# OLD

# 2016

## Part-II 3-Tier

## MATHEMATICS

## (General)

### PAPER—II

*Full Marks* : 90

*Time* : 3 Hours

*The figures in the right-hand margin indicate full marks.*

*Candidates are required to give their answers in their own words as far as practicable.*

### Group—A

### (*Differential Calculus*)

### [ Marks : 45 ]

**1.** Answer any *one* question : 1×15

(a) (i) Prove that $\sqrt{2}$ is an irrational number. 4

(ii) Show that the set

$$\left\{1,\ 1+\frac{1}{2},\ 1+\frac{1}{2}+\frac{1}{2^2},\ldots,\ 1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}},\ldots\right\}$$ is

bounded both above and below but the least upper bound is not in the set. 1+1+1

(iii) Prove that $f(x) = |x|, \forall x \in [-1,1]$ is continuous but not differentiable in $[-1, 1]$. 2+2

(iv) Show that

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots+\frac{1}{\sqrt{n^2+n}}\right)=1.$$ 4

(b) (i) Evaluate $\lim_{x\to 0}(1+x)^{\frac{1}{x}}$. 4

(ii) Prove that a convergent sequence determines its limit uniquely. 3

(iii) To verify that the harmonic sequence $\left\{\frac{1}{n}\right\}$ converges to zero. 3

(iv) What is monotonic sequence? Prove that the sequence $\{x_n\}$ defined by $x_1 = \sqrt{2}$, $x_{n+1} = \sqrt{2x_n}$ for $n \geq 1$, converges to 2. 2+3

**2.** Answer any *one* question : 1×8

(a) (i) State Cauchy's Mean Value Theorem. Verify Cauchy's Mean Value Theorem for the function $f(x) = e^x$, $g(x) = e^{-x}$ in $[0,1]$. 2+3

(ii) If $r^2 = x^2 + y^2 + z^2$ and $V = r^3$ , prove that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 12r$$ 3

(b) (i) Show that $\frac{x}{1+x} < \log(1+x) < x$ , if $x > 0$. 4

(ii) Find the $12^{th}$ derivative of $\frac{1}{x^2 + a^2}$. 4

3. Answer any *four* questions : 4×4

(a) Find the points on the curve $y = x^2 - 4x + 9$, the tangents at which pass through the origin.

(b) Find at what values of x the function

$f(x) = x^3 - 3x^2 + 6x + 3$ has maximum or minimum.

(c) Find the asymptotes of the curve $xy^2 - yx^2 = x + y + 1$.

(d) Find the radius of curvature and equation of the circle of curvature of the curve $x^2 = 4ay$.

(e) Find the envelop of the family of lines $y = mx + c$ where the parameters are connected by $m^2 + c^2 = 16$

(f) Verify the Schwarz Theorem for the following function at (0,0) :

$$f(x,y)=\begin{cases}\dfrac{x^2y^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

4. Answer any *three* questions : 3×2

(a) Verify Rolle's theorem for $f(x) = (x+2)^3 (x-3)^4$ in $(-2, 3)$.

(b) Expand $e^x$ as an infinite series in powers of x by the use of Maclaurin's Theorem. Convergence is not essential here.

(c) State Cauchy's root test for a convergence of a series.

(d) Give an example of a discontinuous function along with its points of discontinuity.

(e) State Cauchy's General Principle on Convergence.

## Group—B

*( Integral Calculus )*

[ Marks : 27 ]

5. Answer any *one* question : 1×15

A. (a) Evaluate any *two* : 2×4

(i) $\int \frac{(x+1)dx}{x(1+xe^x)^2}$

(ii) $\int \frac{dx}{5+4\sin x}$

(iii) Find f (x) if $f'(x) = e^x(\sin x - \cos x)$ and f (0) = 1

(b) Find the reduction formula for the integral $\int_0^{\frac{\pi}{2}} \cos^n x\, dx$, hence find the value of the integral $\int_0^{\frac{\pi}{2}} \cos^7 x\, dx$. 7

B. (a) (i) Define Gamma function. Show $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. 1+2

(ii) Prove that $\int_0^{\infty} e^{-x^4} x^2 dx \times \int_0^{\infty} e^{-x^4} dx = \frac{\pi}{8\sqrt{2}}$. 4

(b) Answer any *two* : 2×4

(i) Evaluate $\int_2^4 f(x)\, dx$ where $f(x) = |x-2| + |x-3|$.

(ii) $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx$

(iii) Find the value of $\lim\limits_{n\to\infty}\sum\frac{n^2}{\left(n^2+r^2\right)^{3/2}}$.

6. Answer any *one* question : 1×8

a. (i) Find the length of perimeter of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$.

(ii) Find the area included between the curve $y^2(2a-x)=x^3$ and its asymptote.

b. (i) The circle $x^2+y^2=a^2$ revolves about the x-axis. Show that the surface area of the sphere thus generated is $4\pi a^2$.

(ii) Find the volume generated by the revolution of $r=2a\cos\theta$ about the initial line.

7. Answer any *one* question : 1×4

(a) Evaluate $\iint\left(x^2+y^2\right)dxdy$ over the region enclosed by the triangle having vertices (0,0), (1,0) and (1,1).

(b) Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}xyz\,dxdydz$.

## Group—C

(*Differential Calculus*)

[Marks : 18]

**8.** Answer any *two* questions : 2×8

a. (i) Solve $(x + 2y - 3)\,dy = (2x - y + 1)\,dx$.

(ii) Solve $p^2 + 2xp - 3x^2 = 0$ where $p = \frac{dy}{dx}$. 4+4

b. (i) Solve $\cos^2 x \frac{dy}{dx} + y = \tan x$.

(ii) Solve $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 2\log x$. 4+4

c. (i) Find the eigen values and eigen functions of the differential equation $\frac{d^2v}{du^2} + \mu v = 0 \ (\mu > 0)$, satisfying the boundary conditions $v(0) = 0 = v(1)$.

(ii) Solve the following simultaneous equations

$\frac{dy}{dx} + 2y - 3z = x$ and $\frac{dz}{dx} + 2z - 3y = e^{2x}$. 4+4

**9.** Answer any *one* question : 1×2

(a) **Find the differential equation of the system of circles of constant radius 'a' with their centres on the x-axis.**

(b) **Find the integrating factor of the differential equation**

$$\frac{dy}{dx}+\frac{4x}{1+x^2}y=\frac{1}{\left(x^2+1\right)^3}$$

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