2018

M.Sc.

1st Semester Examination

MATHEMATICS

PAPER—MTM-103

Full Marks : 50

Time : 2 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answers wherever necessary.

The symbols have their usual meaning.

ODE and Special Functions

1. Answer any *four* questions : 4×2

(a) Define fundamental set of solutions for system of ordinary differential equation.

(b) Find all the singularities of the following differential equation and then classify them :

$$(z - z^2)\,\omega'' + (1 - 5z)\omega' - 4\omega = 0.$$

(c) Find the value of $P_n(1)$.

(d) Show that $J_n(z)$ is an odd function of z if n is odd.

(Turn Over)

(e) Define orthogonal functions associated with Sturm-Liouville problem.

(f) Deduce the symmetric property of Gauss's hypergeometric function.

(g) Show that $J_0(z) + 2J_2(z) + 2J_4(z) = 1$.

(h) Show that $\int_{-1}^{1} P_n(z)dz = \begin{cases} 0, & n \neq 0 \\ 2, & n = 0 \end{cases}$

2. Answer any *four* questions : 4×4

(a) Show that $J_0^2(z) + 2\sum_{n=1}^{\infty} J_n^2(z) = 1$ and prove that for real z, $|J_0(z)| < 1$ and $|J_n(z)| < \frac{1}{\sqrt{2}}$, for all $n \geq 1$.

(b) Using Green's function method, solve the following differential equation $y''(x) = 0$, subject to boundary conditions $y(0) = y'(1)$, $y'(0) = y(1)$.

(c) Deduce the integral formula for the Gauss hypergeometric function.

(d) If $z > 1$, then prove that $P_n(z) < P_{n+1}(z)$.

(e) Find out the generating function for Bessel function of integral order.

(f) Show that $1 + 3P_1(z) + 5P_2(z) + 7P_3(z) + \ldots + (2n + 1)P_n(z)$

$= \frac{d}{dz}\left[P_{n+1}(z) + P_n(z)\right]$, where $P_n(z)$ denotes the Legendre's Polynomial of degree n.

(g) If the vector functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ defined as follows :

$$\varphi_1 = \begin{bmatrix} \varphi_{11} \\ \varphi_{21} \\ \cdot \\ \varphi_{n1} \end{bmatrix}, \varphi_2 = \begin{bmatrix} \varphi_{12} \\ \varphi_{22} \\ \cdot \\ \varphi_{n2} \end{bmatrix}, \ldots \varphi_n = \begin{bmatrix} \varphi_{1n} \\ \varphi_{2n} \\ \cdot \\ \varphi_{nn} \end{bmatrix}$$

be n solutions of the homogeneous linear differential equation $\frac{dx}{dt} = A(t)x(t)$ in the interval $a \le t \le b$, then these n solutions are linearly independent in $a \le t \le b$ iff Wronskian $W[\varphi_1, \varphi_2, \ldots, \varphi_n] \ne 0 \ \forall t$, on $a \le t \le b$.

(h) Consider the boundary value problem

$$\frac{d^2y}{dx^2} + \lambda y = 0, \ 0 \le x \le \pi$$

subject to $y(0) = 0$, $y'(\pi) = 0$. Find the eigen values and eigen functions of the problem.

3. Answer any *two* questions : 2×8

(a) (i) All the eigen values of regular SL problem with $r(x) > 0$, are real.

(ii) Find the general solution of the homogeneous system

$$\frac{dx}{dt} = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ where } X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

(b) (i) Establish generating function for Legendre polynomial. Use it to prove that

$$(2n+1)zP_n(z) = (z+1)P_{n+1}(z) + nP_{n-1}(z).$$

(ii) Deduce the confluent hypergeometric differential equation from hypergeometric differential equation.

(4+2)+2

(c) (i) Prove that if $f(z)$ is continuous and has continuous derivatives in $[-1, 1]$ then $f(z)$ has unique Legendre series expansion is given by where P_n's are Legendre Polynomials

$$C_n = \frac{2n+1}{2}\int_{-1}^{1} f(z)P_n(z)dz, \ n = 1, 2, 3...$$

(ii) Prove that $\frac{d}{dz}\left[J_0(z)\right] = -J_1(z)$. 6+2

(d) (i) Find the general solution of the ODE

$2zw''(z) + (1 + z)w'(z) - kw = 0$ (where k is a real constant) in series form. For which values of k is there a polynomial solution ?

(ii) Show that $\sqrt{\frac{\pi z}{2}}\, J_{3/2}(z) = \frac{1}{z}\sin z - \cos z$. 5+3

[Internal Assessment — 10 Marks]

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