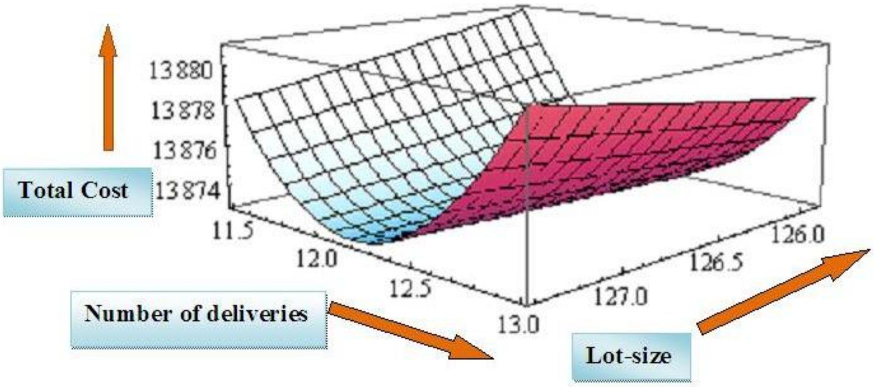

Figure 5.5: Total cost versus number of deliveries per production and lot size when reliability is fixed

5.5 Sensitivity analysis

Effects of changes in parameters such as $C_o, S_0, HC_b, HC_s, K, V_c$, and C_d on the total cost is calculated here. The sensitivity analysis is performed by changing each of the parameters by $-50\%, -25\%, +25\%$, and $+50\%$ taking one parameter at a time while keeping the remaining parameters unchanged.

From Table 5.1 , the discussion of sensitivity analysis of the key parameters are as follows:

- If the ordering cost increases, then material handalling cost, shipping cost, placing order's cost increase; as a result the total relevant cost increases. From the above table, one can conclude that total cost is minor sensitive to changes in ordering cost.
- If the setup cost increases, then the total cost also increases. Negative change in setup cost reduce more in total cost than the positive change in it
- Increasing value of holding cost increases the total cost. From Table 5.1, one can see that the negative and positive change in holding cost gives approximately same amount of change in the total cost function.

Table 5.1: Sensitivity analysis for the key parameters

Parameters	Changes(in %)	$Z(t_1^*, T^*)$
C_o	-50%	-00.53
	-25%	-00.26
	$+25\%$	$+00.26$
	$+50\%$	$+00.52$
S_0	-50%	-04.75
	-25%	-02.22
	$+25\%$	$+2.00$
	$+50\%$	$+03.83$
HC_b	-50%	-01.75
	-25%	-0.83
	$+25\%$	$+0.77$
	$+50\%$	$+1.49$
HC_s	-50%	-06.60
	-25%	-03.03
	$+25\%$	$+02.69$
	$+50\%$	$+05.13$
K	-50%	-03.01
	-25%	-01.37
	$+25\%$	$+01.21$
	$+50\%$	$+02.30$
V_c	-50%	-32.46
	-25%	-16.23
	$+25\%$	$+16.23$
	$+50\%$	$+32.46$
C_d	-50%	-01.72
	-25%	-00.79
	$+25\%$	$+00.69$
	$+50\%$	$+01.32$

- If the transportation cost is increased then the total cost increases.

- If the unit variable cost for order handling and receiving increases while all the other parameters remain unchanged, the expected total cost tends to increase. From Table 5.1, it can be concluded that the negative and positive change in it gives same amount of change in total cost. This is the most sensitive cost than others in this model.

- Increase in deterioration cost indicates increase in total deteriorate items. Therefore increasing deterioration cost increase the total cost.

5.6 Conclusions

This model discussed the effect of reliability on setup cost and deterioration rate. An algebraical procedure was used to minimize the cost for the entire SCM model and obtained a closed-form solution. The main contribution of the model was to obtain the minimum cost with integer number of deliveries, optimal lot size, and reliability by using algebraical procedure. The proposed procedure for the computation of the total cost of the SCM can be easily done without any tedious calculation. An illustrative numerical example and a numerical comparison of this model with that of Sarkar (2013) were provided. Some graphical representations were considered to illustrate the model. It has proved that this model gave more savings than Sarkar (2013). The model is useful where the reduction of setup cost is possible and deterioration is present. This model can be further extended to a multi-item production process with variable transportation cost, demand, and deterioration rate.

Chapter 6

Optimal buffer inventory and inspection errors in an imperfect production system with preventive maintenance

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6.1 Introduction

In existing literature, it is considered that production system always remains in *in-control* state and products produced by the machinery system are of perfect quality. This assumption may not be true in every production system. In many practical situations, during production system, the machinery system goes through a long-run process, it may shift from an *in-control* state to an *out-of-control* state, where the production system produces defective/imperfect quality items. In this direction, Rosenblatt and Lee (1986) developed an economic manufacturing quantity (EMQ) model, where the production process is subject to a random movement from an *in-control* to an *out-of-control* state. Further, they studied constant, linear, and exponential deteriorating models for the case, when an elapsed time to the *out-of-control* state is exponentially distributed. Lee and Rosenblatt (1987) studied an exponentially distributed deteriorating EMQ model to obtain jointly the production cycle and inspection schedule. They proved that the inspection interval is equally distributed for the case, when elapsed time until shift is exponentially distributed. Kim and Hong (1999) extended Rosenblatt and Lee's (1986) model by assuming that elapsed time until the process shift is arbitrarily distributed. Further, Chung and Hou (2003) extended Kim and Hong's (1999) model with shortages. Sarkar (2012b, 2012c) developed two inventory models under the presence of imperfect items during *out-of-control* state. Sarkar *e*t al. (2014) discussed an EMQ model with price and time dependent demand under effect of reliability and inflation.

Sana *e*t al. (2007a, 2007b) extended two inventory models in an imperfect production system, in which the defective item are sold at reduce price. Cárdenas-Barrón (2009a) developed an EPQ model with planned backorders to determine the economic production quantity for single-type-of products. Sana (2010c) developed an imperfect production model to determine the optimal product reliability and production rate to obtain maximum profit. Sana (2010d) discussed an imperfect production model by assuming that the production system may shift from an *in-control* state to an *out-of-control* state at any random time. He considered that a certain percent of total product is defective in *out-of-control* state, which depends production rate and production run time. Pal *e*t al. (2013b) developed a manufacturing system for unreliable machine with random defective units. Pal *e*t al. (2014) considered an imperfect production model to obtain joint pricing and ordering policy, where perfect products are sold with actual price and reworked products with a discount price. Sarkar and Moon (2014) developed the probability of shifting from *in-control* state to *out-of-control* state to maintain the quality of products. Sarkar *e*t al. (2016) derived retailer's optimal strategy for fixed lifetime products. Sarkar *e*t al. (2016) considered an inventory model for deteriorating items with preservation technology and stock-dependent demand.

In today's competitive marketing environment, during long-run production process, failure cannot be tolerable. Due to process failure, a certain percentage of defective items produces. To minimize such failures, management should apply maintenance policy. Preventive maintenance (PM) policy is one of the mostly used policy in large systems such as production systems and transport systems. After regular production, a preventive maintenance of machinery system may reduce the number of defective items in the next production cycle. In this direction, Tseng (1996) discussed a deteriorating production system to determine the optimal preventive maintenance policy. Salameh and Ghattas (2001) developed an inventory model to determine optimum just-in-time inventory buffer level such that sum of holding cost and shortage cost is minimum. Zequeira *e*t al. (2004) examined an inventory model to determine optimal operational times and buffer inventories to satisfy all demands during the maintenance action. Sarkar *e*t al. (2010b, 2010c) developed two inventory models based on corrective and preventive maintenance by considering safety stock to save the lost sales costs. Halim and Tang (2009) derived a confidence-interval of the optimal preventive maintenance interval under two common replacement policies, namely age replacement policy and block replacement policy. Lee

(2009) determined the optimal production run length and scheduled maintenance inspection policy for a deteriorating system in which the products are sold with free minimal repair warranty. Sarkar *e*t al. (2014) extended the single-stage production model with random defective production rate in an imperfect production system. Sana (2012) considered an imperfect production system for products sold with free minimal warranty to determine optimal buffer inventory with regular preventive maintenance. Pal *e*t al. (2013a) considered an EPQ model to determine the optimal buffer inventory for stochastic demand in the market during preventive maintenance.

Recently, warranty policy is one of the most commonly used policy to attract customer. A deteriorating production system produces a certain percentage of defective items. This defective items may be sold with free minimal repair warranty. Therefore, post-sale warrant cost should incorporate in EMQ model to reflect the real life situation. In this direction, Yeh *e*t al. (2000) and Wang (2004) developed two imperfect production models to obtain optimal production run length for products sold with warranty. They assumed two-state continuous-time Markov chain to characterize the deteriorating production system. Wang (2005) developed a product inspection policy for a deteriorating production system, where non-inspected items are sold with post-sale warranty and the inspected items are salvaged before shipping. Chen and Lo (2006) developed an imperfect production system with allowable shortages for products sold with a free minimal repair warranty. Yeh and Chen (2006) investigated a deteriorating production system to determine jointly the optimal lot size and product inspection policy for sold products with free minimal repair warranty.

In real life situation, it is not possible to produce 100% perfect items. During production a certain percentage of items may be of imperfect quality due to machinery problem. Therefore, product inspection policy is necessary before delivery of products. However, in most of the exiting model authors considered inspection policy, but they ignored the human factors during product inspection. In this direction, Raouf *e*t al. (1983) developed an inventory model and suggested that during inspection two types of human errors may occur, one is Type I error (classifying a non-defective item as defective) and and another is Type II error (classifying a defective item as non-defective). Wang (2007) developed an inventory model with two types of inspection errors in order to facilitate the adaptation of economic inspection/disposition model to real world applications. Darwish and Ben-Daya (2007) proposed a production inventory model with the effect of imperfect production processes, preventive maintenance, and inspection errors. Duffuaa and El-Ga'aly (2015) developed an inventory model with more realistic multi-objective optimization model that integrates measurement errors in inspection system. Most recently, Sarkar and Saren (2016) considered a product inspection policy for an imperfect production system with inspection errors and warranty cost. To reduce the inspection cost, they considered the inspection policy at the end of the production cycle and the non-inspected defective items are delivered to the market with warranty. Sett *e*t al. (2016) found an optimal replenishment policy with variable deterioration for fixed lifetime products. Sarkar *e*t al. (2015) discussed a deteriorating inventory model for high-tech products with partial backlogging.

The main purpose of this model is to obtain optimal buffer inventory and inspection policy with preventive maintenance and two types of inspection errors. This model considerers a production system that is subject to a random deterioration from an *in-control* state to an *out-of-control* state with a specific distribution. The production rate of defective item in *in-control* state is lower than in *out-of-control* state. On-line inspection is started after some time to inspect products. To detect the defective items another human-based inspection is considered at the end of production cycle. Inspected items are salvaged at a fixed cost before shipped. Uninspected items are sold to the customer with free minimal warranty. Shortages is considered due to preventive maintenance. Rest of the study is designed as follows: immediate section contains problem definition, and assumptions. In Section 6.3, the proposed model is developed. Solution procedure is given in Section 6.4. Numerical example is given in Section 6.5. Finally, conclusions are made in Section 6.6.

6.2 Problem definition, and assumptions

This section contains definition of the research problem, and several assumptions.

6.2.1 Problem definition

The main focus of this model is to produce a single-type of items in a single-stage production system. During production, the machinery system may shift *in-control* to *out-of-control* state at any random time, which may follow any specific distribution. In both *in-control* and *out-of-control* state, the machinery system produces defective items, but the production rate of defective items in *in-control* state is less than that of *out-of-control* state. As production processes are prone to fail at the end of a production run, on-line inspection is considered after some time of the production and another human-based inspection policy is considered at the end of the production process to detect the defective items. Product inspection starts from $(p\lambda t)$th item until the end of the production lot, and the defective items by the inspector are salvaged at some fixed cost in a parallel system before shipped. The human-based inspection is considered to assure the quality of products, even though the inspection are with errors. The non-inspected items are sold with free minimal warranty. After the end of production run time, the preventive maintenance starts and continues up to a certain time, which follows a specific distribution. Shortages occur, when the buffer inventory goes to zero, but the preventive maintenance of machine is not completed. The management system uses two types of inspections, but as they finds inspection errors by human inspection, thus that outcomes is not used for the rework or warranty, but they use some funds for this human based inspection, thus this cost is incorporated within the system cost.
Decision variables in this study are

B buffer inventory (units) and

λ non-inspected fraction in a batch ($0 \leq \lambda \leq 1$)

6.2.2 Assumptions

The following assumptions are considered to develop this model.

1. At any time, production system, for single-type of products, may shifts *in-control* state to *out-of-control* state until the end of the production-run.
2. Production rate of defective items in *out-of-control* state is grater than *in-control* state.
3. Online inspection is started after some time of production to detect defective units and human-based inspection policy is considered to identify the defective items, where human inspectors are not properly skilled, thus, two types of inspection errors (Type I and Type II) are considered during product inspection.
4. The free minimal repair warranty (FRW) policy is adopted for non-inspected defective items.
5. After the preventive maintenance, it is grunted that the probability of a breakdown of the manufacturing system during the production run time t is zero.
6. At the beginning of any preventive maintenance, the buffer is not subjected to deterioration or obsolescence.
7. Shortages are allowed and are fully backlogged.

