2018

M.Sc.

1st Semester Examination

MATHEMATICS

PAPER—MTM-102

Full Marks : 50

Time : 2 Hours

The figures in the right-hand margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answers wherever necessary.

Complex Analysis

1. Answer any *four* questions : 4×2

(a) Sketch $S = \left\{ z : \left| \frac{z+1}{z-1} \right| < 1 \right\}$ and decide whether it is domain.

(b) Find the length of the curve $C: z(t) = (1 - i)t^2, -1 \le t \le 1$.

(c) Find all the points of discontinuity of the function

$$f(z) = \frac{\tanh z}{z^2 + 1}.$$

(d) Is it possible to evaluate the integral $\int_C f(z)dz$ where

$f(z) = \frac{5z + 2}{z(z - 2)}$ and $C: |z| = 1$ using the single residue

of $\frac{1}{z^2} f\left(\frac{1}{z}\right)$ at $z = 0$? Justify.

(e) Write the Cauchy Integral formula for n^{th} order derivative and then based on this discuss the existence of all order derivatives of a complex function.

(f) Find Res $f(z)$ at $z = 0$ where $f(z) = \frac{z - 3}{z^2} \sin \frac{1}{1 - z}$.

(g) State Rouche's theorem.

(h) Define conformal mapping.

2. Answer any *four* questions : 4×4

(a) Suppose that $f(z) = u(x, y) + iv(x, y)$ and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then prove that the first order partial derivatives of u and v must exist at (x_0, y_0) and they must satisfy the Cauchy Riemann equations :

$$u_x = v_y\,;\ u_y = -v_x \text{ at } (x_0, y_0).$$

Also, prove. $f'(z_0) = u_x + iv_x$ at (x_0, y_0).

(b) Evaluate the integral $\int_C \frac{f(z)+f(-1/z)}{(z-i)^2}\,dz$ where C is the simple closed contour $|z-i| = \frac{1}{2}$, in counter clock-wise sense and $f(z)$ is analytic in $|z-i| \le 1$.

(c) If a function $f(z)$ is continuous on a contour C of length l and if M be the upper bound of $|f(z)|$ on C, prove that

$$\left|\int_c f(z)dz\right| \le Ml.$$

(d) Show that

$$w = \frac{5-4z}{4z-2}$$

transform $|z| = 1$ into a circle in the z-plane and hence find the centre and radius of the circle.

(e) Determine the number of roots, counting multiplicities, of the equation $2z^5 - 6z^2 + z + 1 = 0$ in the annulus $1 \le z \le 2$.

(f) Use the residues to evaluate

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}.$$

(g) If $f(z)$ is an entire function and $|f(z)| \le M|z|^m$, where M is a strictly positive real and m is a positive integer, then show that $f(z)$ is a polynomial of degree less than or equal to m.

(h) State and prove the Jordan's Lemma.

3. Answer any *two* questions : 2×8

(a) (i) Classify the singularity $z = 0$ of the function

$$f(z) = \frac{\cosh(z^3) - 1}{z^7}$$

in terms of removable, pole and essential singularity. In case $z = 0$ is a pole, specify the order of the pole.

(ii) Evaluate the residue of the function

$$f(z) = \frac{\cosh(z^3) - 1}{z^7} \text{ at } z = 0.$$

(iii) Using Part-(ii), evaluate

$$\int_C \frac{\cosh(z^3) - 1}{z^7}\, dz,$$

where $C : |z| = 1$ taken in the positive direction.

4+2+2

(b) (i) Write the Taylor's and Laurent's series representation of a function $f(z)$ by stating necessary condition/s for each of the series. Hence discuss under what condition/s the Laurent's series of the said function reduces to Taylor's series of the said function.

(ii) Find the Laurent series expansion of

$$f(z) = \frac{z}{(z-1)(z-3)}$$

in the domain $0 < |z| < 1$. Finally decide whether the resulting series is Laurent's or Taylor's. 5+3

(c) Let $w = f(z) = u(x, y) + iv(x, y)$ be defined in an open region R such that the partial derivatives of u, v are continuous in R and $\frac{\partial(u, v)}{\partial(x, y)} \neq 0$ in R. If the mapping $w = f(z)$ is conformal in R, then show that $f(z)$ is holomerphic in R and $f'(z) \neq 0$ in R.

(d) (i) Using the method of residues, evaluate

$$\int_0^\infty \frac{x^{-a}}{x+1}\, dx \ (0 < a < 1).$$

(ii) Let $w = f(z) = \frac{az+b}{cz+d}$ is a bilinear transformation.

Then find the inverse of this transformation. Is it a again bilinear ? Also find the determinant of both the transformations. 6+2

[Internal Assessment — 10 Marks]
