2016

MATHEMATICS

[Honours]

PAPER – II

Full Marks : 90

Time : 4 hours

The figures in the right hand margin indicate marks

Candidates are required to give their answers in their own words as far as practicable

Illustrate the answers wherever necessary

[NEW SYLLABUS]

GROUP – A

(*Real Analysis*)

[*Marks* : 35]

1. Answer any *one* question : 15 × 1

(*a*) (*i*) Let $I = (a, b)$ be bounded open interval and $f : I \to R$ be a monotone decreasing

(*Turn Over*)

function on I. If f is bounded above on I then prove that

$$\lim_{x \to a+} f(x) = \sup\{f(x) : x \in I\}.$$ 5

(*ii*) If $f : [a, b] \to R$ be continuous on the closed and bounded interval $[a, b]$, then prove that f is bounded on $[a, b]$. 5

(*iii*) Let $I \subset R$ be an interval and $f : I \to R$. Let there be a positive real number M such that

$$|f(x_1) - f(x_2)| \le M|x_1 - x_2|$$

for any $x_1, x_2 \in I$. Prove that f is uniformly continuous on I. Prove that the function

$$f(x) = \frac{1}{x-1}, x \in (1, 2]$$

is not uniformly continuous on $(1, 2]$.

3 + 2

(*b*) (*i*) Find the upper and lower limits of the sequence

$$\left\{(-1)^n + \sin\frac{n\pi}{4}\right\}.$$ 5

(*ii*) Test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$$ 5

(*iii*) A function $f : R \to R$ is defined by

$$f(x) = x \text{ if } x \text{ is rational}$$
$$= \sin x \text{ if } x \text{ is irrational.}$$

Show that f is differentiable at 0 and $f'(0) = 1$. 5

2. Answer any *two* questions : 8×2

(*a*) (*i*) Use Taylor's theorem to prove that

$$1+\frac{x}{2}-\frac{x^2}{8} < \sqrt{1+x} < 1+\frac{x}{2} \text{ if } x > 0.$$ 4

(*ii*) Let $c \in R$ and f and g be two functions such that $f(c) = g(c) = 0$, $g(x) \neq 0$ in some deleted neighbourhood of c. If f and g are differentiable at c and $g'(c) \neq 0$ then prove that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}.$$ 4

(*b*) (*i*) Evaluate

$$\lim_{n \to \infty} \left(1 - \frac{1}{5n}\right)^{n+1}.$$ 4

(*ii*) Prove that the function

$$f(x) = \sqrt{\sin x},\ x \in [0, \pi]$$

is continuous on $[0, \pi]$. 4

(*c*) (*i*) Examine the convergence of the series

$$x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{x^5}{5} + \cdots,\ x > 0.$$ 4

(*ii*) If

$$\lim_{x \to a} f(x) = l,\ (l \neq 0)$$

then prove that there exists some neighbourhood of 'a', at every point of which $f(x)$ will have the same sign as that of l. 4

3. Answer any *one* question : 4 × 1

(*a*) Deduce the reduction formula for

$$\frac{1}{(a+b\cos x)n},$$

for positive values of n. 4

(*b*) If

$$f(x) = (x-a)^m (x-b)^n$$

where m and n are positive integers, show that c in Rolle's theorem divides the segment $[a, b]$ in the ratio $m : n$. 4

GROUP – B

(*Several Variable and Applications*)

[*Marks* : 20]

4. Answer any *two* questions : 8 × 2

(*a*) (*i*) Verify that the double limit

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{3x^2y^2}{2x^2y^2+5(x-y)^2}$$

does not exist. 4

(*ii*) Show that the function

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

is continuous at (0, 0). 4

(*b*) (*i*) Show that for the function

$$f(x,y)=\begin{cases}\dfrac{x^2y^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0),\end{cases}$$

$f_{xy}^{(0,0)}=f_{yx}^{(0,0)}$, even though the conditions of Schwartz's theorem and also of Young's theorem are not statisfied.