6.3 Model formulation

In this model, it is assumed that at the initial stage, the production system is in *in-control* state and after a random time X it moves to an *out-of-control* state. In this system, three

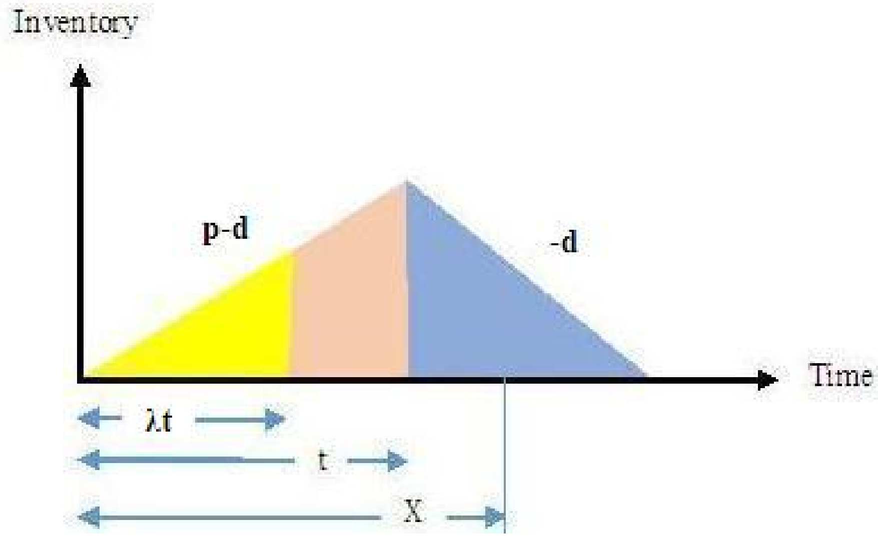

Figure 6.1: Graphical representation of inventory system when $X \geq t$

cases may arise:

Case I $X \geq t$

In this case, the system is in *in-control* state because the random time X occur after the production run time t (See Figure 6.1). Therefore, in this case the production system produces θ_1 percentage of defective items. Again, the whole production interval $[0, t]$ can be divided into two sub-intervals $[0, \lambda t]$ and $[\lambda t, t]$, where λ is the non-inspected fraction in a batch. Further, the produced defective items in the interval $[0, \lambda t]$ are sold with free minimal warranty and the defective items produced in the interval $[\lambda t, t]$ are inspected and then salvaged with some fixed cost in the parallel system before being shipped.

Therefore, the post-sale warranty and salvage cost is

$$C_w \theta_1 p \lambda t + C_s \theta_1 p(t - \lambda t)$$

Case II $\lambda t < X < t$

In this case, the random time X occur between $[\lambda t, t]$ and hence the system is *in-control* state during the time interval $[0, X]$ and in *out-of-control* state during the time interval

$[X, t]$ (See Figure 6.2). Therefore, the production interval $[0, t]$ can be divided into three sub-intervals $[0, \lambda t]$, $[\lambda t, X]$, and $[X, t]$. Following the same procedure as it is discussed in Case I, the total post-sale warranty and salvage cost is

$$C_w\theta_1 p\lambda t + C_w\theta_1 p(X - \lambda t) + C_s\theta_2 p(t - X)$$

Case III $X \leq \lambda t$

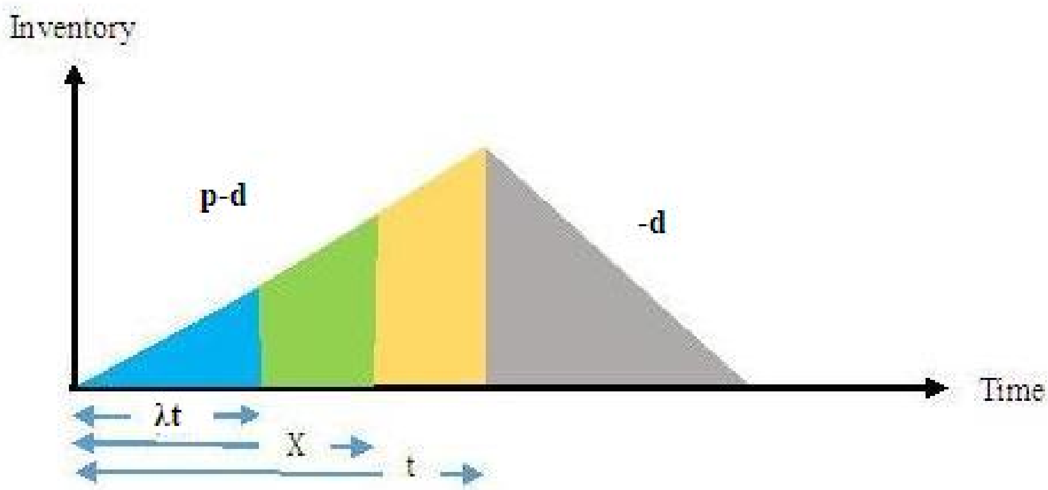

Figure 6.2: Graphical representation of inventory system when $\lambda t \leq X \leq t$

In this case, the random time X occur between $[0, \lambda t]$ and hence the system is *in-control* state during the time interval $[0, X]$ and in *out-of-control* state during the time interval $[X, t]$ (See Figure 6.3). Therefore, the production interval $[0, t]$ can be divided into three sub-intervals $[0, X]$, $[X, \lambda t]$, and $[\lambda t, t]$. Following the same procedure as it is discussed in Case I, the total post-sale warranty and salvage cost is

$$C_w\theta_1 pX + C_w\theta_2 p(\lambda t - X) + C_s\theta_2 p(t - \lambda t)$$

Hence total post-sale warranty and salvage cost within the time interval $[0, t]$ is

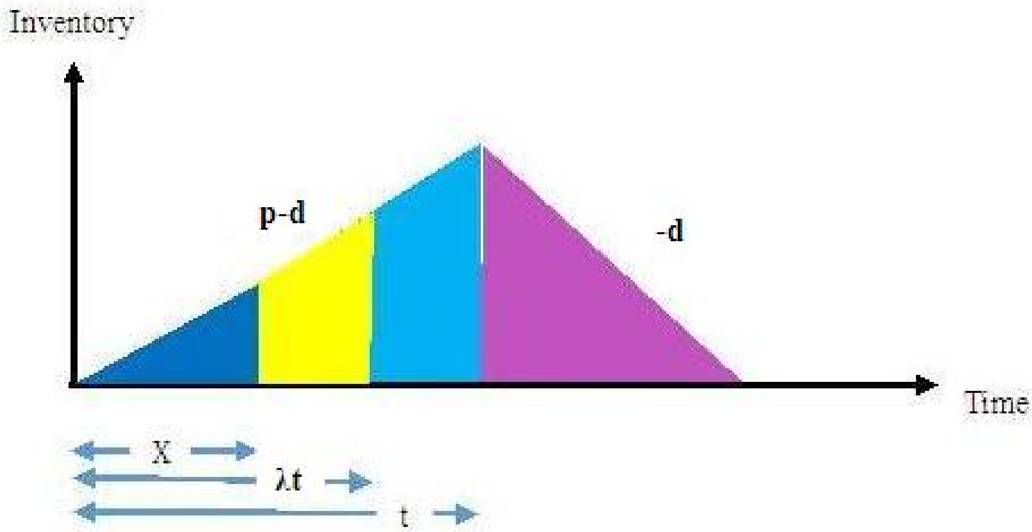

Figure 6.3: Graphical representation of inventory system when $X \leq \lambda t$

$$C_d = \begin{cases} C_w\theta_1 p\lambda t + C_s\theta_1 p(t-\lambda t), & \text{if } X \geq t \\ C_w\theta_1 p\lambda t + C_w\theta_1 p(X-\lambda t) + C_s\theta_2 p(t-X), & \text{if } \lambda t < X < t \\ C_w\theta_1 pX + C_w\theta_2 p(\lambda t - X) + C_s\theta_2 p(t-\lambda t), & \text{if } X \leq \lambda t \end{cases}$$

The expected value of C_d under the inspection policy λ is

$$\begin{aligned} E[C_d] &= \int_0^{\lambda t} [C_w\theta_1 pX + C_w\theta_2 p(\lambda t - X) + C_s\theta_2 p(t-\lambda t)]g(x)dx \\ &+ \int_{\lambda t}^{t} [C_w\theta_1 p\lambda t + C_w\theta_1 p(X-\lambda t) + C_s\theta_2 p(t-X)]g(x)dx \\ &+ \int_t^{\infty} [C_w\theta_1 p\lambda t + C_s\theta_1 p(t-\lambda t)]g(x)dx \end{aligned}$$

After some calculation, one can obtain the expected value of C_d as

$$E[C_d] = C_w\theta_2 p\lambda t + C_s\theta_2 pt(1-\lambda) - (\theta_2-\theta_1)p\left[C_w\int_0^{\lambda t}\overline{G}(x)dx + C_s\int_{\lambda t}^{t}\overline{G}(x)dx\right] \quad (6.1)$$

Inspection errors

During inspection, an inspector may commit two types of inspection errors (Type I and Type II). Due to Type I error, some non-defective items are classified as defective i.e., $(1-\theta_1)m_1$ in *in-control* state and $(1-\theta_2)m_1$ in *out-of-control* state. Again due to Type II error, some defective items are classified as non-defective i.e., $\theta_1 m_2$ in *in-control* state and $\theta_2 m_2$ in *out-of-control* state. Therefore, the inspector rejects some non-defective items and accepts some defective items due to these two types of inspection errors. Three cases occur during inspection.

Case I $X \geq t$

In this case, the system is in *in-control* state and produces θ_1 percentage of defective items. Therefore, within the time interval $[\lambda t, t]$, the inspector accepts $\theta_1 p(t-\lambda t)$ defective items including falsely accepted defective items $p(t-\lambda t)\theta_1 m_2$ and falsely rejected non-defective items $p(t-\lambda t)(1-\theta_1)m_1$.

Therefore, the misclassification cost due to two types of inspection errors is

$$C_R p(t-\lambda t)(1-\theta_1)m_1 + C_A \theta_1 p(t-\lambda t)(1-m_2)$$

Case II $\lambda t < X < t$

In this case, inspection time interval can be divided into two sub-cases as $[\lambda t, X]$ and $[X, \lambda t]$. Within the time interval $[\lambda t, X]$, the system is in the *in-control* state and produces θ_1 percentage of defective items. Therefore, within the time interval $[\lambda t, X]$, the inspector accepts $\theta_1 p(X-\lambda t)$ defective items including falsely accepted defective items $\theta_1 p(X-\lambda t)m_2$ and falsely rejected non-defective items $(1-\theta_1)p(X-\lambda t)m_1$. Within the time interval $[X, t]$, the system is in the *out-of-control* state and produces θ_2 percentage of defective items and the inspector accepts $\theta_2 p(t-X)$ defective items including falsely accepted defective items $\theta_2 p(t-X)m_2$ and falsely rejected non-defective items $(1-\theta_2)p(t-X)m_1$.

Therefore, the misclassification cost due to two types of inspection errors, is

$$p(X-\lambda t)[C_R(1-\theta_1)m_1+C_A\theta_1(1-m_2)]+p(t-X)[C_R(1-\theta_2)m_1+C_A\theta_2(1-m_2)]$$

Case III $X \leq \lambda t$

In this case, the system is in the *out-of-control* state and produces θ_2 percentage of defective items. Therefore, within the inspection time interval $[\lambda t, t]$, the inspector accepts $\theta_2 p(1-\lambda t)$ defective items including falsely accepted defective items $\theta_2 p(1-\lambda t)m_2$ and falsely rejects non-defective items $(1-\theta_2)p(t-\lambda t)$.

Therefore, the misclassification cost due to two types of inspection errors is

$$p(t-\lambda t)[C_R(1-\theta_2)m_1+C_A\theta_2(1-m_2)]$$

Therefore, the total miscalculation cost due to two types of inspection errors within the time interval $[\lambda t, t]$ is

$$C_i = \begin{cases} C_R p(t-\lambda t)(1-\theta_1)m_1 + C_A\theta_1 p(t-\lambda t)(1-m_2), & \text{if } X \geq t \\ p(X-\lambda t)[C_R(1-\theta_1)m_1+C_A\theta_1(1-m_2)] & \\ +p(t-X)[C_R(1-\theta_2)m_1+C_A\theta_2(1-m_2)], & \text{if } \lambda t < X < t \\ p(t-\lambda t)[C_A(1-\theta_2)m_1+C_R\theta_2(1-m_2)] & \text{if } X \leq \lambda t \end{cases}$$

The expected misclassification cost due to two types of inspection errors within the time interval $[\lambda t, t]$ is

$$\begin{aligned} E[C_i] &= \int_0^{\lambda t}(p(t-\lambda t)[C_A(1-\theta_2)m_1+C_R\theta_2(1-m_2)])g(x)dx + \int_{\lambda t}^{t}(p(X-\lambda t)[C_R(1-\theta_1)m_1 \\ &+ C_A\theta_1(1-m_2)]+p(t-X)[C_R(1-\theta_2)m_1+C_A\theta_2(1-m_2)])g(x)dx \\ &+ \int_t^{\infty}[C_R p(t-\lambda t)(1-\theta_1)m_1+C_A\theta_1 p(t-\lambda t)(1-m_2)]g(x)dx \end{aligned}$$

After some calculation, one can obtain the expected value of C_i as

$$\begin{aligned} E[C_i] &= p(1-\lambda)t\left\{C_A\theta_2(1-E[m_2]) + C_R(1-\theta_2)E[m_1]\right\} \\ &+ p(\theta_1-\theta_2)\left\{C_A(1-E[m_2]) - C_R E[m_1]\right\}\int_{\lambda t}^{t}\overline{G}(x)dx \end{aligned} \tag{6.2}$$

Here, the production system runs for a time-period t before preventive maintenance occur. Within the period $[0,t]$, the buffer inventory B builds up at a rate $p-d$ i.e., $B=(p-d)t$. After the time period t, preventive maintenance starts and continues upto time τ, which is a random variable that follows a probability density function $g(\tau)$ (See Figure 6.4).

