2016

M.Sc. Part–I Examination

APPLIED MATHEMATICS WITH OCEANOLOGY AND COMPUTER PROGRAMMING

PAPER—I

Full Marks : 100

Time : 4 Hours

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Illustrate the answers wherever necessary.

Write the answer to questions of each group in Separate answer booklet.

Group—A

(*Real Analysis*)

[Marks : 40]

Answer *Q. No. 1* and any *three* from *Q. No. 2* to *Q. No. 6.*

1. Answer any *one* of the following : 1×1

(a) Define a measurable function.

(b) When a bounded function on [a, b] is said to be Lebesgue integrable on [a, b] ?

(Turn Over)

2. (a) Prove that a function $f : [a, b] \to \mathbb{R}$ is integrable with respect to α on [a, b] if and only if for every $\varepsilon > 0$, there exists a partition P of [a, b] such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$ 6

(b) Consider the sum (2+5+7+9+10+13+15). Show that the above sum can be represented as Riemann-Stieltjes integral. 4

(c) Evaluate : $\int_{-2}^{1} (3x-7)\,d(2|x|-5)$. 3

3. (a) Establish a necessary and sufficient condition for a function $f : [a, b] \to \mathbb{R}$ to be of bounded variation. 6

(b) Let $f(x) = |x-1| + 2,\ x \in [-1, 3]$. Show that f is a function of bounded variation on [–1, 3]. Determine the total variation on [–1, 3] and the variation function on [–1, 3]. 5

(c) Determine the points of discontinuities of the variation function of the following function with bounded variation defined on [–1, 10] : 2

$$f(x) = \begin{cases} 7 + x^2, & -1 \le x < 0 \\ x + 2, & 0 \le x \le 1 \\ 5, & 1 < x < 2 \\ \sin x, & 2 \le x < 3 \\ e^x + 71, & 3 \le x \le 4 \\ x^3 - 2x + 1, & 4 < x \le 10. \end{cases}$$

4. (a) If A and B are measurable subsets of [a, b], then prove that both $A \cup B$ and $A \cap B$ are measurable sets on [a, b]. Also prove that :

$$m(A) + m(B) = m(A \cup B) + m(A \cap B).$$ 5

(b) Prove that every continuous function f(x) defined on [a, b] is a measurable function. 5

(c) Show that,

$$f(x) = 5 \text{ for rational } x \text{ in } [1, 4]$$
$$= 3 \text{ for irrational } x \text{ in } [1, 4]$$

is a measurable function on [1, 4]. 3

5. (a) Let f be bounded and Lebesgue integrable on [a, b] and g be bounded such that f = g a.e. on [a, b]. Then show that g is also Lebesgue integrable and the value of their Lebesgue integral are same. 5

(b) Let A_1 and A_2 be subsets of [a, b]. Then show that $m^*(A_1) + m^*(A_2) \ge m^*(A_1 \cup A_2) + m^*(A_1 \cap A_2)$. 5

(c) Evaluate $\int_{2}^{7} (7x + 5[x])\,dx^3$. 3

6. (a) Prove that if the bounded function f(x) on [a, b] is Lebesgue integrable on [a, b] then $|f(x)|$ is also so on [a, b]. Also show that

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx.$$ 5

(b) Check whether the function f(x) defined by

$$f(x)=\begin{cases}2+\frac{3}{x}, & 0<x\le 1\\ 8, & x=0\end{cases}$$

is Lebesgue integrable on [0, 1].

If the integral exists, then find its value. 5

(c) State the following theorems : 3

(i) Monotone Convergence Theorem ;

(ii) Bounded Convergence Theorem.

Group—B

(Complex Analysis)

[Marks : 30]

Answer *all* questions.