2 + 2 + 2

(*ii*) Let

$$f(x,y)=\begin{cases}x, & |y|<|x|\\ -y, & |y|\geq|x|\end{cases},$$

prove that $f(x, y)$ is not differentiable at (0, 0). 2

(*c*) (*i*) Find the Jacobian $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$ where

$$x = r\cos\theta\cos\phi$$

$$y = r\sin\theta\sqrt{1-m^2\sin^2\phi}$$

$$z = r\sin\phi\sqrt{1-n^2\sin^2\theta} \text{ and}$$

$$m^2 + n^2 = 1$$ 4

(*ii*) Define envelope of a curve. Obtain the envelope of the circles drawn upon the radii vectors of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ as diameter.}$$ 4

5. Answer any *one* question : 4 × 1

(*a*) Determine the pedal equation of the curve

$$r^m = a^m \sin m\theta + b^m \cos m\theta.$$ 4

(*b*) Define curvature of a curve. For the equiangular spiral $r = ae^{\theta\cot\alpha}$, prove that the radius of curvature subtends a right angle at the pole. 4

GROUP – C

(*Analytical Geometry for Two Dimensions*)

[*Marks* : 20]

6. Answer any *two* questions : 8 × 2

(*a*) Prove that three normals can be drawn to a parabola from a givem point and the ordinates of the feet of the normals is zero. Also show that the feet of the normals lie on a rectangular hyperbola. 3 + 2 + 3

(*b*) Reduce the equation

$$x^2 + 4xy + y^2 - 2x + 2y + 6 = 0$$

to its canonical form. Determine the nature of the conic and its eccentricity. 5 + 1 + 2

(*c*) If $lx + my = 1$ is a chord of the circle $x^2 + y^2 = a^2$ which (chord) subtends an angle 45° at the origin, show that

$$4\left\{a^2\left(l^2+m^2\right)-1\right\}=\left\{a^2\left(l^2+m^2\right)-2\right\}^2.$$ 8

7. Answer any *one* question : 4 × 1

(*a*) The polar of the point P with respect to the circle $x^2 + y^2 = a^2$ touches the circle $4x^2 + 4y^2 = a^2$. Show that the locus of P is the circle $x^2 + y^2 = 4a^2$. 4

(*b*) Show that the straight line $r\cos(\theta - \alpha) = p$ touches the conic

$$\frac{l}{r} = 1 + e\cos\theta \text{ if } (l\cos\alpha - ep)^2 + l^2\sin^2\alpha = p^2.$$ 4

GROUP – D

(*Differential Equation* - I)

[*Marks* : 15]

8. Answer any *one* question : 15 × 1

(*a*) (*i*) Solve :

$$xy\left\{\left(\frac{dy}{dx}\right)^2 - 1\right\} = \left(x^2 - y^2\right)\frac{dy}{dx}.$$ 5

(*ii*) Obtain the complete primitives and singular solutions of the Clairaut's equation

$$(y-1)p - xp^2 + 2 = 0$$ 5

(*iii*) If M and N are both homogeneous functions of x and y of same degree n then prove that

$$\frac{Mdx + Ndy}{Mx + Ny}, \ (Mx + Ny \neq 0)$$

satisfy the condition of integrability. 5

Or

Show that the equation

$$(P + Qx)\frac{dy}{dx} = R + Qy$$

where P, R are homogeneous functions in x and y of degree n and Q is a homogeneous functions in x and y of degree m can be solved by the substitution $y = vx$. 5

(*b*) (*i*) State the conditions for which the solution of the differential equation $\frac{dy}{dx} = f(x, y)$ exists uniquely. Show that $\frac{dy}{dx} = 2\sqrt{y}$, $y(0) = 0$ has non unique solution. 2 + 3

(*ii*) Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \frac{e^{-x}}{x^2}$$ 5

(*iii*) What do you mean by eigenvalue of a differential equation ? Find the eigen-values and the eigenfunctions for the differential equation

$$\frac{d^2y}{dx^2} + \lambda y = 0$$

satisfying the boundary condition

$$y(0) = 0 \text{ and } y(1) = 0.$$ 5