During maintenance, the stock out time is

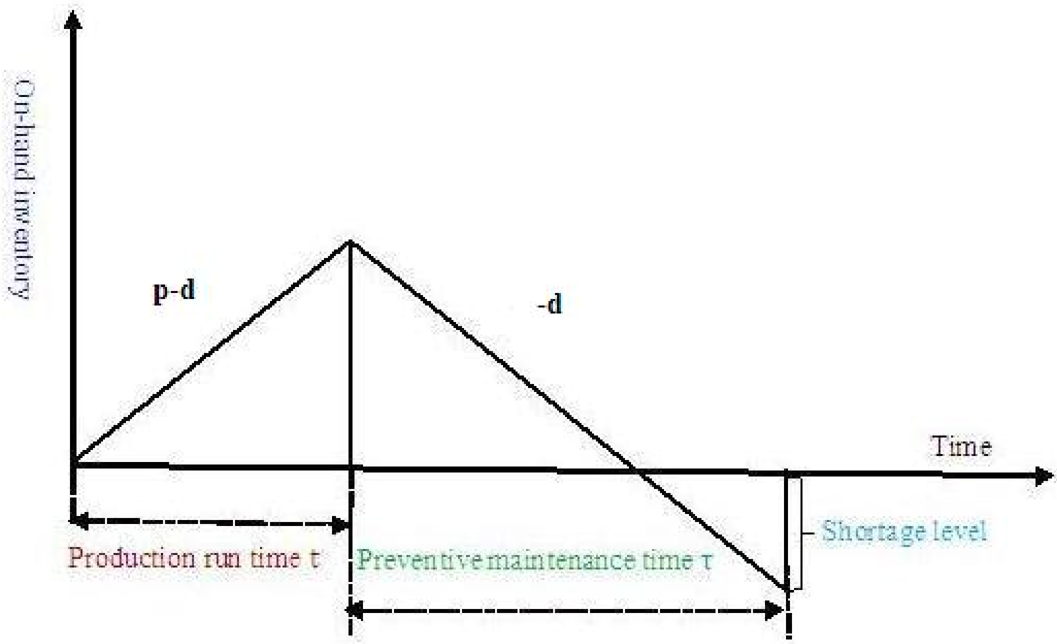

Figure 6.4: Logistic diagram of the proposed model.

$$Y(\tau) = \begin{cases} 0 & \text{if } \tau \le \frac{B}{d}, \\ \tau - \frac{B}{d} & \text{if } \tau \ge \frac{B}{d}. \end{cases}$$

Therefore, the number of shortage units per preventive maintenance cycle is

$$Y(\tau) = \begin{cases} 0 & \text{if } \tau \leq \frac{B}{d}, \\ d\left(\tau - \frac{B}{d}\right) & \text{if } \tau \geq \frac{B}{d}. \end{cases}$$

The expected number of unit shortages is

$$E[Y(\tau)] = d\int_{B/d}^{\infty}\left(\tau - \frac{B}{d}\right)g(\tau)d\tau$$

Therefore, expected shortage cost is

$$ESC = S_c E[Y(\tau)]$$

The setup cost is $SEC = S_0$

The expected holding cost is

$$\begin{aligned} EHC &= C_h\left[(p-d)\int_0^t u du + \int_0^{\frac{(p-d)t}{d}}\{(p-d)t - du\}du\right]\int_0^{\infty} g(x)dx \\ &= \frac{1}{2}C_h(p-d)\frac{pt^2}{d} \end{aligned}$$

Expected maintenance cost is

$$EMC = C_m\int_0^{\infty}\tau g(\tau)d\tau = C_m E[\tau]$$

The variable costs including labor, energy, and material costs is

$$MC = C_m pt$$

The inspection cost is

$$IC = I_c(1-\lambda)pt$$

Therefore, total expected cost per unit item is

$$\begin{aligned}
ETC(B,\lambda) &= \frac{SEC + MC + EHC + ESC + EMC + IC + E[C_d] + E[C_i]}{pt} \\
&= \frac{S_0}{pt} + C_m + \frac{C_h}{2d}(p-d)t + \frac{S_c d}{pt}\int_{B/d}^{\infty}\left(\tau - \frac{B}{d}\right) g(\tau)d\tau + \frac{C_m}{pt}\int_0^{\infty}\tau g(\tau)d\tau \\
&+ I_c(1-\lambda) + \frac{(\theta_1-\theta_2)}{t}\left[C_w\int_0^{\lambda t}\overline{G}(x)dx + C_s\int_{\lambda t}^{t}\overline{G}(x)dx\right] + C_w\theta_2\lambda \\
&+ C_s\theta_2(1-\lambda) + (1-\lambda)\left\{C_A\theta_2(1-E[m_2]) + C_R(1-\theta_2)E[m_1]\right\} \\
&+ \frac{(\theta_1-\theta_2)}{t}\left\{C_A(1-E[m_2]) - C_R E[m_1]\right\}\int_{\lambda t}^{t}\overline{G}(x)dx \qquad (6.3)
\end{aligned}$$

Substituting $t = \frac{B}{p-d}$ in (6.3) and after simplification, one has

$$\begin{aligned}
ETC(B,\lambda) &= \frac{S_0}{B}\left(1-\frac{d}{p}\right) + C_m + \frac{C_h B}{2d} + \frac{C_m}{B}\left(1-\frac{d}{p}\right)\int_0^{\infty}\tau g(\tau)d\tau + \frac{B_c d}{B}\left(1-\frac{d}{p}\right) \\
&\quad \int_{B/d}^{\infty}\left(\tau-\frac{B}{d}\right)g(\tau)d\tau - \frac{(\theta_2-\theta_1)(p-d)}{B}\left[C_w\int_0^{\frac{\lambda B}{p-d}}\overline{G}(x)dx\right] + C_w\theta_2\lambda \\
&- \frac{(\theta_2-\theta_1)(p-d)}{B}\left\{C_s + C_A(1-E[m_2]) - C_R E[m_1]\right\}\int_{\frac{\lambda B}{p-d}}^{\frac{B}{p-d}}\overline{G}(x)dx \\
&+ (1-\lambda)\left\{I_c + C_s\theta_2 + C_A\theta_2(1-E[m_2]) + C_R(1-\theta_2)E[m_1]\right\} \qquad (6.4)
\end{aligned}$$

Main objective is to find the optimal value of B and λ such that the corresponding expected total cost per unit item $TC(B,\lambda)$ is minimum.

6.4 Solution procedure

In order to find the optimal solution, some useful policies are discussed in this section.

6.4.1 Optimal inspection policy λ

This section investigates the optimal product inspection policy λ_s for a given buffer inventory $B > 0$, which minimizes the total cost function given in (6.4). It can be shown that when $\hat{\lambda} \in (0, \infty)$, the optimal inspection policy for a given buffer inventory $B > 0$, which is equal to $\lambda_s = \mathrm{Min}\{1, 0\}$, where

$$\hat{\lambda} = \left(\frac{p-d}{B}\right) \overline{G}^{-1} \left(\frac{\theta_2 - \rho}{\theta_2 - \theta_1}\right)$$

and

$$\rho = \frac{I_c + C_R E[m_1]}{(C_w - C_s - C_A(1 - E[m_2]) + C_R E[m_1])}$$

Proof: Taking first and second order differentiation of (6.4) with respect to λ

$$\begin{aligned} \frac{\partial ETC(\lambda, B)}{\partial \lambda} &= -I_c - C_R E[m_1] - (C_s + C_A(1 - E[m_2]) - C_R E[m_1] - C_w) \\ & \quad \left[\theta_2 - (\theta_2 - \theta_1)\overline{G}\left(\frac{\lambda B}{p-d}\right)\right] \end{aligned} \tag{6.5}$$

and

$$\begin{aligned} \frac{\partial^2 ETC(\lambda, B)}{\partial \lambda^2} &= \{C_w + C_R E[m_1] + C_A E[m_2] - C_s - C_A\} \\ & \quad \left[(\theta_2 - \theta_1)\left(\frac{B}{p-d}\right) f\left(\frac{\lambda B}{p-d}\right)\right] > 0 \end{aligned}$$

As $ETC(B, \lambda)$ is convex on λ, setting (6.5) to zero, one has

$$\begin{aligned}
&\frac{\partial ETC(B,\lambda)}{\partial \lambda} = 0 \\
\Rightarrow\ & -I_c - C_R E[m_1] - \{C_s + C_A(1 - E[m_2]) - C_R E[m_1] - C_w\} \\
& \left[\theta_2 - (\theta_2 - \theta_1)\overline{G}\left(\frac{\lambda B^*}{p-d}\right)\right] = 0 \\
\Rightarrow\ & \theta_2 - (\theta_2 - \theta_1)\overline{G}\left(\frac{\lambda B}{p-d}\right) = \frac{I_c + C_R E[m_1]}{(C_w - C_s - C_A(1 - E[m_2]) + C_R E[m_1])} \qquad (6.6)
\end{aligned}$$

Let us define $\dfrac{I_c + C_R E[m_1]}{(C_w - C_s - C_A(1 - E[m_2]) + C_R E[m_1])} = \rho$

Hence, (6.6) becomes

$$\begin{aligned}
& \theta_2 - (\theta_2 - \theta_1)\overline{G}\left(\frac{\lambda B}{p-d}\right) = \rho \\
\Rightarrow\ & \overline{G}\left(\frac{\lambda B}{p-d}\right) = \frac{\theta_2 - \rho}{\theta_2 - \theta_1} \\
\Rightarrow\ & \hat{\lambda} = \left(\frac{p-d}{B}\right)\overline{G}^{-1}\left(\frac{\theta_2 - \rho}{\theta_2 - \theta_1}\right) \\
& \lambda_s = Min\{1, \hat{\lambda}\} \qquad (6.7)
\end{aligned}$$

Therefore, for given $B > 0$, the optimal inspection interval can be obtain using (6.7).

Properties

- If $\theta_1 < \rho < \theta_2$, then the inspection policy is unique and $\lambda_s = \text{Min}\left\{1, \hat{\lambda}\right\}$.
- If $\rho \leq \theta_1$, then $\lambda_s = 0$ and in this case full inspection is the optimal policy.
- If $\rho \geq \theta_2$, then $\lambda_s = 1$ and in this case no inspection is the optimal policy.

6.4.2 Optimal buffer inventory B

For given inspection policy λ^*, the optimal buffer inventory B^* minimizing $C(B,\lambda)$ exists and is unique.

Proof: The first order derivative of (6.3), when $\lambda = \lambda^*$, is

$$\begin{aligned}
\frac{dC(B|\lambda^*)}{dB} &= \frac{C_h}{2d} - \frac{1}{B^2}\left(1-\frac{d}{p}\right)\left(S_0 + C_m\int_0^\infty \tau g(\tau)d\tau\right) - \frac{S_c d}{B^2}\left(1-\frac{d}{p}\right)\int_{B/d}^\infty \tau g(\tau)d\tau \\
&+ \frac{(\theta_2-\theta_1)(p-d)}{B^2}\Bigg(C_w\int_0^{\frac{\lambda^* B}{(p-d)}} \overline{G}(x)dx + (C_s + C_A(1-E[m_2]) - C_R E[m_1]) \\
&\quad \int_{\frac{\lambda^* B}{(p-d)}}^{\frac{B}{(p-d)}} \overline{G}(x)dx\Bigg) - \frac{(\theta_2-\theta_1)}{B}\Bigg[(C_s + C_A(1-E[m_2]) - C_R E[m_1]) \\
&\quad \left\{\overline{G}\left(\frac{B}{p-d}\right) - \lambda^*\overline{G}\left(\frac{\lambda^* B}{p-d}\right)\right\} + C_w\lambda^*\overline{G}\left(\frac{\lambda^* B}{p-d}\right)\Bigg]
\end{aligned}$$

Let us define $U(B|\lambda^*) = B^2\frac{dC(B|\lambda^*)}{dB}$, then

$$\begin{aligned}
U(B|\lambda^*) &= \frac{C_h B^2}{2d} - S_0\left(1-\frac{d}{p}\right) - C_m\left(1-\frac{d}{p}\right)\int_0^\infty \tau g(\tau)d\tau - S_c d\left(1-\frac{d}{p}\right)\int_{B/d}^\infty \tau g(\tau)d\tau \\
&+ (\theta_2-\theta_1)\Bigg((p-d)\left[C_w\int_0^{\frac{\lambda^* B}{(p-d)}} \overline{G}(x)dx\right] - B\left[C_w\lambda^*\overline{G}\left(\frac{\lambda^* B}{p-d}\right)\right] \\
&+ (C_s + C_A(1-E[m_2]) - C_R E[m_1])\Bigg\{(p-d)\left[\int_{\frac{\lambda^* B}{(p-d)}}^{\frac{B}{(p-d)}} \overline{G}(x)dx\right] \\
&- B\left[\overline{G}\left(\frac{B}{p-d}\right) - \lambda^*\overline{G}\left(\frac{\lambda^* B}{p-d}\right)\right]\Bigg\}\Bigg)
\end{aligned}$$

As

$$\begin{aligned}
\frac{d^2U(B|\lambda^*)}{dB^2} &= \frac{C_h B}{d} + S_c\left(1-\frac{d}{p}\right)\left(\frac{B}{d}\right)g\left(\frac{B}{d}\right) + \frac{(\theta_2-\theta_1)B}{(p-d)}\Bigg[g\left(\frac{\lambda^* B}{p-d}\right)C_w\lambda^{*2} \\
&+ (C_s + C_A - C_R E[m_1] - C_A E[m_2])\left\{g\left(\frac{B}{p-d}\right) - \lambda^{*2}g\left(\frac{\lambda^* B}{p-d}\right)\right\}\Bigg] > 0
\end{aligned}$$

which implies $U(B|\lambda^*)$ is a monotonic increasing function of B and limiting values of $U(B|\lambda^*)$ are

$$\begin{aligned}
\lim_{B\to 0^+} U(B|\lambda^*) &\to \lim_{B\to 0^+} \frac{C_h B^2}{2d} - \left(1-\frac{d}{p}\right)\left(S_0 + C_m \int_0^\infty \tau g(\tau) d\tau\right) \\
&- S_c d\left(1-\frac{d}{p}\right) \lim_{B\to 0^+} \int_{B/d}^\infty \tau g(\tau) d\tau \\
&+ (\theta_2-\theta_1)\Bigg((p-d)\left[C_w \lim_{B\to 0^+}\int_0^{\frac{\lambda^* B}{(p-d)}} \overline{G}(B)dx\right] - C_w\lambda^*\left[\lim_{B\to 0^+} B\overline{G}\left(\frac{\lambda^* B}{p-d}\right)\right] \\
&+ (C_s + C_A(1-E[m_2]) - C_R E[m_1])\Bigg\{(p-d)\left[\lim_{B\to 0^+}\int_{\frac{\lambda^* B}{(p-d)}}^{\frac{B}{(p-d)}} \overline{G}(x)dx\right] \\
&- \lim_{B\to 0^+} B\overline{G}\left(\frac{B}{p-d}\right) + \lambda^* \lim_{B\to 0^+} B\overline{G}\left(\frac{\lambda^* B}{p-d}\right)\Bigg\}\Bigg) \\
&\to -\left(1-\frac{d}{p}\right)\{S_0 + (C_m + S_c d)E[\tau]\} \\
&\to \text{a negative value } < 0
\end{aligned}$$

and

$$\begin{aligned}
\lim_{B\to \infty} U(B|\lambda^*) &\to \lim_{B\to \infty} \frac{C_h B^2}{2d} - \left(1-\frac{d}{p}\right)\left(S_0 + C_m \int_0^\infty \tau g(\tau) d\tau\right) \\
&- S_c d\left(1-\frac{d}{p}\right) \lim_{B\to \infty} \int_{B/d}^\infty \tau g(\tau) d\tau \\
&+ (\theta_2-\theta_1)\Bigg((p-d)\left[C_w \lim_{B\to \infty}\int_0^{\frac{\lambda^* B}{(p-d)}} \overline{G}(x)dx\right] - C_w\lambda^*\left[\lim_{B\to \infty} B\overline{G}\left(\frac{\lambda^* B}{p-d}\right)\right] \\
&+ (C_s + C_A(1-E[m_2]) - C_R E[m_1])\Bigg\{(p-d)\left[\lim_{B\to \infty}\int_{\frac{\lambda^* B}{(p-d)}}^{\frac{B}{(p-d)}} \overline{G}(x)dx\right] \\
&- \lim_{B\to \infty} B\overline{G}\left(\frac{B}{p-d}\right) + \lambda^* \lim_{B\to \infty} B\overline{G}\left(\frac{\lambda^* B}{p-d}\right)\Bigg\}\Bigg) \\
&\to \frac{C_h}{2d}\times\infty - \left(1-\frac{d}{p}\right)\left(S_0 + C_m\int_0^\infty \tau g(\tau)d\tau\right) \\
&\to +\infty
\end{aligned}$$

Hence $U(B|\lambda^*)$ has a unique solution B^* which minimizes $C(B,\lambda)$ is always exists and unique.