7. Answer any *two* questions : 2×2

(a) Find the principal argument (A×Z) of the complex number $-1+i$. 2

(b) Locate and name the singularity of

$$f(z)=\frac{z}{\left(z^2+4\right)^2}$$ 2

(c) If f(z) and g(z) are analytic at z_0 and $f(z_0)=g(z_0)=0$ but $g'(z_0)\neq 0$, then prove that

$$\lim_{z\to z_0}\frac{f(z)}{g(z)}=\frac{f'(z_0)}{g'(z_0)}.$$ 2

8. Answer any *four* questions : 4×5

(a) Show that, under suitable conditions, to be stated by you,

$$f'(a)=\frac{1}{2\pi i}\int_C\frac{f(z)dz}{(z-a)^2},$$

where C is a closed contour surrounding the point $z=a$. 1+4

(b) Show that $u(x, y)=\sin hy \sin y$ is harmonic and then find the harmonic conjugate v, where $f(z)=u+iv$ is analytic. 5

(c) Show that $w=\frac{5-4z}{4z-2}$ transforms $z=1$ into a circle in the w-plane. Find the centre and radius of the circle. 5

(d) Evaluate the following integral : 5

$$I=\int_C\frac{3z^2+2}{(z-1)\left(z^2+9\right)}dz,\ C:|z-2|=2.$$

(e) If $z = re^{i\theta}$ and $f(z) = u(r, \theta) + iv(r, \theta)$, obtain the Cauchy-Riemann relation in terms of r and θ. 5

(f) (i) Prove that $\bar{Z}$ is nowhere differentiable function.

(ii) Prove that Z = 0 is a removable singularity of $\frac{z - \sin z}{z^3}$. 2+3

9. Answer any *one* question : 1×6

(a) Evaluate : $I = \int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$. 6

(b) Evaluate by the method of contour integration :

$\int_0^{\infty} \frac{x^{a-1}}{1+x} dx, \; 0 < a < 1.$ 6

Group—C

(Ordinary Differential Equations)

[Marks : 30]

Answer any *two* questions : 2×15

10. (a) Show that when n is a positive number, $J_n(z)$ is the coefficient of t^n in the expansion of $e^{z/2(t - 1/t)}$ in ascending and descending powers of t. Also show that $J_n(z)$ is the coefficient of t^{-n} multiplied by $(-1)^n$ in the above expansion. Use it or otherwise prove that

$ZJ_n'(Z) = Z J_{n-1}(Z) - n J_n(Z).$ 5+3

(b) Show that :

$$1 + 3P_1 + 5P_2 + 7P_3 + \cdots + (2n+1)P_n = \frac{d}{dz}[P_{n+1} + P_n]$$

where $P_n(Z)$ be the Legendre polynomial of degree n. 4

(c) Define the following terms in connection with the 2nd order differential equation :

(i) Ordinary point ;

(ii) regular singular point. 3

11. (a) Establish Rodrigue's formula for Legendre polynomial. 4

(b) Find the general solution of the equation $2Z(1 - Z)\omega''(Z) + \omega'(Z) + 4\omega(Z) = 0$, by Frobenius method about Z = 0 and show that the equation has a solution which is a polynomial in Z. 6

(c) If m < n show that

(i) $\int_{-1}^{1} Z^m P_n(Z)\, dz = 0$;

(ii) $\int_{-1}^{1} Z^n P_n(Z)\, dz = \frac{2^{n+1}}{\lfloor 2n+1} (\lfloor n)^2$ 5

12. (a) What is Bessel's functions of order n ? State for what values of n the solutions are independent of Bessel's equation of order n. 3

(b) Under a suitable transformation to be considered by you, prove that hypergeometric function can be reduced to confluent hypergeometric function. 4

(c) Find the Legendre polynomial $P_n(z)$ by solving the following differential equation :

$$\left(1-z^2\right)\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+n(n+1)w=0 .$$

Hence show that :

$$4z^3 - 5z^2 - 3z = \frac{8}{5}P_3(z)-\frac{10}{3}P_2(z)-\frac{3}{5}P_1(z)-\frac{5}{z}P_0(z).$$

5+3
