2017

M.Sc. Part–I Examination

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

PAPER—I

Full Marks : 100

Time : 4 Hours

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answers wherever necessary.

Write the answer to questions of each group in Separate answer booklet.

Group—A

(Real Analysis)

[Marks : 40]

Answer *Q. No. 1* and any *three* from *Q. No. 2* to *Q. No. 6.*

1. Answer any *one* of the following : 1×1

(a) Define R-S integral in terms of R-S sum.

(b) Give an example of an unconstable set which is a null set.

2. (a) Let f be R–S integrable w.r.t α on $[a, b]$ and $a < c < b$. Then show that f is R–S integrable on $[a, c]$ w.r.t α. Also, prove that $\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$. 6

(b) Let F : $[a, b] \to R$ be a function defined by

$F(x) = \int_a^x f(t) d\alpha(t), x \in [a, b]$, where f is R–S integrable w.r.t.α on $[a, b]$. Then show that F is a function of bounded variation on $[a, b]$. 4

(c) Evaluate : $\int_2^5 (x + [x]) \, d(x^3)$ 3

3. (a) Prove that the variation function V of a function $f : [a, b] \to R$ of bounded variation is continuous at a point $c \in [a, b]$ if and only if f is continuous at c. 6

(b) Let V be the set of all functions of bounded variations on $[a, b]$. Then show that V is a vector spale w.r.t. pointwise addition and pointwise multiplication. 6

4. (a) Define Contor set. Show that it is unconstable. 6

(b) Let $A_1, A_2 A_3$ .... be pairwise disjoint measurable subsets of [a. b]. Then show that $\bigcup_{n=1}^{\infty} A_n$ is measurable and $m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n)$. 4

(c) Show that every non-empty open set of real numbers has positive measure. 3

5. (a) Let f_1 and f_2 be two measurable functions defined on $[a, b]$ and $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Then show that $\phi[f_1(x), f_2(x)]$ is a measurable function on $[a, b]$. 5

(b) Prove that every bounded Riemann integrable function is Lebesgue integrable and the two integrals are equal. 5

(c) If f and g are Lebesgue integrable on $[a, b]$, then show that $(f-g)$ is Lebesgue integrable on $[a, b]$. 3

6. (a) Define Lebesgue integral for unbounded functions.

Let $f : [0, 1] \to \mathbb{R}$ be a function such that $f(x) = 0$ for every x in the canter set P_o and $f(x) = k$ for x in each of the deleted intervals of length $\frac{1}{3^k}$ in $P_o^c = [0, 1] - P_o$

during the formation of cantor set. Show that f is Lebesgue integrable on [0, 1] and also find its value.

(b) Establish a necessary and sufficient condition for a bounded function f to be Lebesgue integrable on $[a, b]$. 5

Group—B

(Complex Analysis)

[Marks : 30]

Answer *all* questions.

1. Answer any *two* questions : 2×2

(a) Let, $f(z) = Re\ z$, then evalute

$$\int_c f(z)\,dz \quad \text{where } c : y = x,\ 0 \le x \le 1$$ 2

(b) Show that the function $u = \frac{1}{2}\ln\left(x^2 + y^2\right)$ is harmonic. 2

(c) Find the residue of $f(z) = \frac{1}{z^3(z+4)}$ at its poles for $c : |z + 2| = 3$. 2

2. Answer any *four* questions : 4×5

(a) Find Laurent series of the function

$$f(z) = \frac{1}{(z-1)(z-3)}$$

When (i) $D_1 : 0 < |z| < 1$

(ii) $D_2 : 1 < |z| < 3$. 3+2

(b) Evaluate $I = \int_{C:|z|=3} \frac{e^{-z}}{(z-1)^2}\,dz$ 5

(c) $f(z) = (z^2 - 2)\, e^{-x}.\, e^{-iy}$

Check whether $f(z)$ is entire function or not ? 5

(d) Find the residue of $f(z) = e^{1/z^2}$ at its poles and hence evaluate $\int_C f(z)\,dz$, $c : |z| = 1$. 5

(e) Let $f(z)$ be a function defined by

$$f(z) = \begin{cases} \frac{(\bar{z})^2}{z}, & z \ne 0. \\ 0, & z = 0 \end{cases}$$ 5

Show that $f'(0)$ does not exists. 5

(f) Find the image of the interior of the circle $C : |z - 2| = 2$ under the mobius transformation

$$w = f(z) = \frac{z}{2z-8}$$ 5

3. Answer any *one* question : 1×6

(a) Evaluate : $I = \int_0^\infty \frac{x^2 dx}{(x^2+1)(x^2+4)}$. 6

(b) Evaluate $I = \int_0^\pi \frac{\cos 2\theta}{1-2a\cos\theta + a^2}, |a|<1$. 6

Group—C

(*Ordinary Differential Equations*)

[Marks : 30]

Answer any *two* questions :

10. (a) Establish the generating function for Legendre's polynomial. 6

(b) Find the general solution of the ODE

$$(1-z^2)\ w''(z) - zw'(z) + 4w = 0$$

in series form using Frobenius method. Show that the equation has a solution which is a polynomial in z. 5+2

(c) Deduce the integral representation of confluent hypergeometric function. 2

11. (a) If λ_1, λ_2 ... are the positive zeros of Bessel's function $J_n(z)$, then prove that

$$\int_0^1 z\, J_n(\lambda_p z)\, J_n(\lambda_q z)\, dz = \begin{cases} 0 & \text{if } p \neq q \\ \frac{1}{2}\left[J_n^1(\lambda_q)\right]^2 & \text{if } p = q \end{cases}$$ 8

(b) Prove that $(2n+1)P_n(z) = P'_{n+1}(z) - P'_{n-1}(z)$,

The symbols have their usual meaning. 4

(c) Prove that $\log(1+z) = zF(1, 1, 2; -z)$, where F is hypergeometric function. 3

12. (a) Prove the following

(i) $J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\left[-\frac{\cos z}{z} - \sin z\right]$

(ii) $\frac{d}{dz}\left[z^{-n} J_n(z)\right] = -z^{-n} J_{n+1}(z)$ 3+3

(b) For non-negative integer n, prove that

$(z^2-1)\left[Q_n(z)P_n'(z) - P_n(z)Q_n'(z)\right]$ is a constant,

where $P_n(z)$ and $Q_n(z)$ are two solutions of Legendre's differential equation. 4

(c) Deduce confluent hypergeometric differential equations from the hypergeometric differential equation by suitable substitution. 5