6.4.3 Convexity of $ETC(B, \lambda)$

To prove the convexity of $ETC(B,\lambda)$ at the optimal B^* and λ^*, let us define Hessian matrix as follows:

$$H = \begin{pmatrix} \frac{\partial^2 ETC(B^*,\lambda^*)}{\partial {B^*}^2} & \frac{\partial^2 ETC(B^*,\lambda^*)}{\partial B^* \partial \lambda^*} \\ \frac{\partial^2 ETC(B^*,\lambda^*)}{\partial \lambda^* \partial B^*} & \frac{\partial^2 ETC(B^*,\lambda^*)}{\partial {\lambda^*}^2} \end{pmatrix}$$

The solution for optimal B^* and λ^* are determined from the following two equations:

$$\begin{aligned} \left.\frac{\partial ETC(\lambda,B)}{\partial \lambda}\right|_{(B^*,\lambda^*)} &= -I_c - C_R E[m_1] - \{C_s + C_A(1-E[m_2]) - C_R E[m_1] - C_w\} \\ & \left[\theta_2 - (\theta_2-\theta_1)\overline{G}\left(\frac{\lambda^* B^*}{p-d}\right)\right] = 0 \end{aligned}$$

and

$$\begin{aligned} \left.\frac{\partial ETC(\lambda,B)}{\partial B}\right|_{(B^*,\lambda^*)} &= \frac{C_h}{2d} - \frac{1}{{B^*}^2}\left(1-\frac{d}{p}\right)\left(S_0 + C_m\int_0^\infty \tau g(\tau)d\tau\right) \\ &- \frac{S_c d}{{B^*}^2}\left(1-\frac{d}{p}\right)\int_{B^*/d}^\infty \tau g(\tau)d\tau \\ &+ \frac{(\theta_2-\theta_1)(p-d)}{{B^*}^2}\left(C_w\int_0^{\frac{\lambda^* B^*}{(p-d)}} \overline{G}(x)dx + (C_s + C_A(1-E[m_2]) - C_R E[m_1])\right. \\ & \left.\int_{\frac{\lambda^* B^*}{(p-d)}}^{\frac{B^*}{(p-d)}} \overline{G}(x)dx\right) - \frac{(\theta_2-\theta_1)}{B^*}\left[(C_s + C_A(1-E[m_2]) - C_R E[m_1])\right. \\ & \left.\left\{\overline{G}\left(\frac{B^*}{p-d}\right) - \lambda^*\overline{G}\left(\frac{\lambda^* B^*}{p-d}\right)\right\} + C_w\lambda^*\overline{G}\left(\frac{\lambda^* B^*}{p-d}\right)\right] = 0 \end{aligned}$$

Now, the corresponding second-order sufficient conditions are examined for the optimal solutions.

$$\left.\frac{\partial^2 ETC(\lambda, B)}{\partial \lambda^2}\right|_{(B^*,\lambda^*)} = \{C_w + C_R E[m_1] - C_s - C_A(1 - E[m_2])\} \left[\frac{(\theta_2 - \theta_1)B^*}{(p-d)} g\left(\frac{\lambda^* B^*}{p-d}\right)\right] > 0 \tag{6.8}$$

$$\left.\frac{\partial^2 C(B, \lambda)}{\partial B^2}\right|_{(B^*,\lambda^*)} = \frac{1}{B^{*2}}\left[\frac{C_h B^*}{d} + S_c\left(1 - \frac{d}{p}\right)\left(\frac{B^*}{d}\right) g\left(\frac{B^*}{d}\right) + \frac{(\theta_2 - \theta_1)B^*}{(p-d)} \left[g\left(\frac{\lambda^* B^*}{p-d}\right) C_w \lambda^{*2} + (C_s + C_A - C_R E[m_1] - C_A E[m_2]) \left\{g\left(\frac{B^*}{p-d}\right) - \lambda^{*2} g\left(\frac{\lambda^* B^*}{p-d}\right)\right\}\right]\right] > 0 \tag{6.9}$$

$$\left.\frac{\partial^2 C(B, \lambda)}{\partial \lambda \partial B}\right|_{(B^*,\lambda^*)} = \left.\frac{\partial^2 C(B, \lambda)}{\partial B \partial \lambda}\right|_{(B^*,\lambda^*)} = \{C_w + C_R E[m_1] + C_A E[m_2] - C_s - C_A\} \left[\left(\frac{(\theta_2 - \theta_1)\lambda^*}{p-d}\right) g\left(\frac{\lambda^* B^*}{p-d}\right)\right]$$

It can be easily verified that

$$\left|\left.\frac{\partial^2 C(B, \lambda)}{\partial B^2}\right|_{(B^*,\lambda^*)}\right| > \left|\left.\frac{\partial^2 C(B, \lambda)}{\partial \lambda \partial B}\right|_{(B^*,\lambda^*)}\right|$$

and

$$\left|\left.\frac{\partial^2 C(B, \lambda)}{\partial \lambda^2}\right|_{(B^*,\lambda^*)}\right| > \left|\left.\frac{\partial^2 C(B, \lambda)}{\partial \lambda \partial B}\right|_{(B^*,\lambda^*)}\right|$$

Therefore

$$|H| = \frac{\partial^2 C(B,\lambda)}{\partial B^2}\bigg|_{(B^*,\lambda^*)} \times \frac{\partial^2 C(B,\lambda)}{\partial \lambda^2}\bigg|_{(B^*,\lambda^*)} - \left[\frac{\partial^2 C(B,\lambda)}{\partial \lambda \partial B}\bigg|_{(B^*,\lambda^*)}\right]^2 > 0 \qquad (6.10)$$

From (6.8), (6.9), and (6.10), one can easily obtain that (B^*, λ^*) is the global minimum solution.

6.5 Numerical analysis

This section presents a numerical example to illustrate the proposed model. Here, it is considered that the production system and inspection errors follow a Weibull distribution and preventive maintenance time follows an exponential distribution.

The parametric values are taken as S_0 = \$600/setup, p = 500 units/day, d = 450 units/day, h = \$2.5/unit/day, C_m = \$100/item, C_w = \$50/defective item, C_s = \$18/item, I_c = \$1/item, S_c = \$6.5/item, $\theta_1 = 0.15, \theta_2 = 0.35$, M_c = \$100, C_A = \$28/item, C_R = \$10/item, $E[m_1] = \frac{\lambda_1}{2}, E[m_2] = \frac{\lambda_2}{2}, \lambda_1 = \lambda_2 = 0.06$, $G(x) = 1 - e^{-0.5x}, g(\tau) = 0.5e^{-0.5\tau}$. The optimal result is $B^* = 197.72$ units, $\lambda^* = 0.365549$ day, $t^* = 3.9544$ days, $ETC(B^*, \lambda^*)$ = \$116.335/item (See Figure 6.5).

Next, the impact of key cost parameters on optimal solution are studied. The results are given in Table 6.1.

- The buffer stock and production run time often increase due to higher setup costs whereas non-inspected fraction size decreases. Therefore, the expected cost per unit product increases with increasing value of setup cost (See Figure 6.6).

- Table 6.1 indicates that higher holding cost per unit time decreases the optimal buffer inventory and production runtime i.e., when holding cost increases production manager should not continue the production system for long-time and should not store more buffer inventory. On the other hand, the increasing holding cost indicates

Table 6.1: Impact of major key parameters on optimal B, λ, t, and ETC

Parametric values	Optimal buffer inventory (B^*)	Optimal non-inspected fraction batch (λ^*)	Optimal production run time (t^*)	Expected cost $ETC(B^*, \lambda^*)$
Setup cost (S_0)				
300	189.60	0.38	3.79	116.18
400	192.29	0.37	3.84	116.23
500	195.00	0.37	3.90	116.28
600	197.72	0.37	4.01	116.33
700	200.45	0.36	4.06	116.37
800	203.20	0.35	4.06	116.43
900	205.95	0.35	4.12	116.48
Holding cost (C_h)				
1.0	220.15	0.33	4.40	115.99
1.5	211.41	0.34	4.23	116.11
2.0	204.06	0.35	4.08	116.22
2.5	197.72	0.36	3.95	116.33
3.0	192.15	0.38	3.84	116.44
3.5	187.20	0.39	3.74	116.55
4.0	182.74	0.40	3.65	116.65
Warranty cost (C_w)				
49	190.80	0.90	3.82	116.20
50	197.72	0.37	3.95	116.33
51	200.52	0.18	4.01	116.39
52	201.79	0.07	4.03	116.41
53	202.27	0.02	4.04	116.42
Material, labor, and energy cost (C_m)				
70	197.72	0.37	3.95	86.33
80	197.72	0.37	3.95	96.33
90	197.72	0.37	3.95	106.33
100	197.72	0.37	3.95	116.33
110	197.72	0.37	3.95	126.33
120	197.72	0.37	3.95	136.33
130	197.72	0.37	3.95	146.33
Shortage cost (S_c)				
3.5	128.33	0.56	2.57	114.92
4.5	151.00	0.48	3.02	115.48
5.5	174.06	0.41	3.48	115.94
6.5	197.72	0.37	3.95	116.33
7.5	222.11	0.32	4.44	116.68
8.5	247.23	0.29	4.94	116.97
9.5	272.99	0.26	5.46	117.23
Salvage cost (C_s)				
16.0	208.60	0.08	4.17	115.88
16.5	206.33	0.13	4.13	116.00
17.0	203.84	0.18	4.08	116.12
17.5	201.03	0.26	4.02	116.23
18.0	197.72	0.37	3.95	116.33
18.5	193.58	0.54	3.87	116.42
19.0	187.80	0.91	3.76	116.46

Parametric values	Optimal buffer inventory (B^*)	Optimal non-inspected fraction batch (λ^*)	Optimal production run time (t^*)	Expected cost $ETC(B^*, \lambda^*)$
Inspection cost (I_c)				
0.7	201.57	0.12	4.03	116.11
0.8	200.70	0.19	4.01	116.19
0.9	199.44	0.27	3.99	116.27
1.0	197.72	0.37	3.95	116.33
1.1	195.46	0.48	3.91	116.39
1.2	192.53	0.64	3.85	116.44
1.3	188.69	0.87	3.77	116.46
Maintenance cost (M_c)				
70	196.09	0.37	3.92	116.30
80	196.63	0.37	3.93	116.31
90	197.17	0.37	3.94	116.32
100	197.72	0.37	3.95	116.33
110	198.26	0.36	3.97	116.35
120	198.81	0.36	3.98	116.36
130	199.36	0.36	3.99	116.37
Type I error cost C_A				
26	208.34	0.09	4.17	115.89
26.5	206.12	0.13	4.12	116.01
27.0	203.69	0.18	4.07	116.13
27.5	200.93	0.26	4.02	116.24
28.0	197.72	0.37	3.95	116.33
28.5	193.72	0.56	3.87	116.42
29.0	188.22	0.88	3.76	116.46
Type II error cost C_R				
1	200.03	0.18	4.00	116.19
4	199.48	0.24	3.99	116.25
7	198.72	0.30	3.97	116.29
10	197.72	0.37	3.95	116.33
13	196.48	0.44	3.93	116.37
16	194.96	0.52	3.90	116.41
19	193.14	0.62	3.86	116.43

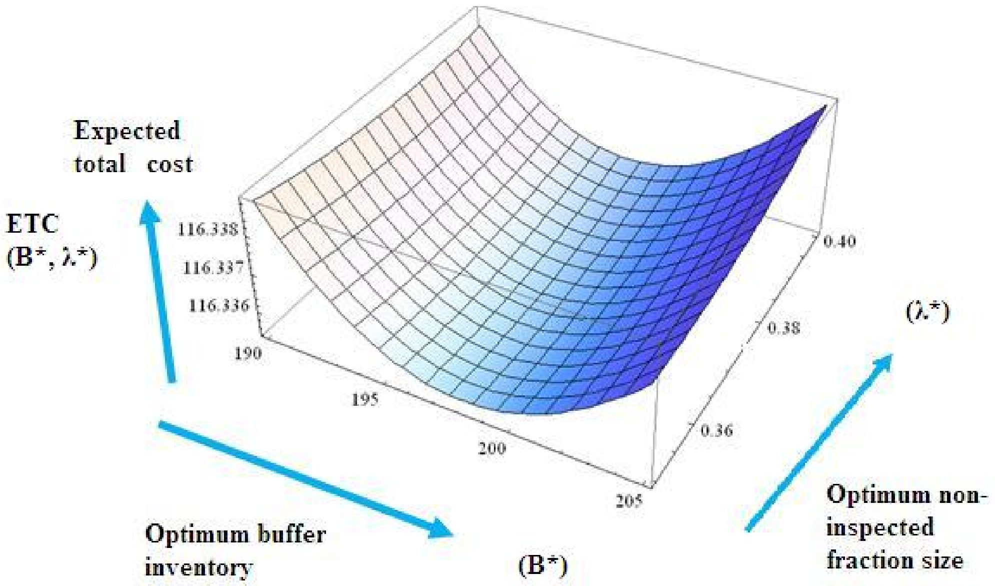

Figure 6.5: Graphical representation of expected total cost versus optimum buffer inventory and optimum non-inspected fraction size.

the increasing value of the non-inspected fraction batch size, which implies manager should left produced products with warranty as much as possible. However, higher holding cost increases the expected total cost per unit product (See Figure 6.7).

- A higher warranty cost increases the buffer inventory, production run time, expected cost per unit item whereas it decreases the non-inspected fraction size. When warranty cost increases, manager should sell less amount of products with post-sell warranty to reduce the total warranty cost as well as total production cost per unit item (See Figure 6.8).
- Effect of changes in material, energy, and labor cost (variable cost) does not effect on optimal buffer inventory, non-inspected fraction size, and production run time, but

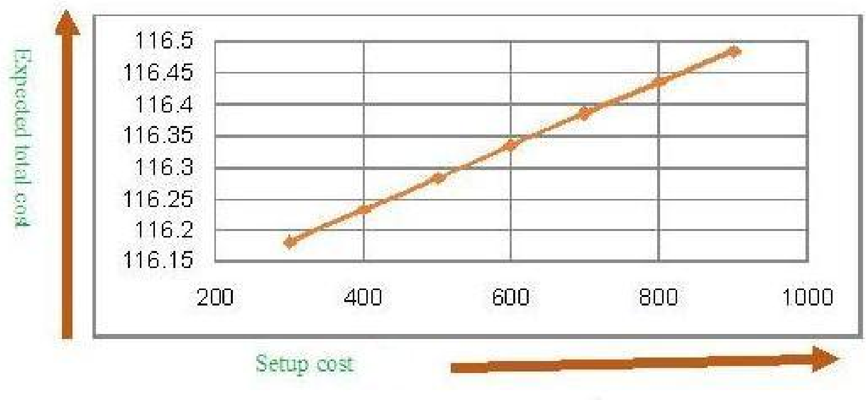

Figure 6.6: Impact of setup cost (S_0) on expected total cost $ETC(B^*, \lambda^*)$

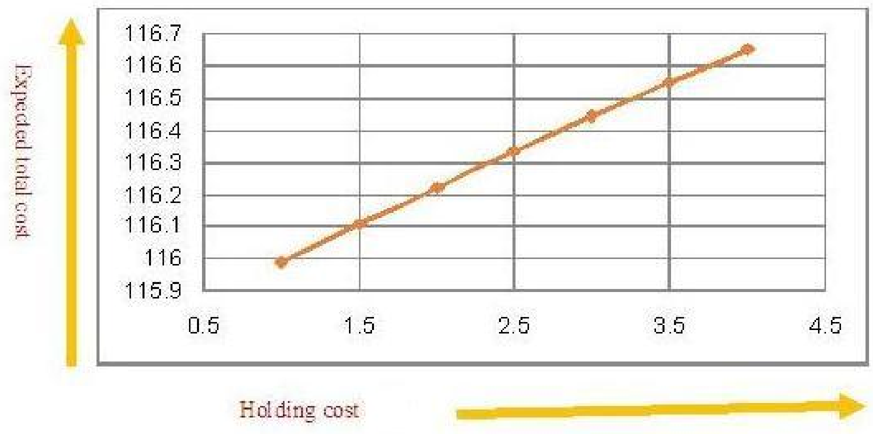

Figure 6.7: Impact of holding cost (C_h) on expected total cost $ETC(B^*, \lambda^*)$

it effects on expected total cost per unit item. Increasing value of material, energy, and labor cost increases the expected total cost per unit product (See Figure 6.9).

- As shortage costs increases, the buffer stock and production run time increase but non-inspected fraction size decreases. Therefore, to avoid shortage manager should continue the production for long time to increase the buffer inventory (See Figure 6.10).

- A higher salvage cost increases the non-inspected fraction size, whereas it decreases the buffer inventory and production run time. Therefore, to decrease the salvage cost

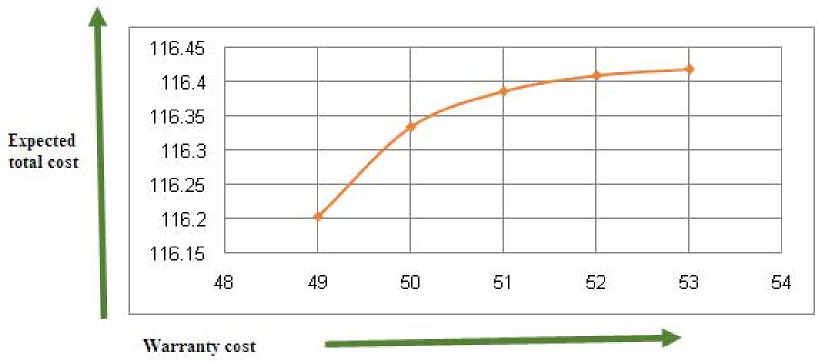

Figure 6.8: Impact of warranty cost (C_w) on expected total cost ETC(B^*, λ^*)

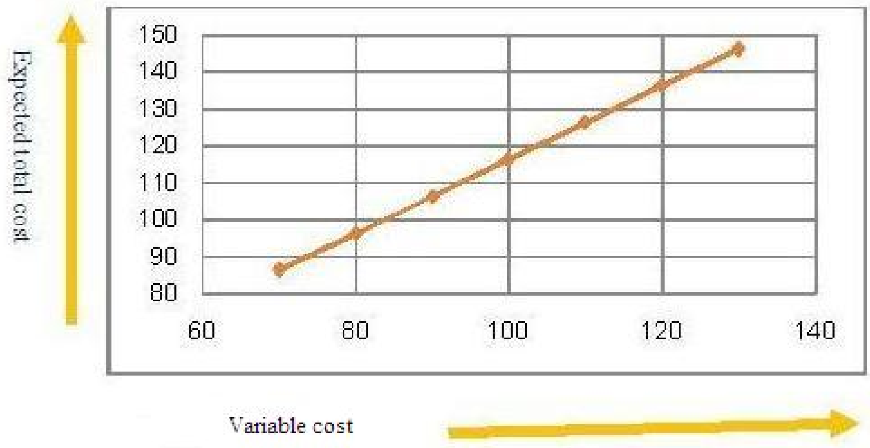

Figure 6.9: Impact of variable cost (C_m) on expected total cost ETC(B^*, λ^*)

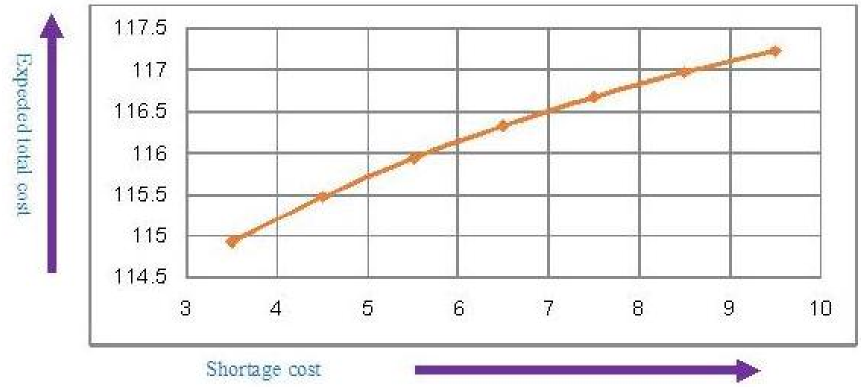

Figure 6.10: Impact of shortage cost (S_c) on expected total cost ETC(B^*, λ^*)

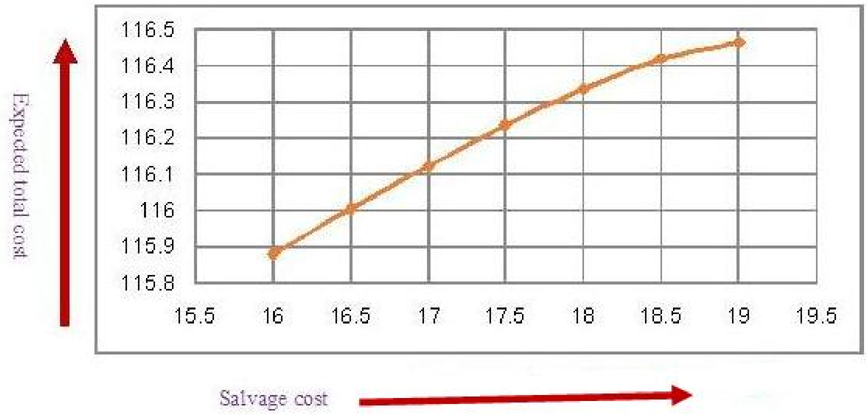

Figure 6.11: Impact of salvage cost (C_s) on expected total cost ETC(B^*, λ^*)

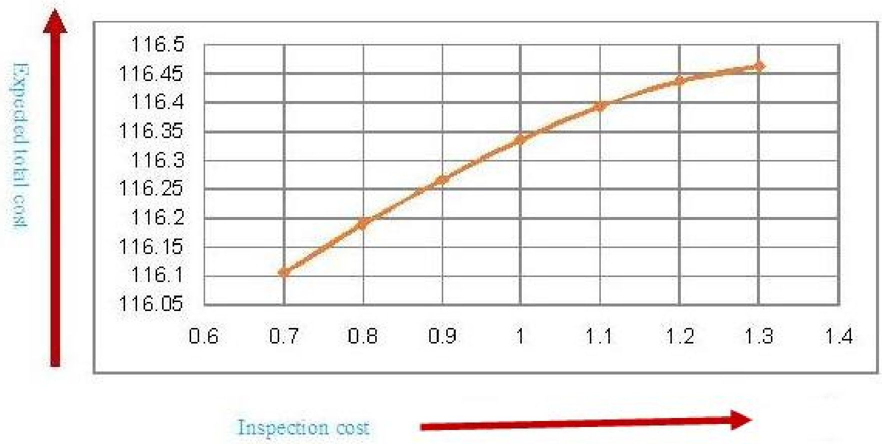

Figure 6.12: Impact of inspection cost (I_c) on expected total cost ETC (B^*, λ^*)

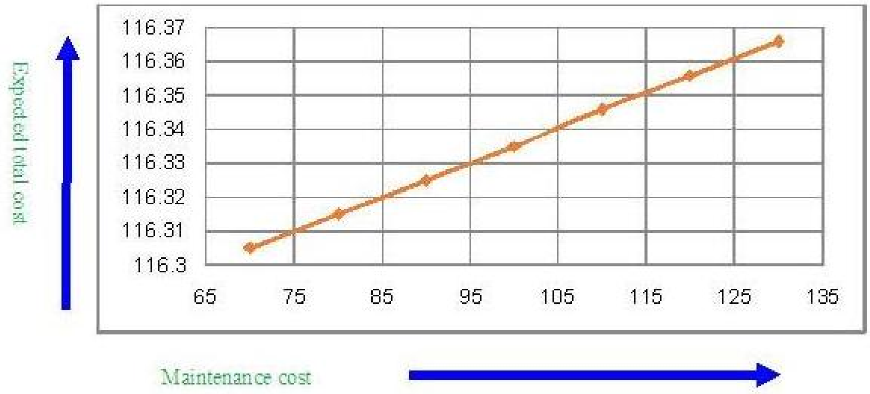

Figure 6.13: Impact of maintenance cost (M_c) on expected total cost ETC (B^*, λ^*)

Figure 6.14: Impact of Type I error cost (C_A) on expected total cost ETC(B^*, λ^*)

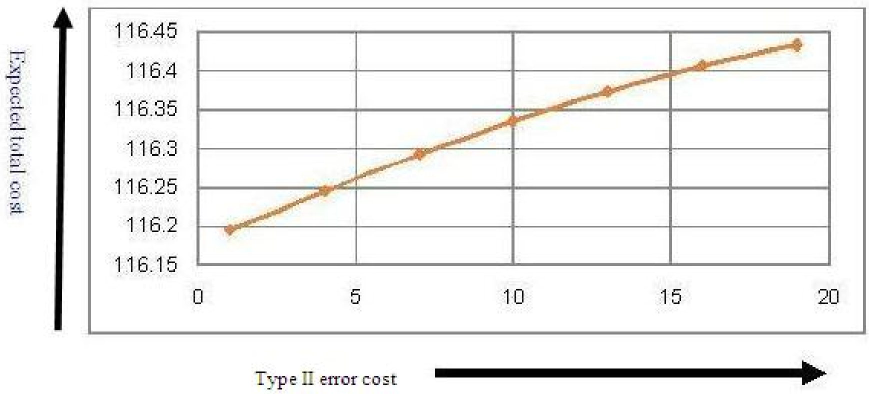

Figure 6.15: Impact of Type II error cost (C_R) on expected total cost ETC(B^*, λ^*)

and corresponding total cost manager should decrease salvaged items (See Figure 6.11).

- When the inspection cost increases, the non-inspected fraction size increases, i.e., when inspection cost increases, manager should inspect less amount of products and should sold more amount of products with warranty. The total cost per item increases with increase in inspection cost (See Figure 6.12).

- If a production manager invests more funds to maintain the machinery system then the system will produce more items which results more buffer inventory as well as less shortage. A well maintain machinery system produces less defective items, i.e., the inspection interval may reduce, which may reduce total inspection cost. The expected total cost per unit item increases with increasing value of maintenance cost (See Figure 6.13).

- Increasing value of Type I error and Type II error cost increases the optimal production run time and expected total, cost whereas it decreases the buffer inventory and non-inspected fraction size (See Figure 6.14 and 6.15).

6.6 Conclusions

This model focused on a deteriorating production process with preventive maintenance and inspection errors. Further, it was considered that during long-run production, the production system moves to an *out-of-control* state after some time. The probability of defective items being produced in *in-control* state is smaller than in the *out-of-control* state. On-line inspection and a human-based inspection were considered to detect the defective items. During inspection, two types of inspection errors were carried out, where human based inspection is not error-free. Inspected items are immediately salvaged at a cost in a parallel system before they were released for sale. Non-inspected items were sold at the market with post-sale warranty. Preventive maintenance of the machinery system was started after the completion of production and was continued up to a random time based on the necessity. During the maintenance period, there was a buffer inventory, which fulfilled the demand during maintenance. The optimal buffer inventory and optimal non-inspected fraction batch were obtained to minimize the total expected cost per unit product. A solution methodology was provided to find the optimal solution. Finally, numerical example and sensitivity analysis of the optimal solution with respect to key parameters were studied to illustrate the model and some managerial insights were provided. This model can be extended by considering the production rate and demand for products as random variables.

Chapter 7

Optimal production run time and inspection errors in an imperfect production system with warranty

Under review in *Journal of Industrial and Management Optimization*

7.1 Introduction

Generally, it is considered that the production system always produces perfect products, which is not always. It may produce defective items during long-run process. This chapter would like to discuss about this research idea.

Many researchers have investigated production models with unreliable machines. Rosenblatt and Lee (1986) initially studied the effects of process deterioration on the traditional economic manufacturing quantity (EMQ) model. In their model, they considered that the elapsed time to the *out-of-control* state is exponentially distributed and concluded that the presence of defective products generates smaller lot sizes than that of the classical economic production quantity (EPQ) inventory model. Porteus (1986) discussed an imperfect production process with significant relationship between quality and lot size, as well as obtained an optimal investment for process-quality improvement and setup cost reduction. Harriga and Ben-Daya (1998) extended Rosenblatt and Lee's (1986) model by assuming a more generalized assumption that an elapsed time, until process shift, is arbitrarily distributed, and provided distribution-based and distribution-free bounds on the optimal cost.

Goyal and Cárdenas-Barrón (2005) developed an EPQ model to determine both the optimal lot size and manufacturing process cost in an imperfect production system. Sana et al. (2007a, 2007b) extended an EMQ model in an imperfect production system in which the defective item are sold at reduce price. Cárdenas-Barrón (2009) proposed an inventory model on optimal batch sizing in a multi-stage production system with rework process. Sana (2011) developed an imperfect production model to determine the optimal product reliability and production rate to obtain maximum profit. Sana (2012) considered an imperfect production system with allowable shortages due to regular preventive maintenance for products sold with free minimal repair warranty. Chung et al. (2011) considered a deteriorating inventory model with stochastic machine unavailability time and shortage. Hsu and Hsu (2013) developed an integrated vendor-buyer inventory model to determine an optimum policy of production, where the vendor's production process is imperfect and produces a certain number of defective items with a known probability density function.

Defective items can be identified by an inspection process, which carries an inspection cost. Chryssolouris and Patel (1987) discussed a production process with imperfect items and perfect full inspection process. Salameh and Jaber (2000) developed an inventory model with 100% inspection and poor-quality items are sold as a single-batch by the end of the 100% screening process. Wang and Sheu (2001) considered a production inventory and product-inspection policy for deteriorating production systems. They did not inspect the first s produced items, but inspect only those items from the $(s + 1)$th till the end of the production run.

Full inspection policy results higher inspection cost and higher expected total cost. To reduce the inspection cost, Wang (2005) developed an inspection policy, where inspection was performed at the end of the production run. Using the same concept of Wang (2005), Wang and Meng (2009) developed another model with offline inspection policy of products. Hu and Zong (2009) proposed an extended product inspection policy for a deteriorating production system, where product inspections are performed at any time of a production cycle. There are many situations, where the organization may have more than one objective functions to optimize. Lee (2009) determined the optimal production run length and scheduled maintenance inspection policy for a deteriorating system in which the products are sold with free minimal repair warranty. To solve this type of problem, Duffuaa and El-Gáaly (2013) developed a multi-objective inventory model using 100% error-free inspection as a means of product control. Sarkar (2016) considered a supply chain coordination model with three-stage inspection to ensure perfect quality products, where the products has special features as fixed lifetime.

In the above mentioned papers, authors considers error-free inspection process to detect the imperfect items. But it is quite unrealistic that during machinery or human inspection process error may not occur. Raouf *e*t al. (1983) suggested that during inspection two types of human error may occur, one is Type I error (classifying a non-defective item as defective) and and another is Type II error (classifying a defective item as non-defective). Duffuaa and Khan (2002) extended Raouf *e*t al. (1983) inspection process for the cases of six types of misclassification errors. Wang (2007) developed an inventory model with two types of inspection errors in order to facilitate the adaptation of economic inspection/disposition model to real world applications. Darwish and Ben-Daya (2007) proposed a production-inventory model with the effect of imperfect production processes, preventive maintenance, and inspection errors.

Wang *e*t al. (2010) considered an inventory model to purchase lot size under a partial inspection policy over commonly used policy for both full and no inspection. Duffuaa and El-G*á*aly (2015) extended their's (2013) model by developing more realistic multi-objective optimization model that integrates measurement errors in inspection system. Most recently, Sarkar and Saren (2016) considered a product-inspection policy for an imperfect production system with inspection errors and warranty cost. To reduce the inspection cost, they considered the inspection policy at the end of the production cycle and the non-inspected defective items are shifted to the market with warranty.

In this proposed model, an effort has been made to obtain optimal production run time and inspection policy for an imperfect production system under an extended inspection policy. Production process is subject to a random deterioration from an *in-control* state to an *out-of-control* state with Weibull distribution. Type I and Type II inspection errors are considered. Defective items are identified through product inspection policy and salvaged with a fixed cost before being shipped. A free minimal warranty for customer is provided for the non-inspected item. Rest of this study is designed as follows: In Section 7.2.1., the problem is defined and Section 7.2.2. describes the mathematical model. In Section 7.2.3., some special cases are discussed. Numerical examples are given in Section

7.3. In 7.4, sensitivity analysis are provided. Finally, conclusions are made in Section 7.5.

7.2 Formulation of the model

This section contains problem definition, and mathematical model.

7.2.1 Problem definition

An imperfect production system for a single-type of item is considered. Production starts from *in-control* state and after a period of operation, the production system may shift to *out-of-control* state until the end of the production-run. At each state, θ_1 and θ_2 represent the percentage of the number of defective items during *in-control* state and *out-of-control* state, respectively with $\theta_1 < \theta_2$. The elapsed time until the production system shifts to the *out-of-control* state is denoted by X, which follows an exponential distribution with $g(x)$ as probability density function, $G(x)$ as distribution function, and $\overline{G}(= 1 - G(x))$ as survival function. The failure rate function of the random variable X is defined as $\phi(x) = g(x)/G(x)$. After completion of a lot, the system is inspected with fixed cost η to obtain the information about the state of the system. If the system is in *out-of-control* state, after completion of the production cycle, the production system brought back to the *in-control* state with an additional restoration cost r. To detect the defective items produced in a produced lot, a product inspection policy is carried out at a fixed cost I_c. The inspection time is considered as negligible. Product inspection policy starts from (pu_1t)th item to (pu_2t)th item, and the defective items from those inspected will be salvaged at some fixed cost C_s before being shipped. After completion of inspection, all produced products during production time u_2t to the end of production are reworked without inspections. During inspection, due to misclassification an inspector classified some non-defective items as defective with a fixed rate m_1 and classified some defective items as non-defective with a fixed rate m_2. The non-inspected defective items are taken as salvageable and those items are sent to the market with post sale (warranty) cost C_w with the assumption $I_c + C_s < C_w$.

7.2.2 Mathematical model

The inventory level starts with $p-d$ rate and depletes with a rate $-d$, where production rate (p) > demand rate (d) > 0. The total produced items are pt during cycle time t and the time duration of a production cycle is pt/d. The production cost per product is C_p and the inventory holding cost per unit per unit time is C_h. Thus, the maximum inventory is $(p-d)t$ and the holding cost is $\frac{1}{2}C_h(pt/d)(p-d)t$. Hence, the holding cost per item is $\frac{C_h(p-d)t}{2d}$. Setup cost for each production-run is S_0. Theretofore, setup cost per item is $\frac{S_0}{pt}$ and hence the system inspection cost per item is $\frac{\eta}{pt}$. If the system moves to *out-of-control* state, then r is the fixed cost to transfer the system back to the *in-control* state. Therefore, the restoration cost per unit item is $\frac{rG(t)}{pt}$.

The number of defective items in the time interval $(0, u_1 t]$ (say $N_1(t)$) are

$$N_1(t) = \begin{cases} \theta_1 pX + \theta_2 p(u_1 t - X), \text{if} & X < u_1 t \\ \theta_1 p u_1 t, \quad \text{if} & X \geq u_1 t \end{cases}$$

Thus, the expected value of $N_1(t)$ is

$$\begin{aligned} E[N_1(t)] &= \int_0^{u_1 t} [\theta_1 pX + \theta_2 p(u_1 t - X)]g(x)dx + \int_{u_1 t}^{\infty} \theta_1 p u_1 t g(x)dx \\ &= \theta_1 p \left[-u_1 t\overline{G}(u_1(t)) + \int_0^{u_1(t)} \overline{G}(x)dx \right] + \theta_2 p u_1 t \left[1 - \overline{G}(u_1(t))\right] \\ &- \theta_2 p \left[-u_1 t\overline{G}(u_1(t)) + \int_0^{u_1 t} \overline{G}(x)dx \right] + \theta_1 p u_1 t\overline{G}(u_1(t)) \\ &= \theta_2 p u_1 t - (\theta_2 - \theta_1)p \int_0^{u_1 t} \overline{G}(x)dx \end{aligned}$$

Similarly, the expected number of defective items in the time interval $[0, u_2t]$ is

$$E[N_2(t)] = \theta_2 p u_2 t - (\theta_2 - \theta_1)p \int_0^{u_2t} \overline{G}(x)dx$$

Hence, expected number of defective items in the time interval $[u_1t, u_2t]$ is

$$E[N_2(t)] - E[N_1(t)] = \theta_2 p(u_2 - u_1)t - (\theta_2 - \theta_1)p \int_{u_1t}^{u_2t} \overline{G}(x)dx$$

During product screening process, the inspectors make Type I and Type II errors. They classify some non-defective item as defective item i.e., $(1 - \theta_1)m_1$ in *in-control* state and $(1 - \theta_2)m_1$ in *out-of-control* state. In another side, the inspector classifies some defective items as non-defective i.e., $\theta_1 m_2$ in *in-control* state and $\theta_2 m_2$ in *out-of-control* state. Therefore, the inspector rejects some non-defective items and accepts some defective items, thus, two cases arise.

Case I $X < u_1t$

During $[0, X]$ inspector accepts $\theta_1 pX$ defective items in which falsely accepted defective items are $\theta_1 pXm_2$ with some fixed cost C_A per unit and falsely rejected non-defective items are $(1 - \theta_1)pXm_1$ with some fixed cost C_R per unit.

Therefore, the defective cost in $[0, X]$ is

$C_R p(1 - \theta_1)Xm_1 + C_A\theta_1 pX(1 - m_2)$

On the other hand, during $[X, u_1t]$ inspector accepts $\theta_2 p(u_1t-X)$ defective items, in which falsely accepted defective items are $\theta_2 p(u_1t-X)m_2$ and falsely rejected non-defective items are $(1 - \theta_2)p(u_1t - X)m_1$.

Therefore, the defective cost in $[X, u_1t]$ is

$C_R p(1 - \theta_2)(u_1t - X)m_1 + C_A\theta_2 p(u_1t - X)(1 - m_2)$.

Hence, the defective cost in $[0, u_1t]$ under the condition $X < u_1t$ is

$C_R p[(1 - \theta_1)X + (1 - \theta_2)(u_1t - X)]m_1 + C_A p[\theta_1 X + \theta_2(u_1t - X)](1 - m_2)$.

Case II $X \geq u_1 t$

In this case, the inspector accepts $\theta_1 p u_1 t$ defective items in which falsely accepted defective items are $\theta_1 p u_1 t m_2$ and falsely rejected non-defective items are $(1-\theta_1) p u_1 t$.

Therefore, the defective cost in $[0, u_1 t]$ under the condition $X \geq u_1 t$ is

$C_R(1-\theta_1) p u_1 t m_1 + C_A \theta_1 p u_1 t (1-m_2)$.

Therefore, total defective cost is

$$C_d = \begin{cases} C_R p[(1-\theta_1)X + (1-\theta_2)(u_1 t - X)] m_1 & \\ +C_A p[\theta_1 X + \theta_2(u_1 t - X)](1-m_2), & \text{if } X < u_1 t \\ C_R(1-\theta_1) p u_1 t m_1 + C_A \theta_1 p u_1 t (1-m_2), & \text{if } X \geq u_1 t \end{cases}$$

The expected value of defective cost within the time interval $[0, u_1 t]$ is

$$\begin{aligned} E[C_{d_1}] &= \Big\{ C_R[(1-\theta_1)X + (1-\theta_2)(u_1 t - X)] p E[m_1] \\ &+ C_A[\theta_1 X + \theta_2(u_1 t - X)] p(1 - E[m_2]) \Big\} \int_0^{u_1 t} g(x)dx \\ &+ \Big\{ C_R(1-\theta_1) p u_1 t E[m_1] + C_A \theta_1 p u_1 t (1 - E[m_2]) \Big\} \int_{u_1 t}^{\infty} g(x)dx \\ &= p(\theta_1 - \theta_2)\{C_A(1 - E[m_2]) - C_R E[m_1]\} \int_0^{u_1 t} x g(x)dx \\ &+ \{C_R p(1-\theta_2) u_1 t E[m_1] + C_A p \theta_2 u_1 t (1 - E[m_2])\} \int_0^{u_1 t} g(x)dx \\ &+ \{C_R(1-\theta_1) p u_1 t E[m_1] + C_A \theta_1 p u_1 t (1 - E[m_2])\} \int_{u_1 t}^{\infty} g(x)dx \\ &= p(\theta_1 - \theta_2)\{C_A(1 - E[m_2]) - C_R E[m_1]\} \left[-u_1 t \overline{G}(u_1 t) + \int_0^{u_1 t} \overline{G}(x)dx \right] \\ &+ \{C_R p(1-\theta_2) u_1 t E[m_1] + C_A p \theta_2 u_1 t (1 - E[m_2])\} \left[1 - \overline{G}(u_1 t) \right] \\ &+ \{C_R(1-\theta_1) p u_1 t E[m_1] + C_A \theta_1 p u_1 t (1 - E[m_2])\} \overline{G}(u_1 t) \\ &= p(\theta_1 - \theta_2)\{C_A(1 - E[m_2]) - C_R E[m_1]\} \int_0^{u_1 t} \overline{G}(x)dx \\ &+ p u_1 t \{C_A \theta_2 (1 - E[m_2]) + C_R(1-\theta_2) E[m_1]\} \end{aligned}$$

Similarly, expected defective cost in the time interval $[0, u_2t]$ is

$$\begin{aligned} E[C_{d_2}] &= pu_2t\Big\{C_A\theta_2(1-E[m_2]) + C_R(1-\theta_2)E[m_1]\Big\} \\ &+ p(\theta_1-\theta_2)\Big\{C_A(1-E[m_2]) - C_RE[m_1]\Big\}\int_0^{u_2t}\overline{G}(x)dx \end{aligned}$$

Hence,the expected defective cost within the interval $[u_1t, u_2t]$ is

$$\begin{aligned} E[C_{d_2}] - E[C_{d_1}] &= p(u_2-u_1)t\left\{C_A\theta_2(1-E[m_2]) + C_R(1-\theta_2)E[m_1]\right\} \\ &+ p(\theta_1-\theta_2)\left\{C_A(1-E[m_2]) - C_RE[m_1]\right\}\int_{u_1t}^{u_2t}\overline{G}(x)dx \end{aligned}$$

Therefore, the expected total warranty, salvage, defective, and rework cost with lot size pt under the inspection policy (u_1, u_2) is

$$C_wE[N_1(t)] + C_s\{E[N_2(t)] - E[N_1(t)]\} + E[C_{d_2}] - E[C_{d_1}] + C_rpt(1-u_2)$$

Now, the expected total cost per item i.e., $C(t, u_1, u_2)$ is the addition of manufacturing cost, holding cost, setup cost, process inspection cost, restoration cost, product inspection cost, warranty cost, salvage cost, defective cost, and rework cost as follows:

$$\begin{aligned} C(t,u_1,u_2) &= C_p + \frac{C_h(p-d)t}{2d} + \frac{K+rG(t)}{pt} + (I_c + C_s\theta_2)(u_2-u_1) + C_w\theta_2u_1 \\ &+ C_r(1-u_2) + (u_2-u_1)\Big[C_A(1-E[m_2])\theta_2 + C_RE[m_1](1-\theta_2)\Big] \\ &+ \frac{(\theta_1-\theta_2)}{t}\left\{C_s + C_A(1-E[m_2]) - C_RE[m_1]\right\}\int_{u_1t}^{u_2t}\overline{G}(x)dx \\ &+ \frac{C_w(\theta_1-\theta_2)}{t}\int_0^{u_1t}\overline{G}(x)dx \end{aligned}$$

where $K = S_0 + \eta$.

The objective is to obtain the optimum value of t, u_1, and u_2 such that $C(t, u_1, u_2)$ is minimum. The optimization of this model is done by numerical example. This model is the extension of several models which are discussed in the next section.

7.2.3 Some special cases

Case I If $u_2 = 1$ and $C_r = C_s$, then this model converges to Sarkar and Saren's (2016) model. In this case, the cost function is

$$\begin{aligned} C(t,u_1) &= C_p + \frac{C_h(p-d)t}{2d} + \frac{K+rG(t)}{pt} + I_c(1-u) + C_w\theta_2 u \\ &+ (1-u)\Big[C_s\theta_2 + C_A(1-E[m_2])\theta_2 + C_R E[m_1](1-\theta_2)\Big] \\ &+ \frac{(\theta_1-\theta_2)}{t}\{C_s + C_A(1-E[m_2]) - C_R E[m_1]\}\int_{u_1 t}^{t}\overline{G}(x)dx \\ &+ \frac{C_w(\theta_1-\theta_2)}{t}\int_0^{u_1 t}\overline{G}(x)dx \end{aligned}$$

which is the same cost function of Sarkar and Saren's (2016) model.

Case II If $C_A = C_R = 0$, and $C_s = C_r$, then the model becomes Hu and Zong's (2009) model. In this case, the cost function is

$$\begin{aligned} C(t,u_1,u_2) &= C_p + \frac{C_h(p-d)t}{2d} + \frac{K+rG(t)}{pt} + (I_c + C_r\theta_2)(u_2-u_1) + C_w\theta_2 u_1 \\ &+ C_r(u_2-u_1) + \frac{(\theta_1-\theta_2)}{t}\left\{(C_w - C_r)\int_0^{u_1 t}\overline{G}(x)dx + C_r\int_0^{u_2 t}\overline{G}(x)dx\right\} \end{aligned}$$

which is the same cost function with Hu and Zong's (2009) model.

Case III If $u_2 = 1$, $C_A = C_R = 0$, and $C_s = C_r$, then the model becomes Wang's (2005) model. In this case, the cost function is

$$\begin{aligned} C(t,u_1,u_2) &= C_p + \frac{C_h(p-d)t}{2d} + \frac{K+rG(t)}{pt} + (I_c + C_r\theta_2)(1-u) + C_w\theta_2 u \\ &+ C_r(1-u) + \frac{(\theta_1-\theta_2)}{t}\left\{(C_w - C_r)\int_0^{u_1 t}\overline{G}(x)dx + C_r\int_0^{t}\overline{G}(x)dx\right\} \end{aligned}$$

which is the same cost function with Wang's (2005) model.

In the next part, the numerical comparison of these three models with our model is discussed.

7.3 Numerical examples

This section presents some numerical examples to demonstrate the present model and the results are compared with the existing models. Here, it is considered that the production system follows a Weibull distribution $G(x) = 1 - e^{-(x/\alpha)^\beta}$, where $\alpha > 0$ is scale parameter and $\beta \geq 1$ is shape parameter and inspection errors (Type I and Type II) follow the uniform distribution with

$$g(m_1) = \begin{cases} 25, & 0 \leq m_1 \leq 0.04 \qquad \text{i.e., } E[m_1] = 0.02 \\ 0, & \text{otherwise} \end{cases}$$

and

$$g(m_2) = \begin{cases} 25, & 0 \leq m_2 \leq 0.04 \qquad \text{i.e., } E[m_2] = 0.02 \\ 0, & \text{otherwise} \end{cases}$$

The parametric values are taken as $K = S_0 + \eta =$ \$20000/setup, $p = 1200$ units/day, $d = 350$ units/day, $C_h =$ \$0.5/unit/day, $C_p = 0.3$, $C_w =$ \$8/non-inspected defective lot, $C_s =$ \$2.6/defective lot, $C_r =$ \$3/defective lot, $I_c =$ \$1.2/ per defective lot, $r =$ \$1500, $C_A =$ \$0.3/ defective lot, $C_R =$ \$0.1/ defective lot, $G(t) = 1 - e^{-(t/0.5)^2}$, $G(x) = 1 - e^{-(x/0.5)^2}$ (See Figure 7.1 and Figure 7.2).

Sarkar and Saren (2016) considered inspection at the end of the production process, whereas this model considers inspection at the arbitrary time of the production process. Form Table 7.1, one can find out that expected total cost per item for this model is $C(t^*, u_1^*, u_2^*) =$ \$8.48. If Sarkar and Saren's (2016) model is applied, then total cost

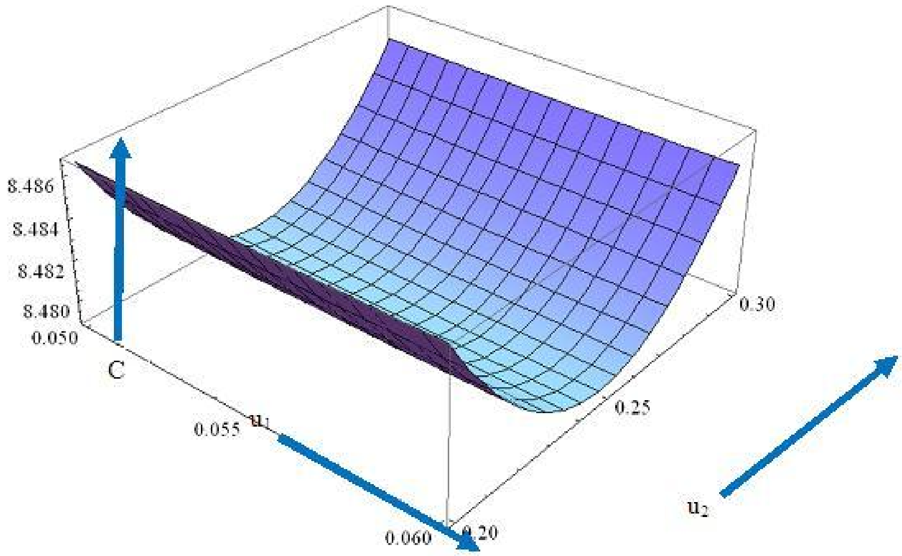

Figure 7.1: Plot of expected total cost $C(u_1, u_2|t^* = 2.19)$

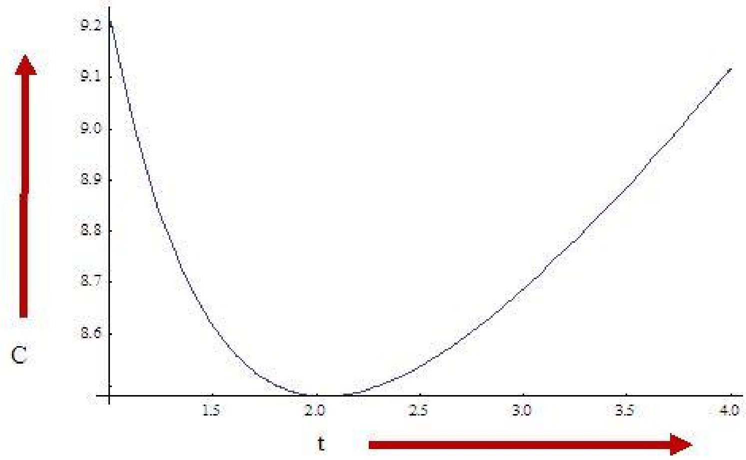

Figure 7.2: Plot of expected total cost $C(t|u_1^* = 0.00132, u_2^* = 0.01023)$

Table 7.1: Summary of numerical results

This model (t^*, u_1^*, u_2^*) $C(t^*, u_1^*, u_2^*)$	Sarkar and Saren (2016) (t^*, u^*) $C(t^*, u^*)$	Hu and Zong (2009) (t^*, u_1^*, u_2^*) $C(t^*, u_1^*, u_2^*)$	Wang (2005) (t^*, u^*) $C(t^*, u^*)$
(2.04,0.058,0.251) \$8.48	(1.85,0.064) \$8.90	(2.05,0.062,0.239) \$8.49	(1.83,0.069) \$8.97

is $C(t^*, u^*) = \$8.90$ with a percentage of cost penalty (PCP) which is computed as $100 \times [C(t^*, u_1^*, u_2^*) - C(t^*, u^*)]/C(t^*, u^*) = 4.71\%$.

Hu and Zong (2009) considered a extended inspection policy, where inspection process was performed at the middle of the production process, but they did not considered the inspection error. This model considers same extended inspection policy with more realistic assumption as two types of inspection errors. Table 7.1 shows that expected total cost per item for this model is $C(t^*, u_1^*, u_2^*) = \$8.48$. If Hu and Zong's (2009) model is applied, then total cost is $C(t^*, u_1^*, u_2^*) = \$8.49$.

Wang (2009) considered an inspection policy, where inspection process was performed at the end of the production process and the inspected items was salvaged. Non-inspected items were sold at market with free minimal warranty. But they did not consider inspection error. From Table 7.1, one can find that expected total cost per item for this model is $C(t^*, u_1^*, u_2^*) = \$8.48$. If Wang's (2005) model is applied, then total cost is $C(t^*, u^*) = \$8.97$ with a percentage of cost penalty (PCP) which is computed as $100 \times [C(t^*, u_1^*, u_2^*) - C(t^*, u^*)]/C(t^*, u^*) = 5.46\%$.

7.4 Sensitivity Analysis

In this section, the effects of changes in major cost parameters such as C_h, C_s, C_w, C_r, I_c and r on optimal solution are studied. The results are presented in Table 7.2. The effect of major cost parameters on the optimal solution are shown in Figures (7.3-7.8).

From Table 7.2, the following observations are made:

* Table 7.2 indicates that increasing value of C_h increases the total cost per item $C(t^*, u_1^*, u_2^*)$. Holding cost can be reduced by reducing the production run length t^*. Reducing value of production run length will reduce the total produced items as well as total holding cost and the total cost per item. Therefore, the production run length decreases as holding cost increases.

Table 7.2: Impact of key parameters on optimal solution

C_h	t^*	u_1^*	u_2^*	$C(\cdot)$
0.1	4.56	0.026	0.112	7.12
0.3	2.64	0.045	0.194	7.92
0.5	2.04	0.058	0.251	8.48
0.7	1.72	0.068	0.296	8.93
0.9	1.52	0.078	0.336	9.32
1.1	1.37	0.086	0.372	9.68
C_s	t^*	u_1^*	u_2^*	$C(\cdot)$
2.0	1.98	0.032	0.382	8.41
2.2	2.01	0.042	0.317	8.44
2.4	2.03	0.050	0.278	8.46
2.6	2.04	0.058	0.251	8.48
2.8	2.05	0.065	0.229	8.49
3.0	2.06	0.072	0.211	8.51
3.2	2.07	0.079	0.196	8.52
C_w	t^*	u_1^*	u_2^*	$C(\cdot)$
6.4	2.014	0.123	0.254	8.446
6.8	2.024	0.105	0.253	8.457
7.2	2.031	0.089	0.252	8.466
7.6	2.037	0.073	0.251	8.473
8.0	2.042	0.058	0.251	8.479
8.4	2.045	0.041	0.250	8.483
8.8	2.047	0.018	0.249	8.486
C_r	t^*	u_1^*	u_2^*	$C(\cdot)$
2.4	2.14	0.0552	0.15	8.00
2.6	2.11	0.0560	0.17	8.16
2.8	2.08	0.0568	0.21	8.32
3.0	2.04	0.0579	0.25	8.48
3.2	1.10	0.0592	0.30	8.62
3.4	1.94	0.0608	0.36	8.76
3.6	1.87	0.0631	0.49	8.88
I_c	t^*	u_1^*	u_2^*	$C(\cdot)$
1.0	2.007	0.003	0.017	8.437
1.2	2.041	0.058	0.251	8.479
1.4	2.068	0.084	0.214	8.511
1.6	2.085	0.105	0.181	8.531
1.8	2.093	0.124	0.151	8.542
2.0	2.094	0.141	0.120	8.543
r	t^*	u_1^*	u_2^*	$C(\cdot)$
1200	1.94	0.061	0.264	8.35
1300	1.97	0.060	0.259	8.40
1400	2.01	0.059	0.255	8.44
1500	2.04	0.058	0.251	8.48
1600	2.07	0.057	0.246	8.52
1700	2.11	0.056	0.243	8.56
1800	2.14	0.055	0.239	8.60

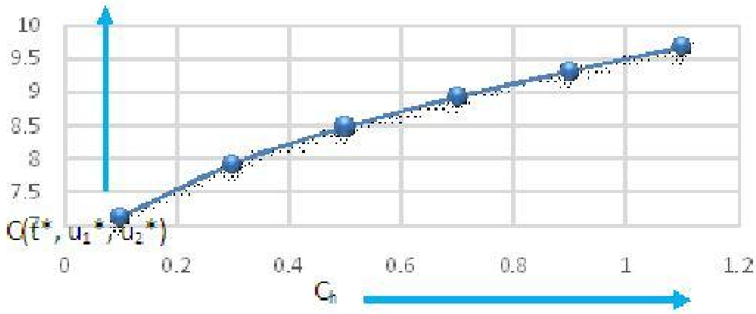

Figure 7.3: Impact of holding cost (C_h) on expected total cost $C(t^*, u_1^*, u_2^*)$

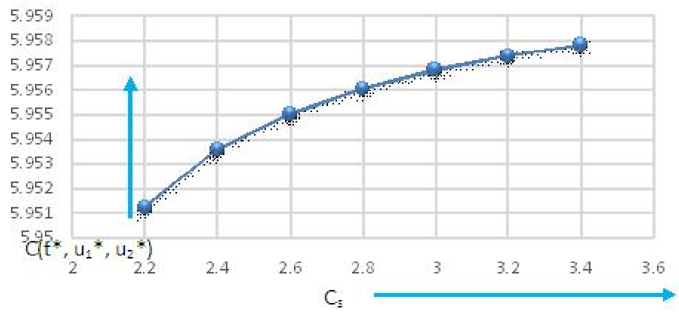

Figure 4. Impact of salvage cost (C_s) on expected total cost $C(t^*, u_1^*, u_2^*)$

Figure 7.4: Impact of salvage cost (C_s) on expected total cost $C(t^*, u_1^*, u_2^*)$

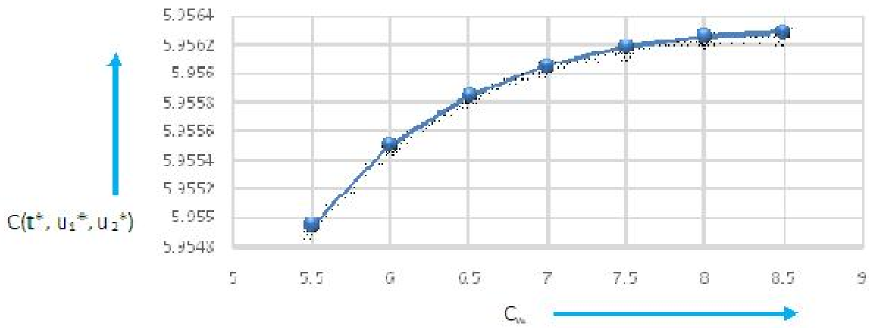

Figure 7.5: Impact of warranty cost (C_w) on expected total cost $C(t^*, u_1^*, u_2^*)$

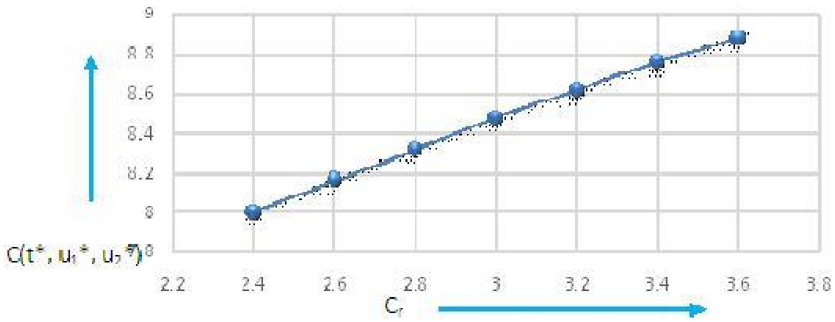

Figure 7.6: Impact of rework cost (C_r) on expected total cost $C(t^*, u_1^*, u_2^*)$

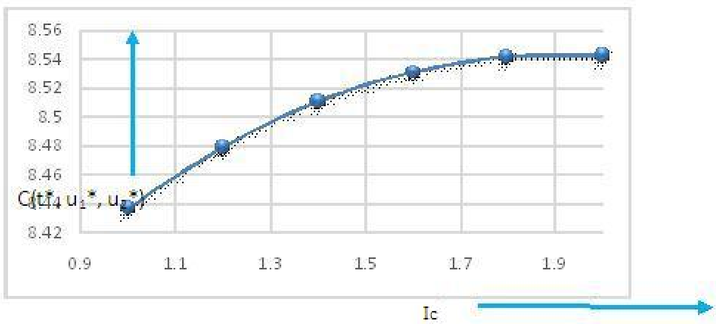

Figure 7.7: Impact of inspection cost (I_c) on expected total cost $C(t^*, u_1^*, u_2^*)$

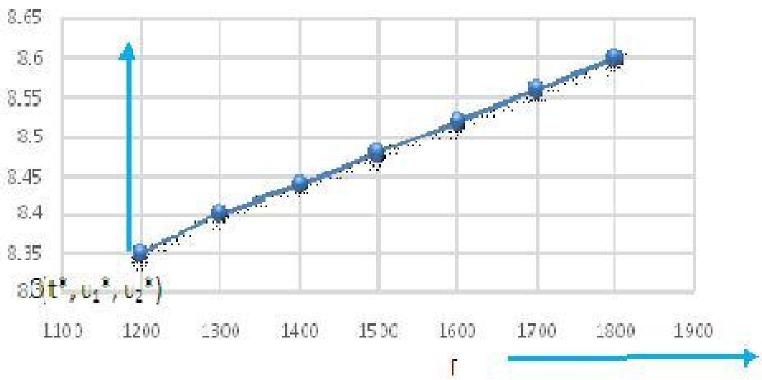

Figure 7.8: Impact of restoration cost r on expected total cost $C(t^*, u_1^*, u_2^*)$

* From Table 7.2, one can observe if C_w increases, then the total cost per item $C(t^*, u_1^*, u_2^*)$ increases. Increasing value of C_w implies more warranty cost, which results in greater total cost per item. Therefore, manufacturer have to decrease the failure rate of the items in the warranty period to avoid the greater warranty cost.

* From Table 7.2, one can find out if salvage cost C_s increases, then total cost per item $C(t^*, u_1^*, u_2^*)$ increases. Increasing value of C_s implies more defective items, which implies more items to inspect, i.e., more inspection cost as well as grater total cost. To reduce the total salvage cost as well as the total cost per item, manufacturer has to reduce the total inspected batch size, i.e., u_2^*. Therefore, increasing value of C_s decreases u_2^*.

* The effect of rework cost C_r on total cost $C(t^*, u_1^*, u_2^*)$ is clear from Table 7.2. Increasing value of C_r indicates more rework cost as well as greater total cost per item. Therefore, rework cost can be decrease by reducing imperfect items. By decreasing the production run time, one can reduce the total number of imperfect items and higher rework cost. Therefore, the production run length t^* decreases as C_r increases.

* When the inspection cost I_c increases, the total cost per item $C(t^*, u_1^*, u_2^*)$ increases. To reduce the total inspection cost, the manufacturer has to reduce the inspected fraction batch u_2^*. Therefore, u_2^* decreases as I_c increases.

* When the restoration cost r increases, the expected total cost per item $C(t^*, u_1^*, u_2^*)$ also increases. Therefore, to avoid the frequent restorations, the production run length should be longer.

7.5 Conclusions

This study deals with an imperfect production system under the presence of Type I and Type II errors. These two types of inspection errors are considered as random variable with known probability density function. Inspection interval was considered as variable. Inspected items are salvaged as some fixed cost before they are released for sale. Non-inspected items are sold at market with post-sale warranty. The total cost per item was minimized by obtaining the production run time and inspection policy. The numerical comparison between other models has been conducted to display the impact of inspection errors on the optimum solutions. Finally, the sensitivity analysis on the optimal solution with respect to key parameters were studied to illustrate the model and some managerial insights were provided. This model can be extended for items having linear increasing demand, price, and advertising-dependent demand or power-demand. This study can be extended further by considering stochastic demand (see for instance Pal, Sana, and Chaudhuri (2013a)). One another extension can be considered by allowing process maintenance during a production run (see for instance Sana (2012)).

Chapter 8

General conclusions and scope of future research

8.1 General conclusions

From the second world war to till date, various research papers are published by several researches on inventory management/control system. Their efforts have enriched the inventory management systems. Nowadays, the inventory management system are used as a decision maker in various fields. It is applied all most all the branches of scientific study, trade and commerce, etc. The main goal of inventory management concerns the satisfactory service-lines between replenishment lead time, carrying costs of inventory, asset management, inventory forecasting, physical inventory, future inventory price forecasting, available physical space for inventory, returns, replenishment, quality management and defective goods and demand forecasting. The aim of inventory management is to balance all these at a time. The present research works on some problems on production planning and inventory management.

The main aim of any company is to meet demand on time and for this purpose the company has to keep inventory in their own warehouse (OW). We know that, the capacity of any warehouse is limited. In many practical situations there exist various factors (like price discounts for bulk purchases or if the items are seasonal product) under consideration that induce the decision maker of the inventory system to order more goods than can be stored in his warehouse. In this case, company will either rent (RW) other warehouse or rebuild a new warehouse. They usually choose RW because it is more economical for any manufacturing system. The company generally wants more profit with the help of production of different types of items which may keep in OW or RW. In **Chapter 2**, an attempt have been made to develop and to solve a two-warehouse inventory model with the assumption of quadratic demand which is useful for the items whose demand increases very rapidly such as newly launched products in the market. In this study two-warehouse model is discussed, which is based on time dependent deterioration rates. The associated cost of this model is obtained by analytical method. A numerical example is given to illustrate the model. The sensitivity analysis is also allowed to check the deviation of the parameters in the range −50% to +50%. This model is applicable in an industry where the production rate is fixed throughout the production-run, demand increases rapidly and the item has a variable deterioration rate.

In this competitive environment, when a new brand of consumer goods are launched, the demand of goods increases quickly to a certain moment and after sometime it stabilizes. Finally, it becomes almost constant. Keeping in mind this type of demand pattern, in **Chapter 3**, demand is considered as a ramp-type function of time. To make the research a more realistic one, four different types of continuous probabilistic deterioration functions are considered here. The associated profit function is maximized at the optimal values of decision variables. A unique solution procedure is provided to find the optimal solution. Some numerical examples, graphical representations, special cases, and sensitivity analysis are given to illustrate the model.

In modern marketing environment, a practical problem is to control the deterioration of items. Some products (e.g., fruits, vegetables, pharmaceutical, volatile liquids, and others) not only deteriorate continuously due to evaporation, obsolescence, spoilage, etc., but also it deteriorates with increase in time (i.e., a deteriorating item has its maximum lifetime). In existing literature, very few researcher considered maximum lifetime of deteriorating items in their model. Keeping this in mind, in **Chapter 4** an inventory model is considered for products with maximum lifetime, time-varying deterioration rate, and ramp-type demand. A simple solution procedure is given and existence and uniqueness of the optimal solutions are obtained analytically. This model minimized the associated cost function at the optimal values of the decision variable. Sensitivity analysis on the optimal solution with respect to key parameters are studied to illustrate the model and some managerial insights were provided. This model used the concept of fixed lifetime of products as in time-varying deterioration rate. Therefore, products which have fixed lifetime and the deterioration rate is time-dependent, the managers of the different industries can follow this strategies.

Chapter 5 discussed the effect of reliability on setup cost and deterioration rate. An algebraical procedure is used to minimize the cost for the entire supply chain management model. Finally, a closed-form solution is obtained. The main contribution of this model is to obtain the minimum cost with integer number of deliveries, optimal lot size, and reliability by using algebraical procedure. The proposed procedure for the computation of the total cost of the supply chain management can be easily done without any tedious calculation.

In most of the inventory model, it is considered so far that the produced products are perfect in nature. But in reality, due to the different types of machinery problems during production run-time, it is often see that some of the items may be defective in nature which are reworked at a cost to make them perfect. In **Chapter 6**, an imperfect production system is considered with preventive maintenance. There is a buffer inventory, which fulfilled the demand during maintenance period. A strategy is formulated, which gives the minimum value of total expected cost per unit product with optimal buffer inventory and optimal non-inspected fraction batch. This strategy is supported by the numerical examples, so from the practical point of view, this strategy is valid and useful to the competitive business.

Chapter 7 is devoted to develop an imperfect production process under the presence of Type I and Type II errors. The *out-of-control* probability of the system as well as Type I and Type II inspection errors are considered as random variable with known probability density function. Variable inspection interval is consider here. Inspected items are salvaged at some fixed cost before they are released for sale. Non-inspected items are sold at market with post-sale warranty. This model minimized the total cost per item by obtaining the production run time and inspection policy. The numerical comparison between other models has been conducted to display the impact of inspection errors on the optimum solutions. Finally, the sensitivity analysis on the optimal solution with respect to key parameters were studied to illustrate the model and some managerial insights are provided.

8.2 Scope of future research

For future research a lot of scope may arise from this thesis.

(*i*) In this thesis, we have taken a two-warehouse inventory model (for single-item) with the assumption that the RW has unlimited capacity but any company has budget constrains, so RW may take limited capacity. One may develop this model for multi-item inventory model.

(*ii*) Since most of the physical events are more or less uncertain, so, the deterministic models are not reflect actually the practical life. Hence the new idea fuzzy demand may consider in these models.

(*iii*) Different deteriorating inventory models are considered here. These studies can be extended further by considering the preservation technology cost for deteriorating products.

(*iv*) In these six models, the production rate and demand rate for products are taken as constant. These models can be extended by considering the production rate and demand rate as random variable.

(*v*) Machine breakdown is not consider in these models. During production run time, machine breakdown can be considered in future.

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